COHOMOLOGY SUPPORT LOCI FOR LOCAL SYSTEMS AND HIGGS BUNDLES

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X will stand for a smooth projective variety over $\mathbb{C}$ throughout these notes. To simplify some statements, it will be convenient (but not essential) to also assume that

$$(*)\ H^2(X,\mathbb{Z}) \text{ is torsion free.}$$

To fix notation recall: The Picard variety $Pic^0(X)$ is the set of line bundles with trivial first Chern class (the Chern class can be taken $H^2(X,\mathbb{C})$, since we are assuming that $H^2(X,\mathbb{Z})$ is torsion free) This is an abelian variety of dimension $q(X) = \dim H^1(X,\mathcal{O}_X)$. Green and Lazarsfeld have introduced the cohomology support loci

$$S^i(X) = \{L \in Pic^0(X) \mid H^i(X,L) \neq 0\}$$

along with certain variants. They proved the following amazing theorem [GL]:

Theorem 0.1 (Green-Lazarsfeld)**.** $S^i(X)$ *is a union of translates of abelian subvarieties.*

Simpson later showed that these are in fact translates by points of finite order.

There are now several proofs of this theorem in addition to the original. I would like to explain a couple of "topological" proofs, due to Simpson [S3] and the speaker [A1]. I also want to say something about the nonabelian version of this in part II.

1. LOCAL SYSTEMS

The first step is to find a topological version of $S^i(X)$. By a *local system*, we mean a locally constant sheaf. Let $\tilde{X} \to X$ denote the universal cover. Given a representation $\rho : \pi_1(X) \to GL(V)$, we have a diagonal action of $\pi_1(X)$ on $V \times \tilde{X}$. Then we can form the local system V_ρ of locally constant sections of $(V \times \tilde{X})/\pi_1(X) \to X$.

We have the standard fact:

Proposition 1.1. *This construction yields an equivalence between the category of finite dimensional representations of* $\pi_1(X)$ *and the category of local systems,*

The set of rank one local systems is parameterized by the set of characters

$$Char(X) = Hom(\pi_1(X),\mathbb{C}^*)$$

Since we are assuming that $H^2(X,\mathbb{Z})_{tors} = H_1(X,\mathbb{Z})_{tors} = 0$. $Char(X)$ is a product of $\mathbb{C}^*$'s or an affine torus (otherwise this is only true for the identity component). The (topological) cohomological support locus is

$$\Sigma^i(X) = \{\rho \in Char(X) \mid H^i(X,\mathbb{C}_\rho) \neq 0\}$$

The topological analogue of theorem 0.1 is

Theorem 1.2. *$\Sigma^i(X)$ is a finite union of translates of subtori.*

It is convenient to state a slight refinement.

Theorem 1.3.

$$\Sigma^i_m(X) = \{\rho \in Char(X) \mid \dim H^i(X, \mathbb{C}_\rho) \geq m\}$$

is a finite union of translates of subtori.

This theorem implies theorem 0.1. I want to outline the argument. Given a character ρ, let $L_\rho = \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}_\rho$. This is a flat line bundle which means that it has locally constant transition functions. Since $c_1(L_\rho)$ can be written in terms of the logarithmic derivatives of these functions, it must vanish, i.e. $L_\rho \in Pic^0(X)$. The map $\rho \mapsto L_\rho$ coincides with the natural map

$$Char^0(X) \cong H^1(X, \mathbb{C}^*) \to H^1(X, \mathcal{O}_X^*) = Pic(X)$$

For now, we will restrict our attention to unitary characters

$$Uchar(X) = Hom(\pi_1(X), U(1))$$

which is a real torus by our assumption (*). From the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \xrightarrow{e^{2\pi i}} & U(1) & \longrightarrow & 1 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \xrightarrow{e^{2\pi i}} & \mathcal{O}^* & \longrightarrow & 1
\end{array}$$

we get

$$\begin{array}{ccccccc}
H^1(X, \mathbb{Z}) & \longrightarrow & H^1(X, \mathbb{R}) & \longrightarrow & Uchar(X) & \xrightarrow{0} & H^2(X, \mathbb{Z}) \\
\downarrow{=} & & \downarrow{\cong} & & \downarrow{L} & & \downarrow{=} \\
H^1(X, \mathbb{Z}) & \longrightarrow & H^1(X, \mathcal{O}) & \longrightarrow & Pic(X) & \xrightarrow{c_1} & H^2(X, \mathbb{Z})
\end{array}$$

The top rightmost horizontal map is zero since $c_1(L_\rho) = 0$. The second vertical map is an isomorphism by Hodge theory. Therefore

$$Uchar(X) \cong Pic^0(X)$$

as real tori. So now we see that $\Sigma^i_m(X) \cap Uchar(X)$ has the required structure, but it is still not clear what this has to do with $S^i(X)$. The missing ingredient, which I will now explain, is Hodge theory with coefficients in a flat unitary bundle.

Lemma 1.4. *There exists a unique differential operator $\nabla : L_\rho \to \Omega^1_X \otimes L_\rho$ such that ∇ is $\mathbb{C}$-linear, satisfies the Leibnitz rule and $\ker(\nabla) = \mathbb{C}_\rho$.*

Locally $\mathbb{C}_\rho \cong \mathbb{C}$ and ∇ is the exterior derivative d. It follows that we can extend this to a resolution

$$0 \to \mathbb{C}_\rho \to L_\rho \xrightarrow{\nabla} \Omega^1_X \otimes L_\rho \xrightarrow{\nabla} \Omega^2_X \otimes L_\rho \ldots$$

Thus we get a spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_X \otimes L_\rho) \Rightarrow H^{p+q}(X, \mathbb{C}_\rho)$$

By slight modification of the usual Hodge theoretic arguments, we have

Theorem 1.5. *When ρ is unitary, this degenerates at E_1 i.e.*

$$H^i(X, \mathbb{C}_\rho) \cong \bigoplus_{p+q=i} H^q(X, \Omega_X^p \otimes L_\rho)$$

Proof. The connection ∇ extends to C^∞ forms $\Gamma(\mathcal{E}^\bullet \otimes L_\rho)$ and its $(0,1)$ part is $\bar{\partial}$. The left and right sides are given by $H^*(\Gamma(\mathcal{E}^\bullet \otimes L_\rho), \nabla)$ and $H^*(\Gamma(\mathcal{E}^\bullet \otimes L_\rho), \bar{\partial})$ respectively. The Kähler identities extend to these operators and guarantee an isomorphism of cohomology. □

Note that this is usually false for nonunitary ρ. We will come back this issue in a bit. Define

$$S_m^{pq}(X) = \{L \in Pic^0(X) \mid \dim H^q(X, \Omega_X^p \otimes L) \geq m\}$$

Corollary 1.6.

$$\Sigma_m^i(X) \cap Uchar(X) = \bigcup_{\sum m_p = m} S_{m_p}^{p,i-p}(X)$$

We are now in a position to prove a stronger form of theorem 0.1 modulo theorem 1.3.

Theorem 1.7 (Green-Lazarsfeld)**.** *All of these sets $S_m^{pq}(X)$ are unions for translates of subabelian varieties.*

Proof. Note that an algebraic subvariety of $Pic^0(X)$ is an abelian subvariety if and only if it is a real subtorus. Therefore is suffices to show any irreducible component V of $S_m^{pq}(X)$ is a translate of a subtorus. Suppose that this is not the case for some V. We can assume that m is maximal i.e. $V \not\subseteq S_{m+1}^{pq}$. Set $i = p + q$. If V is not contained in any other $S_1^{a,i-a}(X)$, with $a \neq p$, then we can find a neighbourhood U of a general point of V disjoint from these $S_1^{a,i-a}(X)$. Then a point $x \in U \cap \Sigma_m^i(X)$ would have to lie in V. Therefore V would be a component of $\Sigma_m^i(X)$, leading to a contradiction. In general, for each a, let $m_a \geq 0$ be the maximal value for which $V \subseteq S_{m_a}^{a,i-a}(X)$. Set $M = \sum m_a$. Then $V \subset \Sigma_M^i(X)$ can again be seen to be component by similar argument. □

2. Higgs line bundles

As we saw unitary characters correspond to elements of Pic^0, but what about arbtrary characters $\rho : \pi_1(X) \to \mathbb{C}^*$? We can decompose ρ as a product of an $\mathbb{R}^+$ character $|\rho|$ and a unitary one $\rho/|\rho|$. Via the isomorphism

$$Hom(X, \mathbb{R}^+) \stackrel{\log}{\to} H^1(X, \mathbb{R}) \cong H^0(X, \Omega_X^1)$$

we see that $|\rho|$ corresponds to a 1-form. Thus we see that a ρ gives rise to a Higgs[1] line bundle, which is an element of the cotangent bundle $T^*Pic^0(X) = Pic^0(X) \times H^0(X, \Omega_X^1)$. This correspondence sets up an isomorphism

$$Char(X) \cong T^*Pic^0(X) \tag{1}$$

as *real* Lie groups, but it is very far from an isomorphism of complex Lie groups. As complex manifolds they are very different. The space on the left is Stein, whereas the space on the right cannot be Stein since it has a nontrivial compact subvariety.

[1]Hitchin first introduced these kinds of objects into mathematics as analogues of the Yang-Mills-Higgs fields from physics. So some authors refer to these as Hitchin pairs.

Thus we have a C^∞ manifold with different complex structures. This is unusual, but not unprecedented. The quaternions $\mathbb{H} = \mathbb{R}+\mathbb{R}i+\mathbb{R}j+\mathbb{R}k$ have several complex structures, such as i and j. And in fact, this gives local model. More precisely, when the tangent spaces of $Char(X)$ are endowed with both complex structures, they become an $\mathbb{H}$-module. Kähler manifolds with this kind of quaternonic structure are called *hyper-Kähler*.

Given a character ρ, let $(L_\rho, \theta = \theta_\rho)$ denote the corresponding Higgs bundle. Since $\theta^2 = 0$, we have complex

$$L_\rho \xrightarrow{\theta} \Omega^1_X \otimes L_\rho \xrightarrow{\theta} \Omega^2_X \otimes L_\rho \ldots$$

There is a very nice generalization of theorem 1.5 to nonunitary bundles.

Theorem 2.1 (Simpson [S2])**.**

$$H^i(X, \mathbb{C}_\rho) \cong \mathbb{H}^i(\ldots \Omega^1_X \otimes L_\rho \xrightarrow{\theta} \Omega^2_X \otimes L_\rho \ldots)$$

Proof. This again comes down to an appropriate extension of the Kähler identities. □

Corollary 2.2. *$\Sigma^i(X)$ is compatible with both complex structures coming from* (1)*.*

Corollary 2.3. *$\Sigma^i(X)$ is invariant under the $\mathbb{C}^*$ action $(L,\theta) \mapsto (L, t\theta)$.*

Both of these properties can be exploited to give a proof of theorem 1.3 by using the either of the next two lemmas.

Lemma 2.4 (Deligne-Simpson)**.** *A closed subset of $Char(X)$ holomorphic for both complex structures is a union of translates of subtori.*

Proof. The key point is that a function f which is holomorphic for both structures is linear. This follows by observing that the second derivative $H = (\partial^2 f/\partial x_i \partial x_j)$ viewed as a quadratic form must vanish identically:

$$ijH(u,v) = iH(u,jv) = H(iu,jv) = jH(iu,v) = jiH(u,v) = -ijH(u,v)$$

It's fairly easy to deduce from this that the pullback of a "doubly holomorphic" subset to the universal cover of $Char(X)$ is union of linear subvarieties. □

For the second proof, note that when the $t \in \mathbb{R}^+ \subset \mathbb{C}^*$ action on $T^*Pic^0(X)$ is transfered to $Char(X) = (\mathbb{C}^*)^n$, it can be written explicitly as $(r_1 e^{is_1}, \ldots) \mapsto (r_1 e^{its_1}, \ldots)$.

Lemma 2.5 (A)**.** *Then a Zariski closed subset of $(\mathbb{C}^*)^n$ invariant under $\mathbb{R}^+$ is necessarily a union of translates of subtori.*

Proof. One checks that Zariski closures of orbits are translates of tori. □

Part II

3. Higher rank bundles

As a first step toward formulating a nonabelian Green-Lazarsfeld theorem, we need to replace $Pic^0(X)$ by the moduli space of vector bundles with trivial Chern classes and rank r. Unfortunately, as Mumford discovered long ago, the moduli space generally won't exist as a scheme when $r > 0$. There are couple of ways around this:

(1) Restrict the class of bundles so as to eliminate the pathologies, or
(2) work with something more general than a scheme like a stack.

Although stacks are no longer as obscure as they once were, I prefer to use option (1). The good bundles are called *stable*. To simply things, I'll concentrate on the relevent case, where the Chern classes are trivial. A vector bundle V with $c_i(V) = 0$ is stable if for any proper coherent subsheaf $W \subset V$, $\deg W < 0$ (the degree is measured with respect to a fixed embedding $X \subset \mathbb{P}^N$). If V is a unitary flat bundle corresponding to an irreducible representation, then V is a stable vector bundle with trivial Chern classes. This follows from the fact that the curvature of V is zero and that it decreases for sub-bundles. This turns out to be the only kind of example:

Theorem 3.1 (Narasimhan-Seshadri (dim = 1), Donaldson, Uhlenbeck-Yau)**.** *A stable vector bundle with trivial Chern classes is unitary and flat.*

The set of irreducbile representations $\pi_1(X) \to U(r)$ modulo conjugacy forms a real algebraic variety $Uchar(X, r)$. This can be given the structure of a complex variety by identifying it with:

Theorem 3.2 (Mumford, Gieseker, Maruyama)**.** *The (coarse) moduli scheme $M(X, r)$ of stable bundles on X with rank r and $c_i = 0$ exists.*

Of course, $M(X, 1) = Pic^0(X)$, and we recover the earlier bijection $Uchar(X, 1) \cong Pic^0(X)$. In general, $M(X, r)$ is a much more complicated space, even locally. The infinitesimal properties can be studied in terms of deformations of V, which can be thought of as a family of vector bundles V_ϵ such that $V_0 = V$. The underlying C^∞ bundle won't change in a deformation. So we can view this as given by a deformation of the Cauchy-Rieman operator

$$\bar{\partial}_\epsilon = \bar{\partial} + \Phi(\epsilon) = \bar{\partial} + \epsilon\Phi'(0) + \dots$$

The first order deformation is determined by the endomorphism valued $(0, 1)$-form $\phi = \Phi'(0)$. Differentiating the integrability equation $\bar{\partial}_\epsilon^2 = 0$ shows that ϕ is $\bar{\partial}$-closed. Therefore it defines a class in $H^1(X, End(V))$, and in fact:

Proposition 3.3. *The Zariski tangent space of $M(X, r)$ at V is $H^1(X, End(V))$.*

For $M(X, r)$ to be smooth, we would need to know that tangent vectors extend to (formal) arcs. In general, there are obstructions. Expanding $\bar{\partial}_\epsilon^2 = 0$ to second order shows that

$$[\phi, \phi] = 0 \qquad (2)$$

It follows that $M(X, r)$ won't be smooth at V unless this holds for all tangent vectors ϕ. In principle, there could be third and higher order obstructions. Remarkably, these vanish:

Theorem 3.4 (Goldman-Millson [GM], Nadel)**.** *$M(X, r)$ has quadratic singularities. More precisely, the local analytic structure of this scheme is given by* (2)*.*

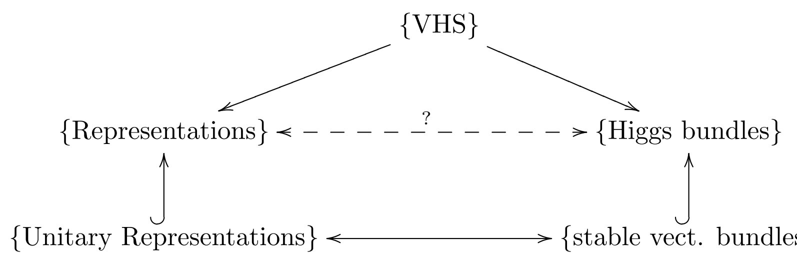
If V is unitary flat, we have an isomorphism

$$H^1(X, End(V)) \cong \overline{H^0(X, \Omega^1_X \otimes V)}$$

by Hodge theory. If ϕ is a first order deformation satisfying (2), then $\theta = \bar{\phi} \in H^0(\Omega^1 \otimes End(V))$ is element satisfying $\theta^2 = 0$. Such a pair (V, θ) is called a Higgs bundle. As natural as this development sounds, this not how they first arose in higher dimensions. Simpson wanted to extend the Narasimhan-Seshadri picture to nonunitary representations. The key examples that gave the clues to such a correspondence came from variations of Hodge structure. Given a family of smooth projective varieties $f : Y \to X$, the associated ith variation of Hodge structure [V] is a package giving rise to the following data:

- A (generally nonunitary) monodromy representation $\pi_1(X) \to Aut(H^i(Y_x))$.
- A vector bundle $V = \bigoplus_{p+q=i} \bigcup_x H^q(Y_x, \Omega^p)$ (more precisely $V = \oplus R^q f_* \Omega^p_{Y/X}$).
- An operator $\theta : H^q(Y_x, \Omega^p) \to H^{q+1}(Y_x, \Omega^{p-1})$ associated to cupping with the Kodaira-Spencer class $\kappa \in H^1(Y_x, T)$. One has $\theta^2 = 0$.

Thus we get a picture



The bottom arrow on the right sends $V \mapsto (V, 0)$. It's not obvious that a dotted arrow should exist, but that's what was proved, at least after imposing stability assumptions. In this setting, the Higgs field θ is exactly the datum encoding the nonunitarity. For Higgs bundles, stability is defined by restricting to θ-invariant subsheaves. A direct sum of stable Higgs bundles is called polystable. The key result is what Simpson calls the nonabelian Hodge theorem:

Theorem 3.5 (Simpson [S2, S4])**.** *The moduli space $M_{Higgs}(X, r)$ of polystable rank r Higgs bundles with trivial Chern classes exists. There is a correspondence between semisimple representations of $\pi_1(X)$ and stable Higgs bundles with trivial Chern classses,. This gives a homeomorphism between $M_{Higgs}(X, r)$ and the space $Char(X, r)$ of conjugacy classes of rank r semisimple representations.*

Remark 3.6. *The special case of unitary rep. $\to$ (stable v.b.$, \theta = 0$) was pretty direct. Unfortunately, in general the correspondence is quite complicated in both directions, since it involves solving certain nonlinear PDEs. In one direction one constructs a harmonic map, and in the other a Hermitian-Einstein metric. This builds on earlier work of Corlette, Donaldson...*

Remark 3.7. *When $r = 1$, we had*

$$M_{Higgs}(X, 1) = T^* Pic^0(X) = Pic^0(X) \times H^0(X, \Omega^1)$$

There are partial analogues of this for $r > 1$. We can define the fake cotangent bundle $T^*M(X,r) \subset M_{Higgs}(X,r)$ as the locus of pairs (V,θ) where V is stable. This maps to $M(X,r)$ and coincides with the cotangent bundle over the smooth locus. There is also a map generalizing the second projection when $r = 1$. This is called the Hitchin map. Given (V,θ) the coefficients of the characteristic polynomial can be interpreted as an element of $H^0(X, S^*\Omega^1_X)$. The Hitchin map

$$h : M_{Higgs}(X,r) \to \bigoplus_{i=1}^{r} H^0(X, S^i\Omega^1_X)$$

is given by the characteristic polynomial.

Theorem 3.8 (Simpson). *h is proper.*

Hitchin introduced and studied this map when X was a curve. The fibre over 0 contains $M(X,r)$, while the general fibres are Jacobians of some other curves called spectral curves. This allows one to get a handle on the structure of this space. This also points the way to the desired result.

4. Cohomology support loci II

We can define $\Sigma^i_m(X,r)$ as the locus

$$\{L \in Char(X,r) \mid \dim H^i(X,L) \geq m\}$$

Let $\Sigma^i_{Higgs,m}(X,r)$ denote the image of this set in $M_{Higgs}(X,r)$.

Theorem 4.1 ([A3]). *Let Σ be an irreducible component of $\Sigma^i_{Higgs,m}(X,r)$ with reduced structure, and let $\tilde{\Sigma} \to \Sigma$ denote its normalization. Then the general fibres of the restriction of h to $\tilde{\Sigma}$ are abelian varieties.*

Remark 4.2. *Note that the theorem applies to the whole of M_{Higgs}.*

In addition to the ideas already discussed, a key tool is a result of Hitchin [H] that h is a Lagrangian fibration when X is a curve. One then argues that this property persists for $h_{\tilde{\Sigma}}$. Thus by symplectic geometry, the nonsingular fibres are tori.

Of course, this is just the first step. There are many more questions.

(1) What can one say about the singularities of these sets? (My conjecture is that they're quadratic c.f. [GM]).
(2) Are there good bounds on dimension similar to the rank one case? In other words, are there good generic vanishing theorems?
(3) The statement involved moving to $M_{Higgs}(X,r)$. Is there a way to formulate things more topologically on $Char(X,r)$?
(4) It would be useful to allow boundary divisors. This is partially understood for $r = 1$ [A2]. What about in general? (I think the time is ripe to work this out, since Mochizuki [M] has made a recent breakthrough in understanding the analytic issues in this situation)
(5) The natural conjecture here, which refines a conjecture of Simpson for M_{Higgs} and also what's known for $r = 1$, is that $\Sigma^i_m(X,r)$ should contain "motivic" points, i.e. points corresponding to direct summands of geometric variations of Hodge structures. This would be very hard!

(6) All of this may be way too analytic for some people's taste. So I would like to point out there is a characteristic p proof of theorem 0.1, and there are some algebraic analogues of Simpson's stuff [F, OV]. Maybe, someone should should study the higher cohomology support loci from this point of view.

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