

FORMAL PROOFS WITH QUANTIFIERS

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An argument in predicate logic is *valid* precisely when the conclusion is true in a model or interpretation whenever the assumptions are. This is similar to the definition of validity for propositional logic using truth tables, with one important difference: the number of rows of a truth table is finite, but there may be *infinitely* models. To show that an argument is not valid it is enough to find one model where the argument fails. But to show that an argument is valid, we really need a *proof*. Either a formal proof as explained here, or a semiformal proof to convince us it is valid in every model.

In this supplement, I'll continue to follow the terminology of Barwise-Etchemenedy rather than Rubin.

1. RULES FOR \forall

We allow all the rules given earlier, but continuing the pattern, we add a pair of rules for \forall . In Rubin these go by the name Universal Specialization and Universal Generalization respectively. The basic ideas are best illustrated first by an example. Suppose we are given the commutative law

$$\forall x \forall y (x + y = y + x)$$

and ask to prove

$$\forall y (1 + y = y + 1)$$

A semiformal proof would proceed as follows. First drop the quantifiers on the first line, keeping in mind that x and y could be anything. Then take $x = 1$ to get $1 + y = y + 1$. Since y is still arbitrary, this must hold for all y . The general rules for adding and dropping \forall are as follows.

Rule 1 (\forall -elimination). *Given a formula $\forall x \phi(x)$, we can deduce $\phi(-)$ for any constant or variable “-” substituted for x .*

Rule 2 (\forall -introduction). *Given a formula $\phi(x)$, we can deduce $\forall x \phi(x)$, provided that x doesn't occur as a free (unquantified) variable in any previous line.*

A few things to keep in mind: The first rule says that \forall 's can be dropped if they occur at the beginning of the formula, but not in general. E.g. for example it would be incorrect to deduce $\neg P(x)$ from $\neg \forall x P(x)$. In the second rule, the proviso about x is meant to avoid the following trap. Suppose earlier we placed some extra condition on x (for example that x is even if we are talking about integers), then it would be wrong to conclude that what we have proved is true for all x .

Example 2. *In the game of chess, each piece is either white or black:*

$$\forall x ((W(x) \vee B(x)) \wedge \neg(W(x) \wedge B(x)))$$

We would like to conclude from this that any piece which is not white is black:

$$\forall x (\neg W(x) \rightarrow B(x))$$

This is intuitively clear, but here's a formal proof:

1	$\forall x((W(x) \vee B(x)) \wedge \neg(W(x) \wedge B(x)))$	Assumption
2	$(W(x) \vee B(x)) \wedge \neg(W(x) \wedge B(x))$	\forall -elim
3	$W(x) \vee B(x)$	\wedge -elim
4	$\neg W(x)$	\rightarrow -intro assumption
5	$W(x)$	\vee -elim, assmp 1
6	<i>Contra</i>	<i>Contra</i> -intro 4,5
7	$B(x)$	<i>Contra</i> -elim 6
8	$B(x)$	\vee -elim, assmp 2
10	$\neg W(x) \rightarrow B(x)$	\rightarrow -intro 4-10
11	$\forall x(\neg W(x) \rightarrow B(x))$	\forall -intro 10

3. RULES FOR \exists

We add two final rules for \exists , called Existential Generalization and Existential Proof in Rubin. Again we start with an algebraic problem which is perhaps more familiar. Suppose you haven't seen complex numbers before (and perhaps you haven't), and are told that

$$\exists x(x^2 = -1)$$

along with the usual rules of algebra, and are asked to prove

$$\exists x(x^2 + 2x + 2 = 0)$$

You can proceed as follows. The first statement is that there is some x satisfying $x^2 = -1$. Let's give this value of x a name i . With some work, we can see that $x = -1 + i$ gives a solution to $x^2 + 2x + 2 = 0$, which proves the second statement. Here's the general scheme:

Rule 3 (\exists -introduction). *Given a formula $\phi(-)$ with " $-$ " a variable or constant, we can deduce $\exists x\phi(x)$ for any variable not in x .*

Rule 4 (\exists -elimination). *Given a formula $\exists x\phi(x)$, we can deduce $\phi(c)$ for some constant not used previously.*

The following are ex. 1 and 4 of p. 173 of Rubin done from this point of view.

Example 4. *Prove $\exists x(\neg Ax)$ given $\forall x(Ax \wedge Bx \rightarrow Cx)$ and $\exists x(Bx \wedge \neg Cx)$.*

1	$\forall x(Ax \wedge Bx \rightarrow Cx)$	Assump
2	$\exists x(Bx \wedge \neg Cx)$	
3	$Bc \wedge \neg Cc$	\exists -elim 2, c temp const.
4	Bc	\wedge -elim 3
5	$\neg Cc$	ditto
6	$Ac \wedge Bc \rightarrow Cc$	\forall -elim 1
7	Ac	\neg -intro temp assmp
8	$Ac \wedge Bc$	\wedge -intro 3,7
9	Cc	\rightarrow -elim 8, 6
10	<i>Contra</i>	5,6
11	$\neg Ac$	\neg -intro 7-10
12	$\exists x(\neg Ax)$	\exists -intro

Example 5. Prove $\exists x(Rx \wedge Sx)$ given $\forall x(Px \rightarrow Qx)$, $\exists x(Rx \wedge \neg Qx)$, $\forall x(Rx \rightarrow Px \vee Sx)$.

I'll give fewer details for this.

1	$\forall x(Px \rightarrow Qx)$	Assump
2	$\exists x(Rx \wedge \neg Qx)$	
3	$\forall x(Rx \rightarrow Px \vee Sx)$	
4	$Rc \wedge \neg Qc$	\exists -elim 2
5	$Pc \rightarrow Qc$	\forall -elim 1
6	$Rc \rightarrow Pc \vee Sc$	\forall -elim 3
7	Rc	4
8	$\neg Qc$	4
9	$Pc \vee Sc$	6,7
10	Pc	\vee -elim 9, assmp 1
11	Qc	5, 9
12	<i>Contra</i>	8,11
13	Sc	<i>Contra</i> -intro
14	Sc	\vee -elim 9, assmp 2
15	Sc	\vee -elim, 10-14
16	$Rc \wedge Sc$	7, 15
17	$\exists x(Rx \wedge Sx)$	\exists -intro

5.1. Homework. On page 173 of Rubin, do problems: A2, A3, A5, A7, B2, B3 using the rules given here and previous supplement.