Chapter 2

Morphisms

2.1 Locally ringed spaces

A ringed space is a pair \((X, \mathcal{O}_X)\) consisting of a topological space and a sheaf of (commutative) rings on it. A locally ringed space is a ringed space such that the stalks, \(\mathcal{O}_{X,x}\), are local rings for all \(x \in X\). We have a basic example

Example 2.1.1. Any affine scheme is a locally ringed space.

There are a couple of important examples outside algebraic geometry.

Example 2.1.2. Let \(X\) be a topological space, and let \(C_X\) be the ring of complex valued (or real valued, it doesn’t really matter) continuous functions. Under pointwise addition and multiplication, this gives the structure of a sheaf of rings, thus we get a ringed space \((X, C_X)\). The ring \(C_{X,x}\) is the ring of germs of continuous functions. Let \(m_x\) be the set of germs of continuous functions with \(f(x) = 0\). Then \(f \notin m_x\) is easily seen to be invertible, so that \(m_x\) is the unique maximal ideal.

Example 2.1.3. Let \(X\) be a \(C^\infty\) manifold, and \(C_X^\infty\) the sheaf of \(C^\infty\) functions. Then \((X, C_X^\infty)\) is a locally ringed space for similar reasons to the last example.

2.2 Morphisms

In order to motivate the definition, consider a continuous map \(F : X \to Y\) between two topological spaces. Let \(f \in C_Y(U)\), then \(F^\# = f \circ F\) gives a continuous function in \(C_X(F^{-1}U)\). Let us record some properties of this operation \(F^\#\).

1. It is a ring homomorphism \(C_Y(U) \to C_X(F^{-1}U)\).

2. If \(x \in X\) and \(y = F(x)\), we get an induced homomorphism \(C_{Y,y} \to C_{X,x}\) which takes \(m_y\) to \(m_x\). (Such a homomorphism is called local.)
We now extend to locally ringed spaces. Given a pair of such spaces \((X, \mathcal{O}_X), (Y, \mathcal{O}_Y)\), a morphism from the first to the second consists of

1. A continuous map \(F : X \to Y\)
2. Ring homomorphisms \(F^\# : \mathcal{O}_Y(U) \to \mathcal{O}_X(F^{-1}U)\) such that

\[
\begin{array}{ccc}
\mathcal{O}_Y(U) & \xrightarrow{F^\#} & \mathcal{O}_X(F^{-1}U) \\
\downarrow & & \downarrow \\
\mathcal{O}_Y(V) & \xrightarrow{F^\#} & \mathcal{O}_X(F^{-1}V)
\end{array}
\]

commutes whenever \(V \subset U\)
3. If \(x \in X\) and \(y = F(x)\), the induced homomorphisms \(\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}\) are local.

This is a lot of information to keep track of. It will help to break it up into parts. Given a pair of (pre)sheaves of groups, rings..., \(\mathcal{F}, \mathcal{G}\) on \(Y\), a morphism \(\eta : \mathcal{F} \to \mathcal{G}\) is collection of homomorphisms \(\eta : \mathcal{F}(U) \to \mathcal{G}(U)\) such that

\[
\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\eta} & \mathcal{G}(U) \\
\downarrow & & \downarrow \\
\mathcal{F}(V) & \xrightarrow{\eta} & \mathcal{G}(V)
\end{array}
\]

commutes whenever \(V \subset U\). We also introduce an operation which takes a (pre)sheaf on \(X\) to a (pre)sheaf on \(Y\). Given a presheaf \(\mathcal{F}\) on \(X\), we define the direct image \(F_*\mathcal{F}\) on \(Y\) by

\[F_*\mathcal{F}(U) = \mathcal{F}(F^{-1}U)\]

**Lemma 2.2.1.** If \(\mathcal{F}\) is a sheaf then so is \(F_*\mathcal{F}\).

**Proof.** Exercise! \(\square\)

Thus a morphism is a pair consisting of a continuous map \(F : X \to Y\) and a morphism of sheaves of rings \(F^\# : \mathcal{O}_Y \to F_*\mathcal{O}_X\).

**Example 2.2.2.** Given a continuous map \(F : X \to Y\) of topological spaces, we get a morphism \((F, F^\#) : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)\) as above.

**Example 2.2.3.** Given a \(C^\infty\) map of manifolds, we get a morphism \((F, F^\#) : (X, \mathcal{C}_X^\infty) \to (Y, \mathcal{C}_Y^\infty)\) in a similar way.

We come to the main example. Let \(\phi : R \to S\) be a ring homomorphism. This induces a continuous map \(F : \text{Spec}\, S \to \text{Spec}\, R\) defined by \(F(p) = \phi^{-1}p\). Given \(r \in R\), \(F^{-1}D(r) = D(\phi(r))\).
Theorem 2.2.4.

(a) There exists a morphism of locally ringed spaces \((F, F^\#) : \text{Spec } S \to \text{Spec } R\) such that \(F^\# : \mathcal{O}(D(r)) \to \mathcal{O}(D(\phi(r)))\) is given by the homomorphism \(R[r^{-1}] \to S[\phi(r)^{-1}]\) induced by \(\phi\).

(b) Any morphism of locally ringed spaces from \(\text{Spec } S \to \text{Spec } R\) is induced by a unique ring homomorphism.