Chapter 4

Fibre products

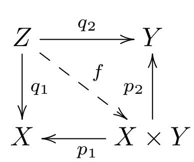
4.1 Universal properties

In classical geometry, we can take the product of varieties simply to be the cartesian product. The identification $\mathbb{A}_k^n \times \mathbb{A}_k^m = \mathbb{A}_k^{n+m}$ shows that this is a reasonable thing to do. However, with schemes we redefine

$$\mathbb{A}_k^n = \operatorname{Spec} k[x_1, \ldots, x_n]$$

and the cartesian product no longer works even as sets!

We have to understand what the product really means. Let us start with sets X, Y, the product is a new set $X \times Y$ with projections $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ which is universal in the sense that given any other set Z with projections $q_1 : Z \to X, q_2 : Z \to Y$, we have a unique map $f : Z \to X \times Y$, namely $f(z) = (q_1(z), q_2(z))$, such that



commutes. This can be used to define the product in any category. Note that there is no guarantee that the product exists, but it will be unique up to isomorphism if it does. Here a few examples.

Example 4.1.1. *The product in the category of groups is simply the usual product.*

Example 4.1.2. *The product in the category of topological spaces is the cartesian product with its product topology.*

There are no surprises here, so this is reassuring. Let us consider a slightly less familiar example. Given a set S, we can form a new category of "sets over

S". The objects are pairs (X, f) consisting of a set and a map $f : X \to S$. The morphisms $(X, f) \to (Y, g)$ are maps $h : X \to Y$ such that $f = g \circ h$. Given (X, f) and (Y, g), their product in this new category is given by fibre product, or pullback,

$$X \times_S Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$$

together with map $(x, y) \mapsto f(x)$. To get a feeling for this, let us look at some special cases. Let $S = \{*\}$ be a one element set. Every set X has a unique map to S, namely the constant map (such an object is called terminal). Then the fibre product is just the cartesian product. Next consider S arbitrary, but $Y = \{*\}$. A map $g : Y \to S$ is determined by choosing the value of $*$, call this s. Then

$$X \times_S Y \cong f^{-1}(s)$$

which is just the fibre of X over s. These example help explain the name. The construction generalizes both the usual product and the fibre of a map.

Let's take a look at the dual notion called the coproduct in some category. This simply the product in the category with arrows reversed. More explicitly, given objects X, Y, the coproduct (if it exists) is an object $X \coprod Y$ with morphisms $p_1 : X \to X \coprod Y$ and $p_2 : Y \to X \coprod Y$ which is universal in the sense that given $X \to Z$ and $Y \to Z$ there exists a unique morphism $X \coprod Y \to Z$ making the obvious diagram commute. Here a few examples.

Example 4.1.3. *The coproduct of sets X and Y, is the disjoint union $X \coprod Y$.*

Example 4.1.4. *The coproduct of groups G, H is the free product $G * H$. For example the free product $\mathbb{Z} * \mathbb{Z}$ is the free group on two letters. This is very different from the product $\mathbb{Z} \times \mathbb{Z}$.*

The dual of the fibre product = pullback is sometimes called the pushout. For groups, the pushout exists and is given by the free product with amalgamation which I won't explain if you don't already know it. Here is the key example.

Proposition 4.1.5. *Let R be a commutative ring, and let S and T be commutative R-algebras. This simply means that we fix ring homomorphisms $R \to S$ and $R \to T$. Then the pushout exists, and is given by tensor product $S \otimes_R T$ viewed as an R-algebra.*

Proof. We have R-algebra homomorphisms $p_1 : S \to S \otimes T$ and $p_2 : T \to S \otimes T$ given by $p_1(s) = s \otimes 1$ and $p_2(t) = 1 \otimes t$. Given an R-algebra P with homomorphisms $q_1 : S \to P$ and $q_2 : T \to P$, we need a unique homomorphism $h : S \otimes R \to P$ such that $h \circ p_i = q_i$. The homomorphism $h : S \otimes R \to P$ given by

$$h(\sum s_t \otimes t_i) = \sum q_1(s_i) \otimes q_2(t_i)$$

does the trick (and is the only one). $\square$

4.2 Fibre products of schemes

Theorem 4.2.1. *Fibre products exist in the category of schemes.*

Before proving this, let us understand some consequences. First of all, it tells us that products exist. Since $\operatorname{Spec}\mathbb{Z}$ is the terminal object in the category of schemes. The product is $X \times Y = X \times_{\operatorname{Spec}\mathbb{Z}} Y$. Secondly, given a point $s \in S$ of a scheme, and a morphism $f : X \to S$, we want to put a natural scheme structure on the preimage $f^{-1}s$, which at moment is just a set. We proceed as follows.The residue field $k(s)$ is the residue field $\mathcal{O}_{S,s}/m_s$ of the local ring at s. There is a canonical morphism $\operatorname{Spec} k(s) \to S$ such that the unique point on the left maps to s. To define it, let $\operatorname{Spec} A$ be an affine open set containing it. Then $\operatorname{Spec} k(s) \to S$ is the composite $\operatorname{Spec} k(s) \to \operatorname{Spec} A \to S$, where the first map corresponds to the ring map $A \to A_s \to k(s)$. Given a morphism $f : X \to S$, the scheme theoretic fibre is the fibre product $X \times_S \operatorname{Spec} k(s)$. It is not yet clear that the underlying set is $f^{-1}s$ so we will have come back to this later.

We outline the main ideas of the proof.

Step1: Since the category of affine schemes is equivalent to the opposite of the category of commutative rings, then fibre products exist in the first category because pushout exists in the second. More explicitly, if $X = \operatorname{Spec} A, Y = \operatorname{Spec} B$, and $S = \operatorname{Spec} C$, then $X \times_S Y = \operatorname{Spec} A \otimes_C B$. This already tells us that

$$\begin{aligned}\mathbb{A}^n_k \times_{\operatorname{Spec} k} \mathbb{A}^m_k &= \operatorname{Spec} k[x_1, \ldots, x_n] \otimes k[y_1, \ldots, y_m] \\ &= \operatorname{Spec} k[x_1, \ldots, x_n, y_1, \ldots y_m] = \mathbb{A}^{m+n}_k\end{aligned}$$

as we hoped.

Step 2: Suppose that Y, S are affine and that X is an open subscheme of an affine scheme $\bar{X}$. This means that $X \subset \bar{X}$ is an open subset and the structure sheaf of X is the restriction of $\mathcal{O}_{\bar{X}}$. Then $\bar{X} \times_S Y$ exists by step 1. Let X' denote the preimage of X under the projection $\bar{X} \times_S Y \to \bar{X}$. This is an open set which can regard as an open subscheme. We have morphisms $X' \to X$ and $X' \to Y$ which, as can be checked, satisfies the universal property of $X \times_S Y$. We can conclude that $X \times_S Y$ exists and is an open subscheme of $\bar{X} \times_S Y$.

Before giving the third step, we have to explain the idea of gluing. Given two sets U_1 U_2, with subsets $U_{12} \subset U_1$, $U_{21} \subset U_2$ and a bijection $\phi : U_{12} \to U_{21}$, we glue the sets as follows. Define the relation $\sim$ on disjoint union $U_1 \coprod U_2$ by $x \sim y$ if $\phi(x) = y$. This generates an equivalence relation $\approx$. More explicitly, if we define $\phi_{12} = \phi, \phi_{21} = \phi^{-1}$ and $\phi_{11} = id$, $\phi_{22} = id$. Then given $x \in U_i, y \in U_j$, $x \approx y$ if and only if $y = \phi_{ij}(x)$. We can now glue by

$$U_1 \cup_\phi U_2 = U_1 \coprod U_2 / \approx$$

In principle, we can glue 3 or more sets U_i. The data we would need is a collection of subsets $U_{ij} \subset U_i$ and bijections $\phi_{ij} : U_{ij} \to U_{ji}$. We want to identify $x \in U_i$ with y in U_j if and only if $y = \phi_{ij}(x)$ (which means in particular that $x \in U_{ij}$). This forces the following identities called cocycle conditions $\phi_{ii} = id$,

$\phi_{ji} = \phi_{ij}^{-1}$, and $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ where we are suppressing the restrictions symbols. Given data satisfying the last condition, we can glue to form

$$X = \coprod_i U_i / \approx$$

as before. This construction is quite flexible. It works if U_i is a collection of topological spaces and $\phi_{ij} : U_{ij} \to U_{ji}$ are homeomorphisms between open subsets. Then X will be a topological space, with the quotient topology. It also works if we have locally ringed spaces $(U_i, \mathcal{O}_{U_i})$ and isomorphisms

$$(\phi_{ij}, \phi_{ij}^{\#}) : (U_{ij}, \mathcal{O}_{U_i}|_{U_{ij}}) \stackrel{\sim}{\to} (U_{ji}, \mathcal{O}_{U_j}|_{U_{ji}})$$

The construction produces a new locally ringed space $(X, \mathcal{O}_X)$. Now comes the key observation, which is simply a restatement of the definition, any scheme can be produced by applying the gluing construction to a collection of affine schemes.

We need one more fact, whose proof will be left as an exercise.

Lemma 4.2.2. *Let $f : X' \to X$, $X \to S$ and $Y \to S$ be morphisms in some category. Then there is a morphism $f \times_S id : X' \times_S Y \to X \times_S Y$ such that*

$$\begin{array}{ccccc} X' & \longleftarrow & X' \times_S Y & \longrightarrow & Y \\ \downarrow f & & \downarrow & & \downarrow id \\ X & \longleftarrow & X \times_S Y & \longrightarrow & Y \end{array}$$

commutes.

We now return to the outline of the proof.

Step 3: We now suppose that S and Y are affine, but X is arbitrary. Let X be obtained by gluing affine schemes U_i via $\phi_{ij} : U_{ij} \cong U_{ji}$ as above. By step 1 and 2, both $U_i \times_S Y$ and $U_{ij} \times_S Y$ exist and the latter is an open subscheme of the former. Then $U_i \times_S Y$ together with $\phi_{ij} \times_S id : U_{ij} \times_S Y \to U_{ji} \times_S Y$ gives gluing data. Gluing these together yields the scheme $X \times_S Y$.

Steps 4, 5: Further gluing allows us to construct $X \times_S Y$ in general.

4.3 Fibres

Fix $f : X \to S$, recall that the fibre over $s \in S$ is $X \times_S \operatorname{Spec} k(s)$. We often write this as X_s. We should think of the morphism $f : X \to S$ as the family of the fibres X_s as s varies over S. We can replace S by a neighbourhood of s, so we may as well take $S = \operatorname{Spec} R$ affine. Then s is a prime ideal of R, and $k(s) = R_s / sR_s$. If $X = \operatorname{Spec} A$, then the scheme theoretic fibre is the spectrum of $A \otimes_C k(s)$. As a set this consists exactly of the prime ideals of A that contract to s in C. More generally if X is obtained by gluing affine schemes U_i, the scheme theoretic fibre over s, is the union of the fibres of U_i over s.

Let $X = \operatorname{Spec} R[x_1, \ldots, x_n]/(f_1(x), \ldots)$. Then $X_s = \operatorname{Spec} k(s)[x_1, \ldots, x_n]/(\bar{f}_i(x))$, where $\bar{f}_i(x)$ denotes the image of $f_i(x)$. To give a better sense of this, let $R = k[y_1, \ldots, y_m]$ where k is algebraically closed. We can think of f_i as polynomial in the x's and y's. A maximal ideal s (= closed point of $\operatorname{Spec} R$) is given by $(y_1 - a_1, \ldots, y_m - a_m)$ and thus corresponds to a point $a \in \mathbb{A}^m_k$. Then $X_s = k[x_1, \ldots, x_n]/(f_i(x, a))$.

Given a ring R, we define

$$U_i = \operatorname{Spec} R[\frac{x_0}{x_i}, \ldots \frac{x_n}{x_i}]$$

$$U_{ij} = D(\frac{x_j}{x_i}) = \operatorname{Spec} R[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}, (\frac{x_j}{x_i})^{-1}]$$

The rings defining U_{ij} and U_{ji} are really the same. And thus we get isomorphisms $U_{ij} \cong U_{ji}$. So we can glue these together to get a scheme which is of course $\mathbb{P}^n_R$. The fibres of U_i over $s \in \operatorname{Spec} R$ are isomorphic to the affine schemes $\mathbb{A}^n_{k(s)}$. Gluing these yields the projective space $\mathbb{P}^n_{k(s)}$.