Chapter 5

Flat familes

5.1 Some pathologies

As we explained a morphism of schemes $f : X \to S$ should be viewed as a family of schemes X_s parameterized by S. However, it is possible that various fibres can have very different properties. The simplest measure of the complexity of a scheme is its dimension:

$$\dim X = \sup\{n \mid X_n \supsetneq X_{n-1} \supsetneq \dots X_0 \text{ is a chain of irreducible closed sets}\}$$

Note that $\dim \operatorname{Spec} R$ is exactly the Krull dimension of R. So by algebra, this implies that $\dim \mathbb{A}^n_k = \dim k[x_1, \dots, x_n] = n$ as we would expect. Consider the following example.

Example 5.1.1. *Consider the projection* $\operatorname{Spec} k[x, y, z]/(y - zx) \to k[x, y]$. *The fibre over* $(x - a, y - b)$ *is*

$$\begin{cases} \operatorname{Spec} k = \textit{point} & \textit{when } a \neq 0 \\ \operatorname{Spec} 0 = \emptyset & \textit{when } a = 0, b \neq 0 \\ \operatorname{Spec} k[z] = \mathbb{A}^1_k & \textit{when } a = b = 0 \end{cases}$$

This means that the dimension can jump between $-\infty, 0$ *and* 1.

This is some sense typical. But our goal is to understand reasonable conditions where the dimension of the fibres don't jump.

5.2 Flatness

We start with some algebra. Let R be a commutative ring, and M, N R-modules. We can form a new module $M \otimes_R N = M \otimes N$, the pairing $(m, n) \mapsto m \otimes n$ is the universal R-bilinear form. In general $- \otimes N$ is right exact which means that if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact, then

$$M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$$

is exact and likewise if the order of M_i and N are switched. We say that N is flat if $- \otimes N$ is exact, or equivalently if

$$M_1 \otimes N \to M_2 \otimes N$$

is injective for every sequence. N is faithfully flat if it is flat and if $M \otimes N = 0$ implies that $M = 0$. (There other equivalent formulations in the literature but we won't use them.)

Example 5.2.1. *A free module is faithfully flat. In particular, $R[x_1, \dots, x_n]$ is faithfully flat over R. More generally a projective module (which can be taken to be direct summand of free module) is faithfully flat.*

Example 5.2.2. *The ring $N = S^{-1}R = R[S^{-1}]$ is flat as an R-module because tensoring an inclusion $M_1 \subset M_2$ with N is the same as $S^{-1}M_1 \subset S^{-1}M_2$. With trivial exceptions, N is not faithfully flat.*

At the other extreme.

Example 5.2.3. *If $I \subset R$ is a nonzero proper ideal, then R/I is not flat. In fact, if $0 \neq f \in I$, then $R/ann(f) \xrightarrow{f} R$ becomes 0 after tensoring by R/I.*

Some sense of the geometric significance of faithful flatness can be gleaned from the next lemma.

Lemma 5.2.4. *Let $A \to B$ be ring homomorphism, and suppose that B is flat as an A-module. Then B is faithfully flat if and only if $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.*

Proof. Suppose that B is faithfully flat. Let $p \in \operatorname{Spec} A$. If $B \otimes_A k(p) = 0$ then $k(p) = 0$ by faithful flatness. But this is clearly impossible. Therefore the fibre $\operatorname{Spec} B \otimes_A k(p) \neq \emptyset$.

Conversely, assume that the map on spectra is surjective. Let M be a nonzero A-module. If $p \in \operatorname{Spec} A$ is an associated prime, then M contains a submodule isomorphic to A/p. Let $q \in \operatorname{Spec} B$ lie over p. Then $M \otimes k(p) \neq 0$. Therefore $M \otimes k(q) \neq 0$ and consequently $M \otimes B \neq 0$. □

The proof of the following localization property can be found for example in Matsumura, Commutative Ring Theory. It is essential for transferring this notion to schemes.

Theorem 5.2.5. *An R-module is flat if and only if M_p is flat over R_p for each $p \in \operatorname{Spec} R$.*

Corollary 5.2.6. *If B is a flat A algebra, and $P \in \operatorname{Spec} B$ a prime lying over $p \in \operatorname{Spec} A$. Then $A_p \to B_P$ is flat.*

Proof. If $M_1 \to M_2$ is an injective map of A_p modules, then $M_1 \otimes (A_p \otimes B) \to M_2 \otimes (A_p \otimes B)$ is injective by the theorem. Localizing further to B_P won't affect injectivity. $\square$

Lemma 5.2.7. *If $A \to B$ is local homomorphism of noetherian local rings and B is flat over A, then B is faithfully flat.*

Proof. Let M be an A-module such that $M \otimes_R B = 0$, we have to show that $M = 0$. If $M \neq 0$, then it contains a nonzero finitely generated submodule M'. We have $M' \otimes_R B \subset M \otimes_R B = 0$ by flatness. Thus we can reduce to the case where M is finitely generated. Let $m \subset A$ and $n \subset B$ denote the maximal ideals. We have B/n is a field extension of A/m because the map is local.Then $M \otimes B = 0$ implies that $M \otimes B/n = 0$ and therefore that $M \otimes A/m = 0$. So by Nakayama's lemma $M = 0$. $\square$

The following "going down" theorem for flatness, will be needed later.

Theorem 5.2.8. *Suppose that $A \to B$ is a faithfully flat A-algebra. Given any pair of prime ideals $p \subset q \in \operatorname{Spec} A$ and a prime ideal $Q \in \operatorname{Spec} B$ lying over q, there exists a prime $P \subset Q$ with P lying over p.*

Proof. By the previous results $A_q \to B_Q$ is faithfully flat. Therefore $\operatorname{Spec} B_Q \to \operatorname{Spec} A_q$ is surjective. So we can find a prime Q' lying over pA_q. Then $Q = Q' \cap A$ will do the job. $\square$

5.3 Finite flat maps

An extension of rings $A \to B$ is called integral or finite, if B is finitely generated as an A-*module*. A morphism of schemes $f : X \to Y$ is called finite if there exists an affine cover $\{U_i\}$ of Y such that $f^{-1}U_i$ is affine and the morphism $f^{-1}U_i \to U_i$ is induced by a finite morphism of rings.

Proposition 5.3.1. *If $f : X \to Y$ is finite then the fibres are finite as sets.*

Proof. We can assume that this is induced by finite homomorphism $A \to B$. Then $B \otimes_A k(p)$ is necessarily finite dimensional over $k(p)$ and therefore Artinian. This implies that it has only finitely many prime ideals (cf Atiyah-Macdonald). $\square$

The converse is not true. For example, open immersions (= inclusions of open subschemes) have finite fibres but are almost never finite. The weaker property of having finite fibres is called "quasi-finiteness". The dimension $\dim B \otimes_A k(p)$ can be thought of the number of points of $f^{-1}(p)$ counted with multiplicity.

Theorem 5.3.2. *Let A be noetherian domain. Given a finite flat map $A \to B$, the number of points of the fibres counted with multiplicity is constant.*

The statement is not true without flatness.

Exercise: Let $A = k[x, y]/(y^2 - x^2(x + 1))$ be a node, $B = k[t]$ and $A \to B$ given by $x \mapsto t^2 + 1, y \mapsto z(z^2 - 1)$. Check that this is finite but conclusion of the theorem fails, so this cannot be flat.

The proof will follow from the next proposition. For this, we introduce a bit of homological algebra. We omit proofs which can be found in just about any book on the subject. Given a pair of A-modules M, N, we have a new module $Tor_1^R(M, N) = Tor_1(M, N)$ with the following properties:

1. It is functorial in both variables.
2. It is symmetric $Tor_1(M, N) \cong Tor_1(N, M)$.
3. Given an exact sequence
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
there is an exact sequence
$$\dots Tor_1(M_3, N) \to M_1 \otimes N \to M_2 \otimes N \dots$$
4. N is flat if and only if $Tor_1(-, N) = 0$ identically.

Proposition 5.3.3. *A finitely generated module over a noetherian local ring A is flat if and only if it is free.*

Proof. Let M be a finitely generated flat A-module. Let $\bar{m}_1, \dots, \bar{m}_n$ be a basis for the vector space $M \otimes k$, where k is the residue field. Lift these to elements $m_i \in M$. These generate M by Nakayama's lemma. Therefore we have an exact sequence
$$0 \to S \to R^n \to M \to 0$$
where the second map sends the ith basis vector to m_i. Then
$$Tor_1(M, k) \to S \otimes k \to k^n \to M \otimes k \to 0$$
is exact. Flatness implies that $Tor_1 = 0$. Since the last map is an isomorphism, it follows that $S \otimes k = 0$. Therefore $S = 0$ by Nakayama's lemma.

□

Proof of theorem 5.3.2. By the previous proposition $B \otimes A_p$. Therefore $\dim B \otimes k(p) = \dim B \otimes K$, where K is the field of fractions of A. □

5.4 Dimensions of fibres

A morphism of schemes $f : X \to Y$ is said to be flat if the local ring $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$ for all $x \in X$. It is called faithfully flat if f is also surjective. A scheme is called noetherian if can be covered by finitely many spectra of noetherian rings.

Theorem 5.4.1. *Let $f : X \to S$ be a morphism of noetherian schemes. Then for any $x \in X$ let $s = f(x)$, then we have*

$$\dim \mathcal{O}_{X,x} \leq \dim \mathcal{O}_{S,s} + \dim \mathcal{O}_{X,x} \otimes k(s)$$

Equality holds if f is flat.

In spite of the geometric language, the theorem is really a statement about local rings. It can be restated as follows.

Theorem 5.4.2. *Suppose that $A \to B$ is a local homomorphism of noetherian local rings, with $m \subset A$ and $n \subset B$ the maximal ideals. Then*

1. $\dim B \leq \dim A + \dim B/mB$
2. *If B is flat over A, then equality holds.*

Given a local ring A with maximal ideal m, an ideal I is called m-primary if it contains a power of I. For example, m^n is m-primary. Here is a more interesting example.

Example 5.4.3. *Let A be the localization of the cusp $k[x, y]/(y^2 - x^3)$ at (x, y). Then (x) is m-primary.*

The key result that we need from dimension theory is the following (see Atiyah-Macdonald, theorem 11.14)

Theorem 5.4.4. *If A is a noetherian local ring. If $(x_1, \ldots, x_e)$ is an m-primary ideal, then $e \geq \dim A$. There exists an m-primary ideal with exactly $\dim A$ generators.*

If $(x_1, \ldots, x_d)$ is m-primary with $d = \dim A$, we say that $x_1, \ldots, x_d$ is a system of parameters.

Proof of theorem 5.4.2. Let $x_1, \ldots, x_a \in A$ be a system of parameters of A. Let $y_1, \ldots, y_b \in B$ be elements, whose images give a system of parameters in B/mB. Then $m^M \subset (x_1, \ldots, x_a)$ for some N, and $n^N \subset m + (y_1, \ldots, y_b)$ for some M, N. Thus $n^{MN} \subset (x_1 \ldots, x_a, y_1, \ldots, y_b)$. So these elements generate an n-primary ideal. This proves 1.

Now suppose that B is flat. Let $m = p_a \supset \ldots p_0$ be a strictly decreasing chain of primes of A. Let $P_{a+b} \supset \ldots P_a \supseteq mB$ be strictly decreasing chain of prime ideals of B. By theorem 5.2.8, we have can extend this to a chain

$$P_{a+b} \supset \ldots P_a \supset P_{a-1} \supset \ldots P_0$$

with $P_i \cap A = p_i$ for $i < a$. This proves the opposite inequality $\dim B \geq \dim A + \dim B/mB$. ∎

To appreciate what the theorem tells us, suppose that $A \to B$ is a faithfully flat homomorphism of affine domains over a field k. Let $f : X = \operatorname{Spec} B \to S = \operatorname{Spec} A$ be the corresponding faithfully flat morphism of schemes. Choose $x \in \operatorname{Spec} B$ to be a maximal ideal, then $s = f(x)$ is also maximal by the Nullstellensatz. The theorem together some basic dimension theory tells us that for any irreducible component Y of X_s through x,

$$\dim Y = \dim \mathcal{O}_{X,x} \otimes k(s) = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{S,s} = \dim X - \dim S$$

This proves

Corollary 5.4.5. *Suppose that $f : X \to S$ is a faithfully flat map of varieties (viewed as schemes). Then the irreducible components of the closed fibres all have the same dimension.*

In fact, a much more general result is true. We refer to EGA IV §13 and 14 for the proof which is a lot harder.

Theorem 5.4.6. *Suppose that $f : X \to S$ is a morphism locally of finite type (which means that locally of the form $\operatorname{Spec} A[x_1, \ldots, x_n]/I \to \operatorname{Spec} A$). Then $s \mapsto \dim f^{-1}(s)$ is upper semicontinuous, i.e. $\{s \mid \dim f^{-1}(s) \geq N\}$ is closed. If S is noetherian and f is flat then all fibres have the same dimension.*