Chapter 7

Étale fundamental group

7.1 Review of usual fundamental group and Galois theory

Given nice (= connected locally path connected locally simply connected) topological space X, with a base point $x \in X$, the fundamental group $\pi_1(X, x)$ can be defined in two ways. First, we take the set of homotopy classes of loops starting and ending at x. The second is in terms of covering spaces. A covering space is a topological space Y together with a continuous map $p : Y \to X$ such that each $x \in X$ can carries an open neighbourhood such that $p^{-1}U$ is a disjoint union of open sets each mapping homeomorphically to U. We will also require Y to be connected unless stated otherwise. It turns out that there exists an covering space $P : \tilde{X} \to X$ such that any other covering space factors $\tilde{X} \to Y \to X$. The fundamental group can be identified with the group of homeomorphisms $\tilde{X} \to \tilde{X}$ which commute with the projection P. Thet role of the base point is less evident in the second construction, but it is hidden in the construction of $\tilde{X}$. It turns out that for any other point x', we have an noncanonical isomorphism $\pi_1(X, x) \cong \pi_1(X, x')$, so we often suppress the base point. If $H \subset \pi_1(X)$ is a subgroup, then $\tilde{X}/H$ gives a covering space of Y. Conversely all covering spaces arise this way. Thus

Theorem 7.1.1. *There is a bijection between the set of covering spaces of X and subgroups of $\pi_1(X)$*

This is very similar to the fundamental theorem of Galois theory which we review next. Given a field k, the separable closure k^s is the union of all separable extensions. This is contained in the algebraic closure $\bar{k}$ and it coincides with it when k is perfect. This plays the role of the universal cover. The analogy becomes clearer if we consider if we consider the map of spectra $\operatorname{Spec} k^s \to \operatorname{Spec} k$. The spectrum of any other separable extension factors through this. Even though k^s/k is generally an infinite extension, we still have a Galois group

$$Gal(k) = Gal(k^s/k) = \{\sigma \in Aut(k^s) \mid \sigma|_k = id\}$$

where $Aut(k^s)$ is the group of field automorphisms. If L/k is finite Galois extension, any $\sigma \in Gal(k)$ restricts to an automorphism of L. Thus we get a homomorphism $r_L : Gal(k) \to Gal(L/k)$. If L/k is a smaller finite Galois extension then we get a commutative diagram

$$\begin{array}{ccc} Gal(k) & \xrightarrow{r_{L'}} & Gal(L/k) \\ & \searrow^{r_L} & \downarrow r_{L/L'} \\ & & Gal(L'/k) \end{array} \tag{7.1}$$

where the vertical map is again restriction. We can define the inverse limit

$$\varprojlim Gal(L/k) = \{\sigma_L \in \prod Gal(L/k) \mid r_{L/L'}(\sigma_L) = \sigma_{L'}\}$$

This can also be characterized by a universal property. The commutativity of the above diagram says that we have a homomorphism

$$r : Gal(k) \to \varprojlim Gal(L/k)$$

Proposition 7.1.2. *r is an isomorphism.*

Proof. Any element of k^s is contained in finite Galois extension. Therefore if $r(\sigma) = 1$ then $\sigma = 1$.

Suppose $(\sigma_L) \in \varprojlim Gal(L/k)$. If $x \in k^s$, we can define $\sigma(x) = \sigma_L(x)$ for some or any, it doesn't matter, finite Galois extension L containing x. This gives an element of $Gal(k)$ which maps to (σ_L). □

The product in (7.1) carries the product topology, where each finite group is given the discrete topology. This is a compact totally disconnected topological space, so it looks like a Cantor set. If we give $Gal(k)$ the induced topology, it becomes a topological group, i.e. the group operations are continuous. More generally, a group is called profinite if it is an inverse limit of finite groups. Such a group can always be topologized in this way. We can now state the fundamental theorem of infinite Galois theory.

Theorem 7.1.3. *There is an inclusion reversing bijection between the set of closed (resp. closed normal) subgroups of $Gal(k)$ and the set of intermediate (Galois) extensions $k \subset L \subset k^s$. A field L gives rise to the closed subgroup $Gal(L) \subset Gal(k)$. A closed subgroup H corresponds to the fixed field $(k^s)^H = \{x \in k^s \mid \forall \sigma \in H, \sigma(x) = x\}$.*

7.2 Normal varieties

Given an integral domain A with a field of fractions K and extension $K \subseteq K$, the integral closure of A in L is the set of elements of L which satisfy a monic polynomial with coefficients in A. This is a ring. A is integrally closed if A coincides with its integral closure in K. Let us say that it is normal if it is a finite product of integrally closed rings. Many standard examples can be produced using the following easy result.

Theorem 7.2.1. *A unique factorization domain is integrally closed.*

Proof. Suppose A is UFD which is not integrally closed. So there is a reduced fraction $p/q \notin A$ but integral over A. Then

$$(p/q)^n = -a_{n-1}(p/q)^{n-1} - \dots$$

Clearing denominators shows that q divides p. □

A ring is normal if and only if all its local rings are. Thus we can extend this to schemes. A scheme X is normal if all of its local rings are integrally closed. A noetherian ring or scheme is regular if all its local rings are regular.

Theorem 7.2.2. *A regular noetherian scheme is normal.*

Proof. It is enough to prove that a regular noetherian local ring R is integrally closed. We give the proof when R contains its residue field. We know that $\hat{R}$ is a power series ring which is known to be a UFD and therefore integrally closed. Suppose that $f, g \in R$ and f/g is integral over R. Then it is integral over $\hat{R}$, which implies that $h = f/g \in \hat{R}$. Then $f = gh$. Write $h = \lim h_n$ with $h_n \in R$, and $h - h_n \in m^n$. Then $f = gh_n + g(h - h_n)$ shows that $f \in \bigcap_n (gR + m^n) = gR$ by Krull's intersection theorem (cf. Atiyah-Macdonald) applied to $R/(g)$. Thus $h \in R$.

□

So normallity is weaker than regularity. The following basic theorem of Serre gives some insight into the geometric significance of normallity.

Theorem 7.2.3 (Serre's normality criterion)**.** *A noetherian scheme X is integrally closed if and only if both conditions hold:*

- *(R_1) If $x \in X$ has codimension one i.e. if $\dim \mathcal{O}_{X,x} = 1$, then it is regular (of dimension one) or equivalently a discrete valuation ring. (This should be understood as say thing the singular set has codmension at least two.)*
- *(S_2) If $x \in X$ has codimension at least two, and if $f \in \mathcal{O}(U - \overline{\{x\}})$ then f extends across x after shrinking U if necessary.*

Theorem 7.2.4. *Suppose that $X \to Y$ is an étale cover of noetherian schemes.*

- *(a) If Y is regular then so is X.*
- *(b) If Y is normal then so is X.*

Proof. We can assume that $Y = \operatorname{Spec} A$, where A is a noetherian local ring with maximal ideal m. For (a) we suppose that $X = \operatorname{Spec} B$ is also local with maximal ideal n. Then we proved earlier that $\dim B = \dim A$ because B/A is finite and flat, and also that $mB = n$ because it's unramified. Thus n is generated by exactly $\dim A$ elements.

For (b), we can assume by earlier results, that $B = A[x]/(f(x))$ where f is a degree n monic polynomial such that f' is a unit in B. Let K be fraction field of A. Then $L = K[x]/(f(x))$ gives the fraction field of B. Let C be the integral closure of B. By assumption $C \cap K = A$. Since L is separable, we can choose n distinct embeddings of $\sigma_i : L \to \bar{K}$ into the algebraic closure. The $Norm(y) = \prod_i \sigma_i(y)$ takes C to $C \cap K = A$.

Choose $\lambda \in C$. Since it lies in L, we may write

$$\lambda = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}, a_i \in K$$

Thus we get n equations

$$\sigma_i(\lambda) = a_0 + a_1 \sigma_i(x) + \ldots + a_{n-1} \sigma_i(x)^{n-1}$$

These elements are integral over A. Cramer's rule gives

$$a_i = \frac{N_i}{D}$$

where

$$D = \begin{vmatrix} 1 & \sigma_1(x) & \ldots \\ 1 & \sigma_2(x) & \ldots \\ & \ldots & \end{vmatrix}$$

and N_i is obtained by replacing the ith column by $(\sigma_1(x), \sigma_2(x), \ldots)^T$. D^2 is the discriminant of f, which can be identified with $\pm Norm(f'(x))$. Therefore it is a unit in A. The expression $N_i D = a_i D^2$ lies in K and is integral over A, so it lies in A. Therefore $a_i \in A$. This implies that $\lambda \in B$. □

Fix X to be a noetherian scheme which is irreducible as a topological space. X is called integral (normal) if it can be covered by a finite union $\operatorname{Spec} A_i$, with A_i a (resp. integrally closed) integral domain. X is called of finite type over a field k if the A_i can be taken to be finitely generated k-algebras. We can redefine a variety over k to be an integral separated scheme of finite type over k. This includes all quasi-projective varieties, and some new examples as well. The fractions fields of the A_i are all the same. This common field is the called the function field of X, let's write it as $K = k(X)$. In the case of variety K has finite transcendence degree over k. Let L be a finite extension of K.

Proposition 7.2.5. *Suppose that X is integral. There exists a normal scheme Y with a morphism $\pi : Y \to X$ such that Y is obtained by gluing $\operatorname{Spec} B_i$, where B_i is the integral closure of A_i in L. When X is a variety, so is Y and the morphism $\pi : Y \to X$ is finite.*

Y is called the normalization of X in L, or simply the normalization if $L = K$. Given a normal scheme X, define the maximal unramified extension $K^{un} \supset K$ (with respect to X) as the union $\bigcup L$, as L varieties over all finite separable extensions in a fixed algebraic closure $\bar{K}$, such that the normalization $\pi : Y \to X$ is étale. The étale fundamental group

$$\pi_1^{et}(X) = Gal(K^{un}/K)$$