Math 385 Handout 4: Uncountable sets

Uncountable sets

One might get the feeling that one infinite set is as big as any other, but in fact:

Theorem. *(Cantor) The set of real numbers $\mathbb{R}$ is uncountable.*

Before giving the proof, recall that a real number is an expression given by a (possibly infinite) decimal, e.g. $\pi = 3.141592\ldots$. The notation is slightly ambigous since

$$1.0 = .9999\ldots$$

We will break ties, by always insisting on the more complicated nonterminating decimal.

Proof It suffices to prove that $\mathbb{R}$ has an uncountable subsets. We we work with numbers in the interval $I = \{x \in \mathbb{R} \,|\, 0 \leq x \leq 1\}$. We give a proof by contradiction. Suppose that I was countable, and let's say that $f : \mathbb{N} \to I$ is a one to one correspondence, say:

$$f(0) = .\mathbf{1}23\ldots$$

$$f(1) = .4\mathbf{5}6\ldots$$

$$f(2) = .78\mathbf{9}\ldots$$

$$\ldots$$

Then mark the numbers down the diagonal, and construct a new number $x \in I$ whose $n+1$th decimal is different from the $n+1$decimal of $f(n)$. Then we have found a number not in the image of f, which contradicts the fact f is onto.

Cantor originally applied this to prove that not every real number is a solution of a polynomial equation with integer coefficients (contrary to earlier hopes). We expand on this idea as follows. Say that a number is describable if there is a name (such as 5, π), or formula $1 + \sqrt{2}/3$, or perhaps a computer program, for obtaining it. The point is that the description should involve a finite number of symbols in a fixed finite alphabet. Since the set of such descriptions is countable (by hw), we obtain.

Cor. *There are real numbers which cannot be described (and in particular computed).*

This is the starting point for Cantor's theory of *transfinite* numbers. The cardinality of a countable set (denoted by the Hebrew letter $\aleph_0$) is at the bottom. Then we have the cardinallity of $\mathbb{R}$ denoted by $2^{\aleph_0}$, because there is a one to one correspondence $\mathbb{R} \to P(\mathbb{N})$. Taking the powerset again leads to a new transfinite number $2^{2^{\aleph_0}}$. This process goes on forever thanks to:

Theorem.(Cantor) For any set X, there does not exist a one to one correspondence from X to $P(X)$. *In particular, the power set $P(\mathbb{N})$ is uncountable.*

Proof We prove this by contradiction. Suppose that $f : X \to P(X)$ is a one to one correspondence. Define $C = \{x \,|\, x \notin f(x)\}$. Note that C is an element of $P(X)$, it is given as $f(y)$ for some y. Either $y \in C$ or $y \notin C$. If $y \in C$, then $y \notin f(y) = C$ which is a contradiction. So $y \notin C = f(y)$, this implies that $y \in C$, which is again a contradiction. The only way to avoid a contradiction is for f not to exist.

Homework

1. Prove that the set of all irrational real numbers is uncountable.

2. Prove that the set of functions $f : \mathbb{N} \to \mathbb{N}$ is uncountable.