Chapter 2

A crash course in homological algebra

By the 1940's techniques from algebraic topology began to be applied to pure algebra, giving rise to a new subject. To begin with, recall that a category $\mathcal{C}$ consists of a set or class[1] of objects (e.g. sets, groups, topological spaces) and morphisms (e.g. functions, homomorphisms, continuous maps) between pairs of objects $Hom_{\mathcal{C}}(A, B)$. We require an identity $1_A \in Hom(A, A)$ for each object A, and associative composition law.

In this section, we will focus on one particular example. Let R be an associative but possibly noncommutative ring with 1, and let Mod_R be the category of left R-modules and homomorphisms. We write $Hom_R(-, -)$ for the morphisms. It is worth noting that $Mod_{\mathbb{Z}}$ is the category of abelian groups. These categories have the following features

1. $Hom_R(-, -)$ is an abelian group, and composition is distributive.
2. There is a zero object 0 such that $Hom_R(0, M) = Hom_R(M, 0) = 0$.
3. Every pair objects A, B has direct sum $A \oplus B$ characterized by certain universal properties.
4. Morphisms have kernels and images, characterized by the appropriate universal properties.

We will encounter other categories satisfying these conditions later on. Such categories are called abelian. We have been a bit vague about the precise axioms; see Weibel's Homological Algebra for this.

[1] I will mostly ignore set theoretic issues, but you should at least be aware that forming the *set* of all sets or groups or spaces leads immediately to paradoxes. The way around this in Gödel-Bernays set theory is to allow two types of constructions, sets and classes. Sets are classes, but there are classes which too big to be sets...

2.0.7 Diagram Chasing

A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is called *exact* if $\ker g = \operatorname{im} f$. A useful skill in this business is to be able to prove things by diagram chasing.

Exercise 1. *Given a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

Show that g is an isomorphism if f and h are isomorphisms.

Theorem 2.0.8 (Snake Lemma)**.** *If*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & &
\end{array}$$

is a commutative diagram with exact rows, then there is an exact sequence

$$0 \to \ker f \to \ker g \to \ker h \xrightarrow{\partial} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h$$

The only part of the sequence which is not obvious is the so called connecting map ∂. This will be explained in class.

2.0.9 Hom Functors

A (covariant) *functor* F from one category to another is a function taking objects to objects and morphisms to morphism such that if $f : A \to B$ then $F(f) : F(A) \to F(B)$, $F(1_A) = 1_{F(A)}$ and $F(f \circ g) = F(f) \circ F(g)$. A contravariant functor reverses direction in the sense that $F(f) : F(B) \to F(A)$, $F(1_A) = 1_{F(A)}$ and $F(f \circ g) = F(g) \circ F(f)$. Here are two basic examples: If $M \in Mod_R$, then $F(-) = Hom_R(M, -)$ is a covariant functor from Mod_R to $Mod_{\mathbb{Z}}$.

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\uparrow{\scriptstyle g} & \nearrow{\scriptstyle F(f)=f\circ g} & \\
M & &
\end{array}$$

When R commutative, $F(-)$ is naturally an R-module, but not otherwise. Similarly, $Hom_R(-, M)$ is a contravariant functor from Mod_R to $Mod_{\mathbb{Z}}$ (or Mod_R when R is commutative).

Lemma 2.0.10. *Suppose that*

$$0 \to A \to B \to C \to 0$$

is exact. Then

(a)
$$0 \to Hom(M, A) \to Hom(M, B) \to Hom(M, C)$$

(b)
$$0 \to Hom(C, M) \to Hom(B, M) \to Hom(A, M)$$

are both exact.

The proof is straightforward and will be omitted.

Exercise 2. *Prove the lemma.*

Exercise 3. *Prove that*

$$0 \to Hom(M, A) \to Hom(M, B) \to Hom(M, C) \to 0$$

and
$$0 \to Hom(C, M) \to Hom(B, M) \to Hom(A, M) \to 0$$

are exact when the sequence

$$0 \to A \to B \xrightarrow{p} C \to 0$$

is split exact. This means that there exists a map $s : C \to B$, called a splitting, such that $p \circ s = 1_C$.

A (contravariant) functor is called exact if it preserves exact sequences. The lemma says that the *Hom* functors have the weaker property left exactness. They are not exact, in general:

Example 2.0.11. *Let $R = \mathbb{Z}$ $M = \mathbb{Z}/2$. Note that $Hom(M, \mathbb{Z}) = 0$ and $Hom(M, M) = \mathbb{Z}/2$. So $Hom(M, -)$ applied to*

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

yields the sequence
$$0 \to 0 \to 0 \to \mathbb{Z}/2$$

The last map is certainly not onto.

Exercise 4. *Find an example for which $Hom(-, M)$ isn't exact.*

Lemma 2.0.12. *If M is a free module, then $Hom(M, -)$ is exact.*

Proof. Let $M = \bigoplus_S R$, where S might be infinite. Given $f : B \to C$ surjective, We have

$$\begin{array}{ccc} Hom(M,B) & \longrightarrow & Hom(M,C) \\ \downarrow = & & \downarrow = \\ \prod_S B & \xrightarrow{\prod f} & \prod_S C \end{array}$$

The horizontal map on the bottom is clearly surjective.

□

Given a module M, let

$$R^{(M)} = \bigoplus_{m \in M} R$$

This is a very big free module which maps onto M by sending the 1 in the mth copy of R to m. Let $K(M)$ be the kernel. We have a *canonical* exact sequence

$$0 \to K(M) \to R^{(M)} \to M \to 0 \tag{2.1}$$

2.0.13 Ext

Inductively, define

$$Ext^1_R(M,N) = \text{coker}[Hom(R^{(M)},N) \to Hom(K(M),N)]$$

$$Ext^{i+1}_R(M,N) = Ext^i_R(K(M),N)$$

This is not the way these groups are usually defined, but we will get to that later. These are clearly covariant functors in the second variable.

Theorem 2.0.14. *$Ext^i(-,N)$ is a covariant functor in the first variable. Given a short exact sequence*

$$0 \to A \to B \to C \to 0$$

we have an infinte long exact sequence

$$0 \to Hom(C,N) \to Hom(B,N) \to Hom(A,N) \to Ext^1(C,N) \to Ext^1(B,N) \to \dots$$

Proof. We prove the second part about the exact sequence. The first part is similar. One constructs a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K(A) & \longrightarrow & K(B) & \longrightarrow & K(C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^{(A)} & \longrightarrow & R^{(B)} & \longrightarrow & R^{(C)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Note that the middle column is split exact. Hom the top two rows into N to get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Hom(R^{(C)}, N) & \longrightarrow & Hom(R^{(B)}, N) & \longrightarrow & Hom(R^{(A)}, N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Hom(K(C), N) & \longrightarrow & Hom(K(B), N) & \longrightarrow & Hom(K(A), N) & &
\end{array}$$

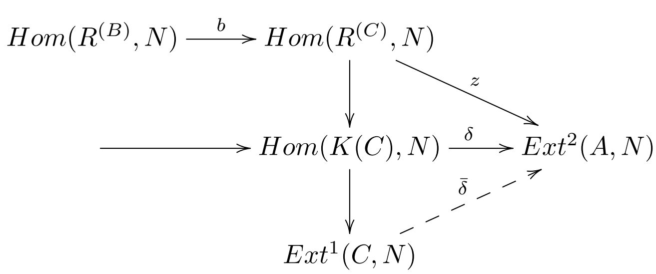
We use the split exactness to see that the top row is exact. Now the snake lemma gives the first 6 terms of the exact sequence. Applying this to

$$0 \to K(A) \to K(B) \to K(C) \to 0$$

yields

$$\ldots Hom(K(C), N) \xrightarrow{\delta} Ext^2(A, N) \ldots$$

We need to show that this map factors through $Ext^1(C, N)$. To see this, use the fact that z below is zero because b is surjective.



It follows that the original 6 term sequence can be continued to a 9 term sequence. This can be continued indefinitely.

□

Lemma 2.0.15. *If M is free then $Ext^1(M,N) = 0$ for any N.*

Proof. If M is free, then we can choose a basis m_i. Let $s : M \to R^{(M)}$ be the homomorphism which sends m_i to 1 in the m_ith copy of R. This gives a splitting of (2.1). It follows that

$$Hom(R^{(M)}, N) \to Hom(K(M), N)$$

is surjective. Therefore Ext^1 vanishes. □

Lemma 2.0.16. *If*

$$0 \to K \to F \to M \to 0$$

is exact, then

$$Ext^1(M,N) \cong \text{coker}[Hom(F,N) \to Hom(K,N)]$$

Proof. This follows from the previous lemma and theorem 2.0.14. □

This is useful for doing computations.

Exercise 5. *Using the sequence*

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$$

compute $Ext^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m)$.

2.0.17 Ext via free resolutions

We can now compare our definition with the more conventional one. A free resolution of M is a possibly infinite exact sequence

$$\dots F_2 \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0 \to M \to 0 \tag{2.2}$$

where the F_i are all free. Exactness implies that $\partial^2 = 0$. So by functoriality, if we Hom this into another module N, we still get a complex

$$Hom(F_0, N) \to Hom(F_1, N) \to \dots$$

Theorem 2.0.18. $Ext^i(M,N) \cong H^i(Hom(F_\bullet, N))$

Proof. We break (2.2) into short exact sequences

$$0 \to K_0 \to F_0 \to M \to 0$$

$$0 \to K_1 \to F_1 \to K_0 \to 0$$

etc. Using the left exactness of Hom, we can see that

$$\text{im}[Hom(F_0,N) \to Hom(F_1,N)] \cong \text{im}[Hom(F_0,N) \to Hom(K_0,N)]$$

$$\ker[Hom(F_1,N) \to Hom(F_2,N)] \cong \ker Hom[(F_1,N) \to Hom(K_1,N)] \cong Hom(K_0,N)$$

The theorem for $i = 1$ follows from this and lemma 2.0.16.

We have a free resolution

$$\dots F_2 \to F_1 \to K_0 \to 0$$

Denote the "tail" by $F_{\geq 1}$. The previous case implies that

$$Ext^2(M,N) \cong Ext^1(K_0,N) \cong H^1(Hom(F_{\geq 1},N)) = H^2(Hom(F_\bullet,N))$$

This implies the theorem for $i = 2$ etc. □

Note that in the usual approach, it is not a priori clear that $H^i(Hom(F_\bullet,N))$ is well defined. This would require proof. Here is an example which shows the utility of this description.

Exercise 6. *Let $R = k[x,y]$ and $M = R/(x,y) \cong k$. Construct a free resolution of M (which is a special case of the* Koszul complex*)*

$$0 \to R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R \to M \to 0$$

where the maps are given by the indicated matrices. Using this, we can see that

$$Ext^i(M,M) = \begin{cases} k^2 & \text{if } i = 1 \\ k & \text{if } i = 2 \\ 0 & \text{if } i > 2 \end{cases}$$

This description can be also be used to clarify meaning of connecting maps. We merely outline the idea. Suppose that we have an exact sequence

$$0 \to A \to B \to C \to 0$$

Then we can construct free resolutions fitting into a commutative diagram

$$\begin{array}{ccccc} \dots\mathcal{F}_1 & \xrightarrow{f} & \mathcal{F}_0 & \longrightarrow & A \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v} & & \downarrow \\ \dots\mathcal{G}_1 & \xrightarrow{g} & \mathcal{G}_0 & \longrightarrow & B \end{array}$$

We build a new complex called the mapping cone of v

$$\mathcal{C}(v) = \dots \mathcal{G}_2 \oplus \mathcal{F}_1 \to \mathcal{G}_1 \oplus \mathcal{F}_0 \to \mathcal{G}_0$$

with maps

$$(a,b) = (g(a) \pm v(b), f(b))$$

We can map $\mathcal{C}_0(v) = \mathcal{G}_0$ to C by composing $\mathcal{G}_0 \to B$ and $B \to C$. It can be shown that this gives a free resolution of C. Note that by construction, we have a map $\mathcal{C}_i(v) \to \mathcal{F}_{i-1}$ which induces a map on cohomology

$$H^i(Hom(\mathcal{C}_\bullet(v),N)) \to H^{i+1}(Hom(\mathcal{F}_\bullet,N))$$

which is precisely the connecting homomorphism.