# Chapter 3

# Sheaves

Let $X$ be a topological space. A *presheaf* of abelian groups $\mathcal{F}$ on $X$ is an assignment of

1. an abelian group $\mathcal{F}(U)$ for every open set $U \subseteq X$,
2. a homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ to every inclusion $V \subseteq U \subset X$ such that
   (a) $\rho_{UU} = 1$
   (b) if $U \subseteq V \subseteq W$
   $$\begin{tikzcd} \mathcal{F}(W) \arrow[r, "\rho_{WV}"] \arrow[dr, "\rho_{WU}"'] & \mathcal{F}(V) \arrow[d, "\rho_{VU}"] \\ & \mathcal{F}(U) \end{tikzcd}$$
   commutes.

Elements of $\mathcal{F}(U)$ are called sections. $\rho_{UV}$ is referred to as restriction and often denoted by $|_V$.

**Example 3.0.19.** *Let $X$ be an arbitrary space, and $A$ an arbitrary group. For every $U$, let $\mathcal{F}(U) = A$ with $\rho_{UV} = 1$.*

**Example 3.0.20.** *Let $X, A$ be as above. Let $\mathcal{F}(U)$ be the group of $A$-valued functions on $U$. Restriction has the usual meaning.*

**Example 3.0.21.** *Let $C(U)$ be the ring of continuous real (or complex) valued functions on $U$. Restriction has the usual meaning.*

**Example 3.0.22.** *Let $X \subset \mathbb{R}^n$ be an open subset. Let $\mathcal{E}^p(U)$ be the group of $p$-forms on $U$. Restriction is restriction of coefficient functions.*

The last four examples satisfy a stronger property that sections are determined by local conditions. A presheaf $F$ is called a *sheaf* if for every open cover $\{U_i\}$ of $U \subset X$, given sections $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_j} = f_j|_{U_i}$, there exists a unique section $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$.

**Exercise 7.** *Let $X$ be a topological space possessing at least two disjoint nonempty open sets. Let $R(U)$ denote the ring of constant real valued functions on $U$. Show that $R$ is not a sheaf. However, show that modifying this by letting $R^+(U)$ to be ring of* locally *constant, we do get a sheaf. Later on we denote $\mathbb{R}_X = R^+$ and called it the constant sheaf associated to $\mathbb{R}$. This construction works for any abelian group.*

### 3.0.23 Morphisms

A morphism between presheaves $\eta : \mathcal{F} \to \mathcal{G}$, is a collection of homomorphisms $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$ such that

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\rho_{UV}"] \arrow[d, "\eta_U"'] & \mathcal{F}(V) \arrow[d, "\eta_V"] \\ \mathcal{G}(U) \arrow[r, "\rho_{UV}"] & \mathcal{G}(V) \end{tikzcd}$$

commutes. A morphism of sheaves is simply a morphism of presheaves. Thus we can form a category $PSh(X)$ and the subcategory $Sh(X)$ of presheaves and sheaves on $X$.

**Example 3.0.24.** *When $X = \mathbb{R}^n$, the inclusion of $C^\infty(U) \subset C(U)$ defines a morphism. The exterior derivative $d : \mathcal{E}^p \to \mathcal{E}^{p+1}$ is a morphism.*

**Example 3.0.25.** *The map $R \to R^+$ of exercise 7 is a morphism.*

**Theorem 3.0.26.** *The category of presheaves $PSh(X)$ is abelian.*

*Outline.* We have to show that

1. $Hom(\mathcal{F}, \mathcal{G})$ is an abelian group: Given $\eta, \xi \in Hom(\mathcal{F}, \mathcal{G})$, define $(\eta+\xi)_U = \eta_U + \xi_U$.

2. There is a zero presheaf: $0(U) = 0$

3. The sum of two presheaves is given by $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$.

4. Given a morphism $\eta : \mathcal{F} \to \mathcal{G}$, we have a kernel defined by $(\ker \eta)(U) = \ker \eta_U$ and an image $(\operatorname{im} \eta)(U) = \operatorname{im} \eta_U$.

□

**Corollary 3.0.27.** *A sequence of presheaves*

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

*is exact in $PSh(X)$ if and only if*

$$\mathcal{A}(U) \to \mathcal{B}(U) \to \mathcal{C}(U)$$

*is exact in the usual sense for every $U$.*

We also have

**Theorem 3.0.28.** *The category of sheaves $Sh(X)$ is abelian.*

All the steps in the previous proof work except for the last one. Given a morphism of sheaves $\eta : \mathcal{F} \to \mathcal{G}$, the image $\mathcal{I}(U) = \operatorname{im} \eta_U$ in the category of presheaves is generally **not** a sheaf, so this step needs to be modified. The key construction we need is the notion of sheafification or associated sheaf $\mathcal{F}^+$.

**Theorem 3.0.29.** *If $\mathcal{F}$ is a presheaf, there exists a sheaf $\mathcal{F}^+$ and morphism $\mathcal{F} \to \mathcal{F}^+$ which is universal in the sense that a morphism $\mathcal{F} \to \mathcal{G}$ to another sheaf factors uniquely through $\mathcal{F}^+$. It follows that $\mathcal{F}^+$ is uniquely determined.*

In outline, sections of $\mathcal{F}^+(U)$ can be given explicitly as suitable equivalence classes of collections of sections $\{f_i \in \mathcal{F}(U_i)\}$ satisfying the patching condition $f_i|_{U_{ij}} = f_j|_{U_{ij}}$, where $\{U_i\}$ is an open cover of $U$, and $U_{ij} = U_i \cap U_j$. We won't go into details, because this particular construction is kind of messy. We will another construction later.

Returning to the proof of the previous theorem, given a morphism of sheaves $\eta : \mathcal{F} \to \mathcal{G}$, the sheafification of the presheaf image $\mathcal{I}^+$ gives the correct image in $Sh(X)$. The conclusion is that even though $Sh(X)$ is a subcategory of $PSh(X)$, *the notion of exact sequences are different.* Exactness in $Sh(X)$ is the more important and useful notion. Here is an explicit description:

**Proposition 3.0.30.** *A sequence of sheaves*

$$\mathcal{A} \xrightarrow{\eta} \mathcal{B} \xrightarrow{\xi} \mathcal{C}$$

*is exact in $Sh(X)$ if and only if or every open $U \subset X$*

*(a) if $\alpha \in \mathcal{A}(U)$ then $\xi \circ \eta(\alpha) = 0$*

*(b) if $\beta \in \mathcal{B}(U)$ such that $\xi(\beta) = 0$, there exists an open cover $\{U_i\}$ of $U$ and $\alpha_i \in \mathcal{A}(U_i)$ such that $\beta|_{U_i} = \eta(\alpha_i)$*

The key point in (b) is that $\beta$ is *locally* the image of something in $\mathcal{A}$.

## 3.1 The sheaf de Rham complex

To convince ourselves that this complicated notion of exactness of sheaves is useful, let us return to the problem that got us started. Suppose that $X \subset \mathbb{R}^n$ is an open set (or more generally a manifold). We have the sheaf $\mathbb{R}_X$ of locally constant real valued functions. This includes into the sheaf of $C^\infty$ functions which is the same as 0-forms $\mathcal{E}^0_X$. We can differentiate this repeatedly and thus get a sequence of sheaves

$$0 \to \mathbb{R}_X \to \mathcal{E}^0_X \xrightarrow{d} \mathcal{E}^1_X \to \dots \mathcal{E}^n_X \to 0$$

which we will refer to as the sheaf de Rham complex.

**Theorem 3.1.1.** *This is an exact sequence of sheaves.*

*Proof.* The composition of adjacent maps is zero because either $d^2 = 0$ or the derivative of a locally constant function is zero. Now suppose that $\beta \in \mathcal{E}^p(U)$ satisfies $d\beta = 0$. If $p = 0$, then $\beta$ is necessarily locally constant. Now suppose that $p > 0$. Cover $U$ by open balls $U_i$. Each ball is diffeomorphic to $\mathbb{R}^n$, therefore by Poincaré's lemma (theorem 1.0.2) we can find $\alpha_i \in \mathcal{E}^{p-1}(U_i)$ such that $\beta|_{U_i} = d\alpha_i$. □

This is generally not exact in the category of presheaves.

**Example 3.1.2.** *Let $X = \mathbb{R}/\mathbb{Z}$ be the circle. Then the de Rham complex is just*

$$0 \to \mathbb{R}_X \to C^\infty_X \xrightarrow{f \to f'} C^\infty_X \to 0$$

*However,*

$$0 \to \mathbb{R} \to C^\infty_X(X) \to C^\infty_X(X) \to 0$$

*is not exact at the last step because the integral of constant function 1 is not periodic.*

## 3.2 Stalks

Given a presheaf $\mathcal{F} \in PSh(X)$ and a point $x \in X$, we define an equivalence class on sections defined in a neighourhood of $x$. $f_i \in \mathcal{F}(U_i)$ have the *same germ at $x$* if $f_1|_U = f_2|_U$ for some $x \in U \subset U_1 \cap U_2$. The equivalence class of $f_1$ is called its germ at $x$ and we will denote it by $\gamma_x(f)$. The stalk $\mathcal{F}_x$ is the set of germs of sections defined in a neighourhood of $x$. Alternatively, this is the direct limit (or inductive limit or colimit)

$$\mathcal{F}_x = \varinjlim_{x \in U} \mathcal{F}(U)$$

Since the category of abelian groups possesses direct limits, we see that $\mathcal{F}_x$ is an abelian group. More generally, if $\mathcal{F}$ were a presheaf of rings, say, $\mathcal{F}_x$ would also be a ring.

**Exercise 8.** *Suppose that $\mathcal{F}$ is a sheaf, and $f \in \mathcal{F}(X)$ has $\gamma_x(f) = 0$ for all $x \in X$. Prove that $f = 0$. Given an example of a presheaf, where this fails.*

We can use the stalk to give another construction of sheafification. Let

$$\mathcal{F}^+(U) = \{(f_x \in \prod_{x\in U} \mathcal{F}_x) \mid \exists \text{ an open cover } U_i, f_i \in \mathcal{F}(U), \forall x \in U_i, f_x = \gamma_x(f_i)\}$$

We define a morphism $\eta : \mathcal{F} \to \mathcal{F}^+$ sending $f \to (\gamma_x(f))_{x\in U}$. One can verify that this construction has the desired properties.

Given a morphism $\eta : \mathcal{F} \to \mathcal{G}$, we get a homomorphism $\mathcal{F}_x \to \mathcal{G}_x$ sending the germ $\gamma_x(f) \mapsto \gamma_x(\eta(f))$. Thus we can view $\mathcal{F} \mapsto \mathcal{F}_x$ as a functor from $PSh(X) \to Ab$.

**Proposition 3.2.1.** *A sequence of sheaves*

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

*is exact if and only if for every $x \in X$*

$$\mathcal{A}_x \to \mathcal{B}_x \to \mathcal{C}_x$$

*is exact.*

*Proof.* We will prove "if" direction. So we assume that the sequences

$$\mathcal{A}_x \to \mathcal{B}_x \to \mathcal{C}_x$$

are exact. Suppose that $\alpha \in \mathcal{A}(U)$, and let $\epsilon \in \mathcal{C}(U)$ be its image. Then the germs of $\gamma_x(\epsilon)$ are all zero. Therefore $\epsilon = 0$ by the previous exercise. Let $\beta \in \mathcal{B}(U)$ map to 0 in $\mathcal{C}(U)$. Then for each $x \in X$, we can find a germ $\gamma_x(\alpha) \in \mathcal{A}_x$ which maps to $\gamma_x(\beta)$. This means that on some neigbourhood $U_x \subset U$ of $x$, the restriction of $\alpha$ maps to the restriction of $\beta$. This implies the exactness of sheaves. □

**Exercise 9.** *Prove the other direction.*

## 3.3 Manifolds via sheaf theory

A ringed space is a topological $X$ together with a sheaf of rings on it. To simplify the story, we will focus on the case where these sheaves of functions, and refer such spaces as concrete ringed spaces. Here some examples.

**Example 3.3.1.** *(Topology.) Let $X$ be any topological space and let $C_X$ denote the sheaf of rings of continuous real valued functions.*

**Example 3.3.2.** *(Differential topology.) Let $X \subset \mathbb{R}^n$ be an open subset. Let $C^\infty_X$ denote the sheaf of rings of $C^\infty$ functions.*

**Example 3.3.3.** *(Complex analysis.) Let $X \subset \mathbb{C}^n$ be an open set. Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions. (If you don't what it means in several variables, assume $n = 1$.)*

**Example 3.3.4.** *(Algebraic geometry.) Let $k$ be a field. Let $X = k^n$ (which algebraic geometers usually write as $\mathbb{A}_k^n$ and called affine $n$ space). Given a polynomial $f \in R = k[x_1, \ldots, x_n]$, let $D(f) = \{a \in \mathbb{A}_k^n \mid f(a) \neq 0\}$. This forms a basis of a topology called the Zariski topology. Let $\mathcal{O}(D(f)) = \{g/f^n \mid g \in R, n \in \mathbb{N}\}$. We will come to this example and prove that $\mathcal{O}_X$ extends to a sheaf of rings.*

To explain the power of the language, let us defined manifolds from this point of view. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ denote two concrete ringed spaces of $k$-valued functions. We say that they are isomorphic if there exists a homeomorphism $F : X \to Y$ such that $f \in \mathcal{O}_Y(U)$ if and only if $f \circ F \in \mathcal{O}_X(F^{-1}U)$ A $C^\infty$ $n$-manifold is a metrizable[1] space $X$ with a sheaf of rings of $\mathbb{R}$-valued functions $C_X^\infty$, such that $X$ is locally isomorphic to $(\mathbb{R}^n, C_{\mathbb{R}^n}^\infty)$. The last condition means that that there is an open cover $\{U_i\}$ of $X$ such that $(U_i, C_X^\infty|_{U_i}) \cong (\mathbb{R}^n, C_{\mathbb{R}^n}^\infty)$. The restriction of the sheaf means that we only take sections on $U \subseteq U_i$.

**Exercise 10.** *If you already knew what a manifold was, then compare your definition to this one.*

A complex $n$-manifold is defined the same way except that we require it to be locally isomorphic to $(B^n, \mathcal{O}_{B^n})$, where $B^n \subset \mathbb{C}^n$ is the unit ball. Note that complex dimension is half of the real dimension. For historical reasons, one dimensional complex manifolds are often called Riemann surfaces.

[1]The key point is that $X$ is Hausdorff and paracompact. We will say more about this later.