Chapter 5

Sheaf cohomology

5.1 Global sections

Let X be a topological space. Recall that in the category $Sh(X)$ of sheaves of abelian groups, we have a notion of exactness which amount to exactness at stalks. We observe that

Lemma 5.1.1. *If $\mathcal{F}$ is a sheaf and $f \in \mathcal{F}(X)$ satisfies $\gamma_x(f) = 0$ for all x, then $f = 0$.*

Proof. The assumption means that there is a cover $\{U_i\}$ such that $f|_{U_i} = 0$. Therefore $f = 0$ be the sheaf axiom. $\square$

Exercise 14. *Show that this is not true for all presheaves.*

Given a sheaf $\mathcal{F} \in Sh(X)$, let $\Gamma(X, \mathcal{F}) = \mathcal{F}(X)$. This defines a functor from $Sh(X)$ to the category Ab of abelian groups.

Proposition 5.1.2. Γ *is left exact, i.e. if*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is exact in $Sh(X)$, then

$$0 \to \Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{B}) \to \Gamma(X, \mathcal{C})$$

is exact.

Proof. Let $\eta : \mathcal{A} \to \mathcal{B}$ denote the map. Let $\alpha \in \mathcal{A}(X)$ map to 0 under η. Since $\mathcal{A}_x \to \mathcal{B}_x$ is injective, $\alpha_x = 0$. Therefore $\alpha = 0$ by the previous lemma.

Suppose that $\beta \in \mathcal{B}(X)$ maps to 0 in $\mathcal{C}$. Then there exists $\alpha_x \in \mathcal{A}_x$ such that $\gamma_x(\beta) = \eta(\alpha_x)$. Therefore there exists an open cover $\{U_i\}$ and $\alpha_i \in \mathcal{A}(U_i)$ such that $\beta|_{U_i} = \eta(\alpha_i)$. Note that α_i and α_j agree on intersections, they patch to a global section $\alpha \in \mathcal{A}(X)$. We must have $\beta = \eta(\alpha)$ (why?).

$\square$

The functor Γ is not exact however. We gave an example before of the de Rham complex on the circle $X = \mathbb{R}/\mathbb{Z}$

$$0 \to \mathbb{R} \to C^\infty \to C^\infty \to 0$$

for which

$$C^\infty(X) \to C^\infty(X)$$

is not surjective. Here is another important example.

Example 5.1.3. *Let $X = \mathbb{P}^1_{\mathbb{C}}$ which is nothing but the Riemann sphere $\mathbb{C}\cup\{\infty\}$. Here we view it as a complex manifold, with $\mathcal{O}_X$ the sheaf of functions. Let $S = \{p_1, \ldots, p_n\} \subset X$ denote a nonempty finite set of points. We define the sky scraper sheaf*

$$\mathbb{C}_{p_i}(U) = \begin{cases} \mathbb{C} & \text{if } p_i \in U \\ 0 & \text{otherwise} \end{cases}$$

We define a morphism $\mathcal{O} \to \mathbb{C}_{p_i}$ which sends $f \in O(U)$ to $f(p_i)$ when $p_i \in U$ and necessarily to 0 otherwise. Then we can add these to get a morphism $\mathcal{O}_X \to \mathbb{C}_S \to 0$ where $\mathbb{C}_S = \bigoplus \mathbb{C}_{p_i}$. The kernel is the ideal sheaf $\mathcal{I}_S(U) = \{f \in \mathcal{O}(U) \mid f(p_i) = 0\}$. Then we have an exact sequence of sheaves

$$0 \to \mathcal{I}_S \to \mathcal{O}_X \to \bigoplus \mathbb{C}_{p_i} \to 0$$

On global sections, we have

$$\begin{tikzcd} \Gamma(\mathcal{O}_X) \arrow[r] \arrow[d, "="] & \Gamma(\mathbb{C}_S) \arrow[d, "="] \\ \mathbb{C} \arrow[r] & \mathbb{C}^n \end{tikzcd}$$

This cannot be surjective when $n > 1$.

One has a similar example using $\mathbb{P}^1_k$ for any field k. We general this as follows. Let $Y \subset X$ be a subvariety of another variety. A regular function on X restricts to a regular function on Y. This gives a morphism $\mathcal{O}_X \to \mathcal{O}_Y$. The kernel is

$$\mathcal{I}_Y = \{f \in \mathcal{O}_X(U) \mid f|_{Y\cap U} = 0\}$$

5.2 Flasque and soft sheaves

We want some natural for when a short exact sequence of sheaves gives a short exact sequence of global sections. We introduce two classes of sheaves. A sheaf $\mathcal{F}$ on a space X is called flasque (sometime translated as flabby) if $\rho_{XU} : \mathcal{F}(X) \to \mathcal{F}(U)$ is surjective for all open $U \subset X$. One can see that sky scraper sheaves are flasque, but there are not very natural examples. Here is a related notion. Given a closed set $Z \subset X$, define

$$\mathcal{F}(Z) = \varinjlim_{U} \mathcal{F}(U)$$

where U runs over open neigbourhoods of Z. For example, $\mathcal{F}(x) = \mathcal{F}_x$. A sheaf is called soft if for any closed $Z \subset X$, $\mathcal{F}(X) \to \mathcal{F}(Z)$ is surjective.

Theorem 5.2.1. *Let*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

denote an exact sequence of sheaves on X. If $\mathcal{A}$ is flasque, then

$$0 \to \Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{B}) \to \Gamma(X, \mathcal{C}) \to 0$$

is exact.

Proof. It is enough to prove surjectivity of the last map. We do this under the extra assumption that X carries a countable dense set (which will hold in all the examples we care about). Given a section $\epsilon \in \Gamma(\mathcal{C})$, we can find a countable open cover $\{U_i\}_{i\in\mathbb{N}}$ such that $\epsilon|_{U_i}$ is the image of $\beta_i \in \mathcal{B}(U_i)$. We will define $\tilde{\beta}_i \in \mathcal{B}(U_0, \cup \ldots U_i)$ by induction, so that $\tilde{\beta}_i$ extends $\tilde{\beta}_{i-1}$ and maps to $\epsilon|_{u_0\cup\ldots U_i}$. Then $\tilde{\beta}_i$ patch to a global section of $\mathcal{B}$ mapping to ϵ.

Set $\tilde{\beta}_0 = \beta_0$. Assume $\tilde{\beta}_i$ is defined, let $(\tilde{\beta}_i - \beta_{i+1})|_{(U_0\cup\ldots U_i)\cap U_{i+1}}$. This maps to 0 in $\mathcal{C}$, so defines a section of $\mathcal{A}$. By assumption, this extends to a global section $\alpha \in \mathcal{A}$. We then restrict this to U_{i+1}. We define

$$\tilde{\beta}_{i+1} = \begin{cases} \tilde{\beta}_i & \text{on } U_0 \cup \ldots U_i \\ \beta_{i+1} + \alpha & \text{on } U_{i+1} \end{cases}$$

These agree on the intersection and therefore define a section on $U_0 \cup \ldots U_{i+1}$ with the required properties. $\square$

Before stating the parallel result for soft sheaves, recall that a topological space X is paracompact if it is Hausdorff and any open cover has a locally finite refinement. Locally finite means that every point has a neigbourhood that meets only finitely many sets of the cover. A theorem of Stone[1] shows that any metric space is paracompact, so this is a very natural condition.

Theorem 5.2.2. *Let*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

denote an exact sequence of sheaves on a paracompact space X. If $\mathcal{A}$ is soft, then

$$0 \to \Gamma(X, \mathcal{A}) \to \Gamma(X, \mathcal{B}) \to \Gamma(X, \mathcal{C}) \to 0$$

is exact.

Proof. We prove this assuming that X is a separable metric space, given $\epsilon \in \Gamma(\mathcal{C})$, we can find a countable locally finite open cover $\{V_i\}_{i\in\mathbb{N}}$ such that $\epsilon|_{U_i}$ is the image of $\beta_i \in \mathcal{B}(V_i)$. We can choose a new open cover $\{U_i\}$ satisfying the same assumptions and such that $\bar{U}_i \subset V_i$. The rest of the proof proceeds as above. $\square$

[1] Stone, Paracompactness and product spaces, Bull AMS (1948)

Now we want construct some natural examples of soft sheaves.

Theorem 5.2.3. *The sheaf of real valued continuous functions on a metric space X is soft. The sheaf C^∞ functions on a C^∞ manifold is soft.*

Sketch. Given a closed set $Z \subset X$ and $\epsilon > 0$, construct a continuou cutoff function which is 1 in an ϵ-neighbourhood of Z and 0 away from Z. Given any continuous function f defined in a neigbourhood of Z, ρf gives an extension to $C(X)$.

The second statement is proved the same way using C^∞ cuttoff function. □

Given a sheaf of commutative rings $\mathcal{R}$ on X, a sheaf of $\mathcal{R}$-modules or simply and $\mathcal{R}$-module is sheaf $\mathcal{M} \in Sh(X)$ such that for each U, $\mathcal{M}(U)$ is an $\mathcal{R}(U)$-module, and restriction is linear in the sense that $\rho_{UV}(rm) = \rho(r)\rho(m)$. We can extend the previous result.

Lemma 5.2.4. *Any C^∞-module on a manifold is soft.*

Proof. The proof is the same. □

5.3 Sheaf cohomology

Theorem 5.3.1. *There exists a sequence of functors $H^i(X, -) : Sh(X) \to Ab$ such that*

1. $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$
2. *Given a short exact sequence of sheaves*

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

there is a long exact sequence

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to H^1(X, \mathcal{A}) \to \dots$$

The idea is similar to the way we constructed and proved basic facts about Ext. I won't give the full details which can be found in my book. I will give a construction of cohomology and outline the exactness of the first 6 terms. The role of free modules will be replaced by flasque sheaves. Given a sheaf $\mathcal{F}$, define

$$\mathcal{G}(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x$$

The restrictions sends a collection $(f_x)_{x \in U}$ to $(f_x)_{x \in V}$. This is a flasque sheaf. Furthermore, we have a morphism $\mathcal{F} \to \mathcal{G}(\mathcal{F})$ sending $f \mapsto (\gamma_x(f))_{x \in U}$. This is injective. We define

$$C(\mathcal{F}) = \operatorname{coker}[\mathcal{F} \to \mathcal{G}(\mathcal{F})]$$

$$C^n(\mathcal{F}) = C \circ \dots C(\mathcal{F}) \text{ (n times)}$$

Define

$$H^1(X, \mathcal{F}) = \text{coker}[\Gamma(X, \mathcal{G}(\mathcal{F}) \to \Gamma(X, C(\mathcal{F}))]$$

and

$$H^n(X, \mathcal{F}) = H^1(X, C^{n-1}(\mathcal{F}))$$

Lemma 5.3.2. *Given a short exact sequence as above, we have a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{G}(\mathcal{B}) \arrow[r] \arrow[d] & \mathcal{G}(\mathcal{C}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(\mathcal{A}) \arrow[r] & C(\mathcal{B}) \arrow[r] & C(\mathcal{C}) \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

Now take global sections to get

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(\mathcal{G}(\mathcal{A})) \arrow[r] \arrow[d] & \Gamma(G(\mathcal{B})) \arrow[r] \arrow[d] & \Gamma(\mathcal{G}(\mathcal{C})) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(C(\mathcal{A})) \arrow[r] & \Gamma(C(\mathcal{B})) \arrow[r] & \Gamma(C(\mathcal{C})) \arrow[r] & {}
\end{tikzcd}$$

Note that the top row is surjective at the end because $\mathcal{G}(A)$ is flasque. It follows that from the snake lemma that we get the 6 term exact sequence

$$0 \to H^0(X, \mathcal{A}) \to H^0(X, \mathcal{B}) \to H^0(X, \mathcal{C}) \to H^1(X, \mathcal{A}) \to \dots$$

as promised.

5.4 Flasque resolutions

The definition we gave is not particularly convenient for doing computations.

Proposition 5.4.1. *Suppose that $\mathcal{F}$ is flasque then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. Consider the exact sequence

$$0 \to \mathcal{F} \to \mathcal{G}(\mathcal{F}) \to C(\mathcal{F}) \to 0$$

Since $\mathcal{F}$ is flasque

$$\Gamma(U, \mathcal{G}(\mathcal{F})) \to \Gamma(U, C(\mathcal{F}))$$

is surjective. This implies that

$$H^1(X, \mathcal{F}) = 0$$

From the diagram

$$\begin{tikzcd}
\Gamma(X, \mathcal{G}(\mathcal{F})) \arrow[r, "\rho^1"] \arrow[d] & \Gamma(U, \mathcal{G}(\mathcal{F})) \arrow[d, "\eta"] \\
\Gamma(X, C(\mathcal{F})) \arrow[r, "\rho^2"] & \Gamma(U, C(\mathcal{F}))
\end{tikzcd}$$

we see that ρ^2 is surjective because η is surjective and ρ^1 is surjective since $\mathcal{G}(\mathcal{F}))$ is flasque. So $C(\mathcal{F})$ is flasque. Therefore

$$H^2(X,\mathcal{F}) = H^1(X, C(\mathcal{F})) = 0$$

etc.

□

Let

$$0 \to \mathcal{A} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

be an exact sequence with $\mathcal{F}^\bullet$ flasque. This is called a flasque resolution of $\mathcal{A}$.

Theorem 5.4.2. *Given a flasque resolution as above,*

$$H^i(X,\mathcal{A}) \cong H^i(\Gamma(X,\mathcal{F}^\bullet))$$

Proof. We will be content to prove this when $i = 1$. The other cases are similar. We break the resolution into short exact sequences

$$0 \to \mathcal{A} \to \mathcal{F}^0 \to K^1 \to 0 \tag{5.1}$$

$$0 \to K^1 \to \mathcal{F}^1 \to K^2 \to 0 \tag{5.2}$$

$$0 \to K^2 \to \mathcal{F}^2 \to K^2 \to 0 \tag{5.3}$$

Applying the above proposition and the cohomology sequence to (5.1) gives

$$H^1(X,\mathcal{A}) \cong \mathrm{coker}[\Gamma(X,\mathcal{F}^0) \to \Gamma(X,K^1)]$$

Now apply $\Gamma(X,-)$ to (5.2) (5.3) above to conclude that

$$\Gamma(X,K^1) = \ker[\Gamma(X,\mathcal{F}^1) \to \Gamma(X,\mathcal{F}^2)]$$

Plug this into the previous formula to obtain

$$H^1(X,\mathcal{A}) = H^1(\Gamma(X,\mathcal{F}^\bullet))$$

□

A standard approach (e.g. in Hartshorne) to sheaf cohomology is to use injective resolutions. Without going into details one can show that these are injective resolutions, and there the above theorem applies. The other thing to remark is there is a canonical flasque resolution due to Godement

$$0 \to \mathcal{A} \to \mathcal{G}(\mathcal{A}) \to \mathcal{G}(C(\mathcal{A})) \to \dots$$

As an exercise, I'll let you figure out the whole pattern.

5.5 Soft resolutions and de Rham's theorem

We next establish the corresponding statements for soft sheaves.

Proposition 5.5.1. *Suppose that $\mathcal{F}$ is flasque on a paracompact space, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

Proof. Consider the exact sequence

$$0 \to \mathcal{F} \to \mathcal{G}(\mathcal{F}) \to C(\mathcal{F}) \to 0$$

Since $\mathcal{F}$ is soft

$$\Gamma(Z, \mathcal{G}(\mathcal{F})) \to \Gamma(Z, C(\mathcal{F}))$$

is surjective for closed Z. This implies that

$$H^1(X, \mathcal{F}) = 0$$

Also using the fact that $\mathcal{G}(\mathcal{F})$ is flasque and therefore soft, we can see that $C(\mathcal{F})$ is soft by looking at

$$\begin{tikzcd}
\Gamma(X, \mathcal{G}(\mathcal{F})) \arrow[r, "\rho^1"] \arrow[d] & \Gamma(Z, \mathcal{G}(\mathcal{F})) \arrow[d, "\eta"] \\
\Gamma(X, C(\mathcal{F})) \arrow[r, "\rho^2"] & \Gamma(Z, C(\mathcal{F}))
\end{tikzcd}$$

This implies that get

$$H^2(X, \mathcal{F}) = H^1(X, C(\mathcal{F})) = 0$$

and so on. $\square$

An exact sequence

$$0 \to \mathcal{A} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$$

with $\mathcal{F}^i$ soft is called a soft resolution of $\mathcal{A}$.

Theorem 5.5.2. *Given a soft resolution as above on a paracompact space,*

$$H^i(X, \mathcal{A}) \cong H^i(\Gamma(X, \mathcal{F}^\bullet))$$

Recall that when X is a manifold, the sheaf de Rham complex

$$0 \to \mathbb{R}_X \to \mathcal{E}_X^0 \to \mathcal{E}_X^1 \to \dots$$

is exact. Thus we have soft resolution of the constant sheaf $\mathbb{R}_X$. This implies half of de Rham's theorem

Theorem 5.5.3 (de Rham 1)**.**

$$H^i(X, \mathbb{R}_X) \cong H^i_{dR}(X, \mathbb{R})$$

To finish, we recall we have the singular chain complex $S^p(X, A)$, where $A = \mathbb{Z}, \mathbb{R}, \mathbb{C}$, of A-valued maps from the sets of continuous maps from $\Delta^p \to X$. Fix A. Given a continuous map $f : X \to Y$, and $\alpha \in S^p(Y, A)$, and $g : \Delta^p \to X$, define $\alpha(g) = \alpha(f \circ g)$. Thus this gives a contravariant functor from the category of topological spaces to A-modules. In particular, we have a presheaf $U \mapsto S^p(U, A)$. Let $\mathcal{S}^p_A = \mathcal{S}^p$ denote the sheafification. Then the coboundary operators induce morphisms $\delta : \mathcal{S}^p \to \mathcal{S}^{p+1}$ satisfying $\delta^2 = 0$. Thus we get a complex of sheaves. After taking global sections, we again get a complex. Before sheafifying, the corresponding complex is the singular chain, so we get singular cohomology by definition. This is still true after sheafifying.

Theorem 5.5.4. $H^i(X, A) \cong H^i(\mathcal{S}^\bullet(X))$

A proof can be found in Warner, Foundations of Differentiable manifolds and Lie groups.

Corollary 5.5.5.
$$0 \to A_X \to \mathcal{S}^0 \to \mathcal{S}^1 \ldots$$
is exact.

Proof. If $x \in X$, then we have a neighbourhood U homeomorphic to a ball. Since U is contractible $H^0(U) = A$ and $H^i(U) = 0$ for $i > 0$. This says that
$$0 \to A \to \mathcal{S}^0(U) \to \mathcal{S}^1(U) \ldots$$
is exact. □

Theorem 5.5.6. $H^i(X, A_X) \cong H^i(X, A)$ *when* $A = \mathbb{R}$ *or* $\mathbb{C}$.

The result is still true for $A = \mathbb{Z}$, but the present proof won't work.

Proof. Given what was proved already, the only thing to observe is that each $\mathcal{S}^p(U)$ is a module of over the sheaf of real valued continuous functions. Therefore is soft.

□

So we now have isomorphisms
$$H^i_{dR}(X, \mathbb{R}) \cong H^i(X, \mathbb{R}_X) \cong H^i(X, \mathbb{R})$$
and this completes the proof of de Rham's theorem.

De Rham's theorem gives us access to methods from algebraic topology to compute the above groups. For example, let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the torus. Since T is connected, $H^0_{dR}(T, \mathbb{R})$ is the space of constant functions, which is isomorphic to $\mathbb{R}$. The Hurewicz theorem tells us that
$$H^1(T, \mathbb{R}) = Hom(\pi_1(T), \mathbb{R})$$

where $\pi_1(T)$ is fundamental group which is just $\mathbb{Z}^2$. This is because if $X = \tilde{X}/\Gamma$ where $\tilde{X}$ is simply connected and Γ is a group acting nicely (properly discontinuously and freely), then $\pi_1(X) = \Gamma$. Thus

$$H^1_{dR}(T, \mathbb{R}) = \mathbb{R}^2$$

Finally, we can see that the second cohomology group is nonzero because $dx \wedge dy$ gives a nonzero class. Poincaré duality, which we will discuss next, in fact shows that $H^2_{dR}(T, R) = \mathbb{R}$ and all the other groups are zero.

5.6 Poincaré duality

We want to give one more application of sheaf theory to manifolds. An n-manifold is called orientable if it possesses a nonwhere zero n-form. Manifolds such as the Klein bottle are not orientable, but most familiar manifolds such as spheres and tori are. A choice of a nonzero n-form is (or more accurately determines) an orientation.

Theorem 5.6.1 (Poincaré duality I). *If X is a compact orientable n-manifold,*

$$H^i(X, \mathbb{R}) \cong H^{n-i}(X, \mathbb{R})^*$$

In fact, the proof will yield a stronger result. Let us drop the condition that X is compact, then let $\mathcal{E}^p_c(X) \subset \mathcal{E}^p(X)$ denote the subspace of forms with compact support. This is closed under d, so we can define compactly supported de Rham cohomology by

$$H^i_{cdR}(X, \mathbb{R}) := H^i(\mathcal{E}^\bullet_c(X))$$

Theorem 5.6.2 (Poincaré duality II). *If X is an orientable n-manifold,*

$$H^i(X, \mathbb{R}) \cong H^{n-i}_{cdR}(X, \mathbb{R})^*$$

The advantage of the second formulation is that it makes sense when $X = \mathbb{R}^n$. In this case, it can be checked directly.

Theorem 5.6.3.

$$H^i_{cdR}(\mathbb{R}^n) = \begin{cases} 0 & \text{if } i \neq n \\ \mathbb{R} & \text{if } i = n \end{cases}$$

A proof can be found in Spivak, Intro to Differential Geom vol I for example.

Given $V \subset U$, we have a map on compactly supported form $\mathcal{E}^p_c(V) \subset \mathcal{E}^p_c(U)$ given by extension by zero, Unfortunately, this goes the wrong way to be a presheaf. So we fix this by dualizing. Let

$$D^p(U) = \mathcal{E}^p_c(U)^*$$

The elements are similar to distributions or current in analysis, for people familiar with them, except that there are no continuity conditions.

Proposition 5.6.4. *D^p is a sheaf.*

A proof, which is not difficult, can be found in my book. We define morphisms $\delta : D^p \to D^{p+1}$ dual to d. Clearly $\delta^2 = 0$, so we have a complex.

Fix an orientation on X. This is needed for integration theory to work. Let $\int \in D^0(X)$ be the functional that sends $\alpha \in \mathcal{E}_0^n(X)$ to $\int_X \alpha$. Stokes' theorem implies that

$$\delta I(\alpha) = \int_X d\alpha = 0$$

So now we have a complex

$$0 \to \mathbb{R}_X \to D^0 \to D^1 \dots$$

where the first map sends $1 \mapsto \int$. Theorem 5.6.3 implies that this is exact. Since the sheaves D^p are clearly C^∞-modules, we obtain a soft resolution of $\mathbb{R}_X$. This immediately implies theorem 5.6.1.

5.7 Dolbeault's theorem

We will explain an analogue of de Rham cohomology for the Cauchy-Riemann operator. To simplify matters, let us stick to complex dimension 1. Let $D \subset \mathbb{C}$ be a disk centered at 0 with complex coordinate $z = x + iy$. Then we can introduce the complex valued differential forms $dz = dx + idy$ and $d\bar{z} = dx - idy$. Given a complex valued C^∞ function $f(z)$, set

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

The last two operators can be defined by this equation. A little thought, shows that the Cauchy-Riemann equations are equivalent to

$$\frac{\partial f}{\partial \bar{z}} = 0$$

Theorem 5.7.1 (Dolbeault's lemma)**.** *Given a C^∞ function g on the closed disk $\bar{D}$, the equation*

$$\frac{\partial f}{\partial \bar{z}} = g$$

has C^∞ solution.

Proof. An explicit solution is given by the formula,

$$f(z) = \frac{1}{2\pi i} \int_D \frac{g(w)}{w - z} dw \wedge d\bar{w}$$

See page 5 of Griffiths and Harris for details. □

We define

$$\partial f = \frac{\partial f}{\partial z} dz$$
$$\bar{\partial} f = \frac{\partial f}{\partial \bar{z}} d\bar{z}$$
$$\bar{\partial} f dz = \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz$$

A multiple of dz, $d\bar{z}$ or $dz \wedge d\bar{z}$ is called a $(1,0)$,$(0,1)$ or $(1,1)$ form. For consistency, we call a C^∞ function a $(0,0)$ form. Let $\mathcal{O}(D)$ denote the space of holomorphic functions, and $\mathcal{E}^{(p,q)}$ denote the space of C^∞ (p,q)-forms. We also let $\Omega^1(D)$ denote the space of holomorphic 1-forms, i.e. form $f(z)dz$ with f holomorphic. Then by theorem 5.7.1 we have an exact sequence of vector spaces

$$0 \to \mathcal{O}(D) \to \mathcal{E}^{(0,0)}(D) \xrightarrow{\bar{\partial}} \mathcal{E}^{(0,1)}(D) \to 0$$

and

$$0 \to \Omega(D) \to \mathcal{E}^{(1,0)}(D) \xrightarrow{\bar{\partial}} \mathcal{E}^{(1,1)}(D) \to 0$$

Theorem 5.7.2. *For any Riemann surface X, the sequences of sheaves*

$$0 \to \mathcal{O}_X \to \mathcal{E}_X^{(0,0)} \xrightarrow{\bar{\partial}} \mathcal{E}_X^{(0,1)} \to 0$$

$$0 \to \Omega_X^1 \to \mathcal{E}_X^{(1,0)} \xrightarrow{\bar{\partial}} \mathcal{E}_X^{(1,1)} \to 0$$

are exact.

Corollary 5.7.3.

$$H^0(X, \mathcal{O}_X) = \ker[\mathcal{E}^{(0,0)}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{(0,1)}(X)]$$

$$H^0(X, \Omega_X^1) = \ker[\mathcal{E}^{(1,0)}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{(1,1)}(X)]$$

$$H^1(X, \mathcal{O}_X) = \operatorname{coker}[\mathcal{E}^{(0,0)}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{(0,1)}(X)]$$

$$H^0(X, \Omega_X^1) = \ker[\mathcal{E}^{(1,0)}(X) \xrightarrow{\bar{\partial}} \mathcal{E}^{(1,1)}(X)]$$

The cohomologies for both sheaves vanish for degrees greater than 1.

Dolbeault's theorem shows that on the disk, all of these higher groups are zero. Here is a nontrivial example. Let $E = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}i$. This is a compact Riemann surface which is an example of an elliptic curve. Topologically E is a torus. Differential forms on E can be pulled back to forms on $\mathbb{C}$ which are doubly periodic with respect to Λ. A holomorphic 1-form is given by $f(z)dz$, where f is doubly periodic entire function. This forces f to be constant. We claim that the form $d\bar{z}$ defines a nonzero class in $H^1(E, \mathcal{O}_E)$. To see this, observe that any solution to the equation $\bar{\partial} f = d\bar{z}$ is of the form $\bar{z}$ plus a holomorphic function. There is no way to ever make f doubly periodic. Thus we have proved

$$H^0(E, \Omega_E^1) = \mathbb{C}$$

$$H^1(E, \mathcal{O}_E) \neq 0$$

Similarly

$$H^1(E, \Omega^1_E) \neq 0$$