Chapter 6

Coherent Sheaves

6.1 Finite generation, finite presentation and coherence

Before getting to sheaves, let us discuss some finiteness properties for modules over a commutative ring R. Recall that an R-module M is finitely generated, or of finite type, if we can find a finite set $\{m_1, \ldots, m_n\} \subset M$ such that every element of M is an R-linear combination of the m_i's. This is equivalent to having a surjective R-module homomorphism $R^n \to M$, where the ith standard basis vector maps to m_i. Recall that R is noetherian if every ideal is finitely generated. This implies, and is equivalent to, the fact that a submodule of a finitely generated module is finitely generated. If R is not noetherian, finite generation is often too weak, so we introduce a stronger notion. A module M is finitely presented if the following equivalent conditions hold.

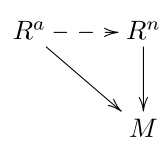
Lemma 6.1.1. *The following are equivalent:*

(a) There is an exact sequence

$$R^a \to R^b \to M \to 0$$

(b) M is finitely generated and the kernel of any surjection $R^n \to M$ is finitely generated.

Proof. Assuming (a), M is clearly finitely generated. Let $R^n \to M$ be another surjection. We can construct the dotted arrow below



inducing a map R^b onto $\ker R^n \to M$, which implies that it is also finitely generated. □

M is coherent if it is finitely generated and all its finitely generated modules are finitely presented. In particular, M is finitely presented. Over a noetherian ring, coherence, finte presentation and finite generation are all the same thing, but not in general. One nice feature of coherence is that it stable under going to submodules. We also have the following important fact.

Theorem 6.1.2 ("2 out of 3" property of coherence)**.** *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence. If any two of M_i are coherent, then so is the third.

Proof. Suppose that M_2, M_3 are coherent. We have to show that M_1 is coherent. Since M_1 is a submodule of M_2, it is enough to show that it is finitely generated. Then we have a surjection $R^a \to M_2$. The kernel K of the composite $R^a \to M_3$ is finitely generated by the previous lemma. We can see that K maps onto M_1 so it is finitely generated.

Suppose that M_1, M_3 are coherent. We can see that if generating $\{m_1, \ldots\}$ and $\{\bar{n}_1, \ldots\}$ are generating sets of M_1 and M_3 respectively, then $\{m_1, \ldots, n_1, \ldots\}$ generate M_2, where the n_i are lifts of $\bar{n}_i$. Therefore M_2 is finitely generated. Let $N \subset M_2$ be a finitely generated submodule. We can assume that it is the image of a map $f : R^n \to M_2$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & R^n & \xrightarrow{=} & R^n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0
\end{array}$$

From the snake lemma, we get an exact sequence

$$0 \to \ker f \to \ker g \to M_1$$

Since M_3 is coherent, $\ker g$ is coherent and in particular finitely generated. Let K denote its image in M_1. The coherence of M_1 implies the coherence of K. Therefore applying the first part of the theorem to

$$0 \to \ker f \to \ker g \to K \to 0$$

shows that $\ker f$ is finitely generated.

The final step is to show that coherence of M_1, M_2 implies coherence of M_3. Since M_3 is a quotient of M_2, it is finitely generated. Given $f : R^n \to M_3$, it can be lifted to a map $g : R^n \to M_2$. We can see that $\ker g$ maps onto $\ker f$. Therefore $\ker f$ is finitely generated. ☐

Corollary 6.1.3. *The direct sum of two coherent modules is coherent.*

Corollary 6.1.4. *If $f : M \to N$ is a homomorphism between coherent modules,* $\operatorname{im} f, \ker f, \operatorname{coker} f$ *are coherent.*

Proof. The image $\operatorname{im} f \subset N$ is coherent because it is a submodule of a coherent module. Now apply the theorem to

$$0 \to \ker f \to M \to \operatorname{im} f \to 0$$

and

$$0 \to \operatorname{im} f \to N \to \operatorname{coker} f \to 0$$

□

Corollary 6.1.5. *If R is coherent, then any finitely presented module is coherent.*

Corollary 6.1.6. *The collection of coherent modules and arbitrary modules forms an abelian subcategory of the category of all modules.*

6.2 Coherent sheaves

Let $(X, \mathcal{R})$ be a ringed space. An $\mathcal{R}$-module $\mathcal{M}$ is coherent if

1. For any point $x \in X$, there exists an open neighbourhood U such that $\mathcal{M}(U)$ is finitely generated. One says that $\mathcal{M}$ is of finite type.
2. If for any open set $U \subset X$ and morphism $\mathcal{R}|_U^n \to \mathcal{M}|_U$, the kernel is of finite type.

Theorem 6.2.1. *The 2 out of 3 property holds for coherent $\mathcal{R}_X$-modules.*

The argument is similar to the previous proof. We get the same corollaries as before. We now turn to some basic examples. Let $X \subset \mathbb{A}_k^n$ be an affine variety, with coordinate ring

$$R = \mathcal{O}(X) \cong k[x_1, \ldots, x_n]/I(X)$$

Given an R-module M, let

$$\tilde{M}(U) = M \otimes_R \mathcal{O}_X(U)$$

This defines an $\mathcal{O}_X$-module. We have

$$\tilde{M}(D(f)) = M[1/f]$$

Since the rings $R[1/f] = \mathcal{O}(D(f))$ are noetherian, we can conclude that:

Example 6.2.2. *If M is finitely generated, then $\tilde{M}$ is coherent.*

It is convenient to define a sheaf of form $\tilde{M}$, where M is not necessarily finitely generated to be quasicoherent.

Theorem 6.2.3. *The operation $M \mapsto \tilde{M}$ induces an equivalence between the category of all (respectively finitely generated) R-modules and the category of quasicoherent (respectively coherent) $\mathcal{O}_X$-modules.*

A proof in the more general setting of schemes can be found in Harshorne, chap II §5. (NB: the definition that Hartshorne gives for coherent sheaves is correct only for noetherian schemes, otherwise you need to use the definition here.) The equivalence is exact, i.e. it takes exact sequences to exact sequences. Let $Y \subset X$ be a subvariety. Then we have an exact sequence of finitely generated modules

$$0 \to I \to \mathcal{O}(X) = R \to \mathcal{O}(Y) \to 0$$

where I is the image of $I(Y)$ in R. Passing to sheaves gives an exact sequence of coherent sheaves

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

The sheaf $\mathcal{I}_Y$ is called the ideal sheaf of Y. $\mathcal{I}_Y(U)$ are regular functions on U vanishing on $Y \cap U$. The sheaf $\mathcal{O}_Y$ is the sheaf of regular functions on Y extended to X. So $\mathcal{O}_Y(U)$ is the ring regular functions on $Y \cap U$. (Using the same symbol $\mathcal{O}_Y$ is a slight abuse of notation, but it won't usually cause a problem.)

Theorem 6.2.4 (Oka)**.** *Let X be a complex manifold, then the sheaf of holomorphic functions $\mathcal{O}_X$ is coherent as an $\mathcal{O}_X$-module.*

A proof can be found in any standard book on several complex variables such as Hörmander. For example, if $Y \subset X$ is complex submanifold. We define the ideal sheaf $\mathcal{I}_Y$ as above, as the collection of holomorphic functions vanishing on Y. Since Y is locally defined by finitely many equations, $\mathcal{I}_Y$ is of finite type.

Corollary 6.2.5. *$\mathcal{I}_Y$ is coherent.*

Corollary 6.2.6. *A locally finitely presented $\mathcal{O}_X$-module is coherent.*

6.3 Locally free sheaves

An $\mathcal{R}$-module $\mathcal{M}$ is called locally free (of rank n) if there exists an open over $\{U_i\}$ such that $\mathcal{M}|_{U_i} \cong \mathcal{R}|_{U_i}^{n_i}$ (with $n = n_i$). If X is disconnected then the ranks can be variable in principle, but we usually won't allow this.

Given C^∞ manifold X, a rank n real vector is manifold V together with a C^∞ map $\pi : V \to X$ such that

1. The fibres $\pi^{-1}(x)$ are n dimensional vector spaces.
2. There exists an open over $\{U_i\}$ and diffeomorphisms

$$\begin{array}{ccc} \pi^{-1}U_i & \xrightarrow{\sim} & U_i \times \mathbb{R}^n \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & U_i & \end{array}$$

which are linear isomorphisms on the fibres. (This data is called a local trivialization.)

Example 6.3.1. *Let $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2\}$ be the 2-sphere. The tangent bundle*

$$T_{S^2} = \{(p,v) \in S^2 \times \mathbb{R}^3 \mid v \cdot p = 0\}$$

which should be thought of as the union of the tangent planes, is a rank 2 vector bundle. (Verifying axiom 2 will be left as an exercise.)

The following is easy.

Lemma 6.3.2. *If $\pi : V \to X$ is C^∞ manifold, then*

$$\tilde{V}(U) = \{v : U \to \pi^{-1}U \mid v\, C^\infty, \pi \circ v = id\}$$

is a locally free C^∞-module.

When X is a complex manifold (or algebraic variety over k), a holomorphic (algebraic) vector bundle can be defined similarly, by replacing $\mathbb{R}$ by $\mathbb{C}$ (or k) and C^∞ by holomorphic (or regular). The above lemma holds, with the obvious modifications, in these cases as well. In these cases, the locally free sheaves are coherent. Here is a very important example.

Example 6.3.3. *Recall that $\mathbb{P}^n_k$ is the set of one dimensional subspaces of $V = k^{n+1}$. Let*

$$L = \{(v,\ell) \in V \times \mathbb{P}^n_k \mid v \in \ell\} \to \mathbb{P}^n_k$$

This is called the tautological line bundle. A line bundle is the same as a rank 1 vector bundle. The sheaf of sections is denoted by $\mathcal{O}_{\mathbb{P}^n}(-1)$.

In the case of affine varieties we have another characterization of locally free sheaves.

Theorem 6.3.4. *Let M be a finitely generated R-module, where X is an affine variety and $R = \mathcal{O}(X)$. The following statements are equivalent*

(a) M is a projective module i.e. a direct summand of a free module.

(b) $Ext^1_R(M,N) = 0$ for all N.

(c) $\tilde{M}$ is locally free

(d) The function $m \mapsto \dim_{k(m)} M_m \otimes_R k(m)$ is constant, as $m \in MaxR$ varies and $k(m) = R/m$ is the residue field.

(I'm too lazy to write this up, but the equivalence of (a), (b), (c) works for affine noetherian schemes discussed briefly in class. You can even drop noetherian provided that you assume M is finitely presented.)

Proof. If $M \oplus M'$ is free, then $Ext^1(M,N) \oplus Ext^1(M',N) = 0$. So (a) implies (b). Suppose (b) holds. Choose a surjection $f : R^n \to M$ and consider the exact sequence

$$0 \to K \to R^n \to M \to 0$$

where K is the kernel. Homing M into this leads to an exact sequence (similar to one we proved)

$$0 \to Hom(M,K) \to Hom(M,R^n) \to Hom(M,M) \to Ext^1(M,K) = 0$$

This means that we can find an element $s : M \to R^n$ such that $f \circ s = id$. In other words, f splits, therefore M is a summand of R^n.

We need to collect a few more facts about Ext. First observe that $Hom_R(M,N)$ has the structure of an R-module and that

$$Hom_R(F^i, N)_f \cong Hom_{R_f}(F^\bullet_f, N_f)$$

where $M_f = M[1/f]$ etc. It follows that if $F^\bullet \to M$ is a free resolution, then

$$Ext^i_R(M,N) = H^i(Hom(F^\bullet, N))$$

is also an R-module. Also localization is exact, so $F^\bullet_f \to M_f$ is again a free resolution. Therefore get the localization property

$$Ext^i_R(M,N)_f \cong Ext^i_{R_f}(M_f, N_f) \tag{6.1}$$

Now suppose (c) holds. We want to show (b) i.e. that $E = Ext^1_R(M,N)$ is zero. Then we can choose an cover $\{D(f_i)\}$ so that M_{f_i} is free. It follows that

$$Ext^1_R(M,N)_f = Ext^1_{R_{f_i}}(M_{f_i}, N_{f_i}) = 0$$

Since $\tilde{E}$ is a sheaf, this forces $E = 0$.

At this point, we need the following fact, which is a consequence of Nakayama's lemma:

Lemma 6.3.5. *Given $m \in MaxR$, $k(m) = R/m$ and K the field of fractions of R_m or equivalently R. Then*

$$\dim M \otimes K = \dim M_m \otimes K \geq \dim M_m \otimes k(m) = \dim M \otimes k(m)$$

with equality if and only of M_m is free.

Using this, we get that (c) implies (d). Conversely, suppose that (d) holds. Let $m \subset R$ be a maximal ideal corresponding to a point $a \in X$. Then from the lemma and (d), we see that M_m is free. Let $m_i/f_i \in M_m$ be a basis. Since this is a finite set, we can choose the same denominator $f = \prod f_i$. If follows that M_f is free. Note that $a \in D(f)$. So by varying a, we can see that M is locally free.

Suppose (a) holds. By assumption $M \oplus M' = R^n$. Then

$$n = \dim M_m \otimes k(m) + \dim M'_m \otimes k(m) \leq \dim M_m \otimes K + \dim M_m \otimes K = n$$

This forces M_m (and M'_m) to be free, so $\dim M_m \otimes k(m) = \dim M \otimes K$. □

6.4 Differentials

Given an affine variety X with coordinate ring R, the R-module of Kähler differentials $\Omega_R = \Omega^1_{R/k}$ is generated by symbols df for $f \in R$ subject to relations

$$d(a_1 f_1 + a_2 f_2) = a_1 df_1 + a_2 df_2, \quad d(f_1 f_2) = f_1 df_2 + f_2 df_1$$

for $a_i \in k, f_i \in R$. This should be viewed as the module of regular 1-forms on X.

Example 6.4.1. *Let $X = \mathbb{A}^n_k$, so that $R = k[x_1, \ldots, x_n]$, then $\Omega_R = \bigoplus Rdx_i$ which is a free module of rank n.*

Example 6.4.2. *Let $X = V(f) \subset \mathbb{A}^n_k$ be hypersurface, then*

$$\Omega_R = \bigoplus Rdx_i / \langle \sum \frac{\partial f}{\partial x_i} dx_i \rangle$$

Let $m = m_a$, then

$$\Omega_R \otimes k(m) \cong k^n / \langle (\frac{\partial f}{\partial x_1}(a), \ldots) \rangle$$

Therefore Ω_R is locally free of rank $n-1$ if and only if the gradient $(\frac{\partial f}{\partial x_1}(a), \ldots) \neq 0$ for all $a \in X$, or in others words if X is nonsingular.

To extend this notion to other varieties, we have to first define the dimension. Given an affine variety, let $\dim X$ denote the transcendence degree of the field of fractions of the coordinate ring R. Alternatively, this can be defined as the Krull dimension of R. This is length of the maximal strictly increasing chain of prime ideals

$$p_0 \subset p_1 \subset \ldots p_n$$

If X is an arbitrary irreducible variety $\dim X$ is the dimension of any nonempty open affine set. The field of fractions of the coordinate ring of a nonempty open affine set is independant of the set, and is called the function field $k(X)$ of X. Thus $\dim X$ is the transcendence degree of $k(X)$. Elements of $k(X)$ are called rational functions on X. They are the algebraic version of meromorphic functions.

Example 6.4.3. $\dim \mathbb{P}^n = \dim \mathbb{A}^n = tr.deg.k(x_1, \ldots, x_n) = n$.

Theorem 6.4.4. *For any algebraic variety, there exists a coherent sheaf Ω^1_X such that $\Omega^1_X|_U = \tilde{\Omega}_R$ for any open affine subset $U \subset X$ with coordinate ring R.*

There a couple of ways to prove this. The most direct method is to choose an affine cover $\{U_i\}$, use the above formula to define $\Omega^1_{U_i}$ and then patch these together. For patching to work, we need isomorphisms $\Omega^1_{U_i}|_{U_i \cap U_j} \cong \Omega^1_{U_j}|_{U_i \cap U_j}$ (subject to some further conditions called cocycle conditions). This is possible because Ω is compatible with localization in the sense that

$$\Omega_{S^{-1}R} \cong S^{-1}\Omega_R$$

(See Matsumura's Commutative Algebra for a proof of the last fact.)

We call X nonsingular, if Ω^1_X is a locally free of rank $\dim X$. For example, $\mathbb{P}^n$ is nonsingular. Here is characterization in terms of local rings.

Theorem 6.4.5. *X is nonsingular if and only if for all $a \in X$, the local ring $\mathcal{O}_{X,a}$ is regular which means that the Krull dimension $\dim O_{X,a}$ equals the vector space dimension $\dim m_a/m_a^2$.*

In outline, we can assume that X is affine with coordinate ring R. By algebra, $\dim R = \dim R_m$ for any maximal ideal $m = m_a$. Given $r \in R_m$, we can consider the image

$$\overline{dr} \in \Omega_{R_m} \otimes k(m) \cong \Omega_R \otimes k(m)$$

If $r \in m^2$, then we can write it as sum $\sum f_i g_i$, with $f_i, g_i \in m$. Therefore

$$\overline{dr} = \sum f_i(a) dg_i + g_i(a) df_i = 0$$

Thus $r \mapsto \overline{dr}$ induces a map $R/m^2 \to \Omega_R \otimes k(m)$. Then the theorem reduces to the following lemma

Lemma 6.4.6. *The restriction of the above map gives an isomorphim $m/m^2 \cong \Omega_R \otimes k(m)$*

Suppose that M is a module over a ring R, then define the exterior power

$$\wedge^p M = \underbrace{M \otimes \ldots \otimes M}_{p} / \langle m_1 \otimes \ldots \otimes m_p - m_1 \otimes \ldots m_{i+1} \otimes m_i \otimes \ldots m_n \rangle$$

The image of $m_1 \otimes \ldots \otimes m_p$ is denoted by $m_1 \wedge \ldots \wedge m_p$. When M is free with basis $m_1, \ldots, m_n$, $\wedge^p M$ is free with basis $\{m_{i_1} \wedge \ldots m_{i_p} \mid i_1 < \ldots i_p\}$. If $\mathcal{M}$ is a sheaf of modules, then $\wedge^p \mathcal{M}$ is the sheafification of the presheaf $U \mapsto \wedge^p \mathcal{M}(U)$. This is locally free if $\mathcal{M}$ is. We define the sheaf of p-forms on a nonsingular variety by $\Omega^p_X = \wedge^p \Omega^1_X$. We can define other linear algebra operations as well. Given two $\mathcal{O}_X$-modules $\mathcal{M}, \mathcal{N}$, define $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sheafification of the presheaf $U \mapsto \mathcal{M}(U) \otimes_{\mathcal{O}(U)} \mathcal{N}(U)$. And $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N})$ as the sheafification of the presheaf $U \mapsto Hom_{\mathcal{O}(U)}(\mathcal{M}(U), \mathcal{N}(U))$.

6.5 Divisors

We discuss the notion of divisors which is an older language than sheaves, and in many ways more intuitive. It is still used very important.

Recall that a ring is a unique factorization domain (UFD), or factorial, if it is an integral domain and any nonzero element is either a unit or a product of irreducible elements, such that the product unique up to reordering and multiplication by units. For example $k[x_1, \ldots, x_n]$ is a UFD. It is convenient to recast this property in terms of prime ideals. A prime ideal p has height 1 if 0 and p are the only prime ideals contained in p. This equivalent to saying the Krull dimension of the quotient is 1.

Theorem 6.5.1. *A noetherian domain is a UFD if and only if all height* 1 *prime ideals are principal.*

Proof. Suppose that $p \subset R$ is a height 1 prime in a UFD. Let $f \in P$ be a nonzero element. Write $f = uf_1f_2\ldots$ be product of a unit times irreducible elements. At least one of these factors lie in p because p is prime. It can't be u, otherwise we would have $p = R$. So $(f_i) \subseteq p$. But (f_i) is prime. Therefore $(f_i) = p$.

The proof of the converse is harder, and can be found in Matsumura's Commutative Algebra. □

An important source of examples is provided by

Theorem 6.5.2 (Auslander-Buchsbaum, Serre)**.** *A regular local ring is a UFD.*

Proof. See Matsumura. □

Let X be a nonsingular algebraic variety. Suppose that $D \subset X$ is an irreducible subset such that $\dim D = \dim X - 1$. D is called an irreducible hypersurface or prime divisor. We can form the ideal sheaf $\mathcal{I}_D$ of regular functions vanishing along D. Just to make life more confusing, this is usually written as $\mathcal{O}_X(-D)$.

Theorem 6.5.3. *Let D be a prime divisor. then $\mathcal{O}_X(-D)$ is locally free sheaf of rank one (also called an invertible sheaf, also often called a line bundle).*

Proof. The local rings are regular and therefore UFDs. The stalk $\mathcal{O}_X(-D_1)_a$ is height 1 prime ideal in $\mathcal{O}_a$. Therefore it is principal, and consequently free of rank 1. □

Given D as above and $a \in D$, the ideal $\mathcal{O}(-D)_a$ is principal. Choose a generator $f \in \mathcal{O}_a \subset k(X)$. This is called a local equation of D. Any function $g \in k(X)^*$ can be written as uf^m, where $u \in \mathcal{O}_a$ is a unit. Set $ord_D(g) = m$ and $ord_D(0) = \infty$. This is independant of f and u and measures the order of zero or pole along D. We define $\mathcal{O}_X(nD)(U) \subset k(X)$ to be set of rational functions which are regular on $U - D$ and satisfy $ord_D(g) \geq -n$. More generally, we introduce a formal finite sum $\sum n_iD_i$, where $n_i \in \mathbb{Z}$ and D_i are prime divisors. This is called a (Weil) divisor. Let $\mathcal{O}(\sum n_iD_i)(U)$ be sheaf of rational functions satisfying $ord_{D_i}(g) \geq -n_i$.

Lemma 6.5.4. *$\mathcal{O}(\sum n_iD_i)$ is a line bundle. There are isomorphisms*

$$\mathcal{O}(D+E) \cong \mathcal{O}(D) \otimes \mathcal{O}(E)$$

and

$$\mathcal{O}(-D) = \mathcal{H}om(\mathcal{O}(D), \mathcal{O})$$

(The last operation is usually denoted by $\mathcal{O}(D)^{-1}$.)

Proof. If f_i is a local equation of D_i. Let $D = \sum n_i D_i$ and $E = \sum m_i D_i$, $\mathcal{O}(D)$ is locally spanned by $\prod f_i^{-n_i}$. Therefore it is a line bundle. The map which sends

$$(\prod f_i^{-n_i}) \otimes (\prod f_i^{-m_i}) \to \prod f_i^{-(n_i+m_i)}$$

and

$$(\prod f_i^{-n_i})^{-1} \to \prod f_i^{n_i}$$

are isomorphisms. □

A central problem in algebraic geometry, that we will come back to, is:

Problem 6.5.5 (Riemann-Roch)**.** *Given a divisor D on a smooth projective variety X, compute* $\dim H^0(X, \mathcal{O}(nD))$ *as a function of n (these numbers are finite).*

The collection of divisors on X forms an abelian group $Div(X)$. Given $f \in k(X)^*$, set

$$div(f) = \sum_D ord_D(f) D$$

where the sum runs over all divisors. In fact, it can be shown to be finite sum, thus it defines an element of $Div(X)$.

Lemma 6.5.6. *ord_D are valuations, and in particular homomorphisms $k(X)^* \to \mathbb{Z}$. Therefore $div : k(X)^* \to Div(X)$ is also a homomorphism.*

The cokernel

$$Cl(X) = Div(X)/\{div(f) \mid f \in k(X)^*\}$$

is called the (divisor) class group of X. We have been tacitly assuming that X is nonsingular. In fact, we can get by with a weaker assumption. A variety is normal if all its local rings are integrally closed. Note that UFDs are integrally closed, so nonsingular implies normal. The class group is defined for normal varieties. We can use this to test the UFD property as follows.

Lemma 6.5.7. *Suppose that X is a normal (e.g. nonsingular) affine variety with $R = \mathcal{O}(X)$. Then R is a UFD if and only if $Cl(X) = 0$.*

Proof. Suppose that R is a UFD. If $D = \sum n_i D_i \in Div(X)$. The ideal of the prime divisors D_i are principal and therefore generated by $f_i \in R$. Then $div(\prod f_i^{n_i}) = D$.

Conversely if $Cl(X) = 0$, we can see that a height one prime must be principal. □

Exercise 15. *Let $X = V(y^2 - x(x-1)(x-2)) \subset \mathbb{A}^2_{\mathbb{C}}$ be an affine elliptic curve. Show that this is nonsingular, but that $R = \mathbb{C}[x,y]/(y^2 - x(x-1)(x-2))$ is not a UFD. (Hint: either use the fact that y^2 has two incompatible factorizations, or show that (x,y) gives a height one nonprincipal prime ideal.) It follows that $Cl(X) \neq 0$. In fact, we will see later that $Cl(X)$ is uncountable.*