

## Chapter 3

# Hilbert and Picard modular surfaces

### 3.1 The Hilbert modular group

Let  $D > 0$  be square free integer ( $D$  is not divisible by any squares other than 1). Let  $K = \mathbb{Q}(\sqrt{D})$  be the corresponding real quadratic field. This is Galois over  $\mathbb{Q}$ , with Galois group generated by the involution  $(a + b\sqrt{D})' = a - b\sqrt{D}$ . The norm  $N(x) = xx'$  and trace  $\text{tr}(x) = x + x'$ . There are two embeddings of fields  $\sigma_i : K \hookrightarrow \mathbb{R}$  given by

$$\sigma_1(a + b\sqrt{D}) = a + b\sqrt{D}$$

$$\sigma_2(a + b\sqrt{D}) = a - b\sqrt{D}$$

This gives an embedding of groups  $SL_2(K) \hookrightarrow SL_2(\mathbb{R})^2$  by  $A \mapsto (\sigma_1(A), \sigma_2(A))$ .

The ring of integers  $\mathcal{O}_K \subset K$  is the integral closure of  $\mathbb{Z}$  in  $K$ . More explicitly,

$$\mathcal{O}_K = \begin{cases} \mathbb{Z} + \frac{1+\sqrt{D}}{2}\mathbb{Z} & \text{if } D \equiv 1 \pmod{4} \\ \mathbb{Z} + \sqrt{D}\mathbb{Z} & \text{if } D \equiv 2, 3 \pmod{4} \end{cases}$$

$\mathcal{O}_K$  is a Dedekind domain, so we can define the class group  $Cl(\mathcal{O}_K)$  in the usual way, as the group of fractional ideals modulo principal ideals. This is a finite group; its cardinality  $h$  is called the class number of  $K$ .

**Lemma 3.1.1.** *If we embed  $\mathcal{O}_K \hookrightarrow \mathbb{R}^2$  by  $\sigma_1 \times \sigma_2$ , then the image is discrete.*

This is false with only a single  $\sigma_i$ .

*Proof.* This is elementary. For example, when  $D \equiv 2, 3 \pmod{4}$ , this follows from the inequality

$$(a + b\sqrt{D})^2 + (a - b\sqrt{D})^2 = a^2 + b^2D \geq 1$$

for a nonzero integer  $a + b\sqrt{D}$ . □

The *Hilbert modular group* (for a given  $K$ ) is  $\Gamma_K = SL_2(\mathcal{O}_K)$ . This embeds into  $SL_2(\mathbb{R})^2$  as above. From the previous lemma, we easily deduce:

**Corollary 3.1.2.** *The image of the Hilbert modular group in  $SL_2(\mathbb{R})^2$  is discrete with respect to the usual topology.*

Given a nonzero ideal  $I \subset \mathcal{O}_K$ , we can define the corresponding principal congruence group as

$$\Gamma_K(I) = \ker[SL_2(\mathcal{O}_K) \rightarrow SL_2(\mathcal{O}_K/I)]$$

This applies, in particular, to an ideal of the form  $(N)$ ,  $N$  where  $N$  is a nonzero integer.

**Lemma 3.1.3.**  $\Gamma_K(N)$  is torsion free when  $N \geq 3$ .

*Proof.* [F, p 42]. □

We will refer to a subgroup containing some  $\Gamma(N)$  as a congruence group.

## 3.2 Hilbert modular surfaces: topology

Fix  $K = \mathbb{Q}(\sqrt{D})$  as before. The group  $\Gamma_K$  acts on  $\mathbb{H}^2$  through its embedding into  $SL_2(\mathbb{R})^2$ . To be more explicit, given  $(z_1, z_2) \in \mathbb{H}^2$  and  $A \in \Gamma_K$ ,  $A \cdot (z_1, z_2) = (A \cdot z_1, A \cdot z_2)$ . The action factors through  $PSL_2(\mathcal{O}_K) = SL_2(\mathcal{O}_K)/\{\pm I\}$ .

**Proposition 3.2.1.** *This action is properly discontinuous.*

*Proof.* This follows from the discreteness of  $\Gamma_K$  by [F, p 21]. □

Let  $\Gamma \subseteq \Gamma_K$  be a congruence subgroup. The quotient  $X^o(\Gamma) = \Gamma \backslash \mathbb{H}^2$ , and various related objects, are called *Hilbert modular surfaces* or sometimes Hilbert-Blumenthal surfaces. The proposition implies that the quotient topology has reasonable properties (e.g. it is Hausdorff). However, the action of  $PSL_2(\mathcal{O}_K)$  is not free. For instance,  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  fixes  $(i, i)$ . If we restrict to a torsion free subgroup such as  $\Gamma_K(3)$ , then the action becomes free. It follows that isotropy subgroup of  $\Gamma$  for any point of  $\mathbb{H}^2$  is finite. In fact, one can show that they are cyclic. We claim that the image of any fixed point in  $X^o(\Gamma)$  is topologically singular in the sense that  $X^o(\Gamma)$  is not even a topological manifold at that point. To see this, we can work locally. A local model for this is given as follows. Choose a finite cyclic subgroup  $G \subset GL_2(\mathbb{C})$ , such that  $\mathbb{C}^2$  has no nonzero invariant vectors. The quotient  $\mathbb{C}^2/G$  is a singular algebraic variety. In the simplest example,  $G = \{\pm 1\}$ ,  $u = x^2, v = y^2, w = xy$  generate the ring of invariant polynomials, and  $\mathbb{C}^2/G = V(uv - w^2)$ . A  $n$ -manifold  $X$  has the property that any point  $x \in X$  has a fundamental system of neighbourhoods  $U$ , such that that  $U - x$  is homotopy equivalent to  $S^{n-1}$ . This is not true for  $\mathbb{C}^2/G$ , with its usual topology.

Finally, we note that  $X^\circ(\Gamma)$  is not compact. We can compactify it, as we did for modular curves, by adding cusps. We embed  $\mathbb{P}^1(K) \rightarrow \mathbb{P}(\mathbb{R})^2$  by  $\sigma_1 \times \sigma_2$ . Using this, we can regard points of  $\mathbb{P}(K)$  as lying on the boundary of  $\mathbb{H}^2$  via

$$\mathbb{P}^1(\mathbb{R})^2 \subset \mathbb{P}^1(\mathbb{C})^2 \supset \mathbb{H}^2$$

A  $\Gamma$ -orbit of point of  $\mathbb{P}^1(K)$  is called a *cusps* with respect to  $\Gamma$ . Given a point  $[a, b] \in \mathbb{P}^1(K)$ , we let  $(a, b)$  denote the fractional ideal generated by these elements. Although this ideal is not well defined, its class in  $Cl(\mathcal{O}_K)$  is.

**Proposition 3.2.2.** *The above map gives a bijection between the set of cusps for  $\Gamma_K$  and  $Cl(\mathcal{O}_K)$ .*

*Proof.* Let  $\phi : \mathbb{P}^1(K) \rightarrow Cl(\mathcal{O}_K)$  denote the above map. It is known that any fractional ideal of  $\mathcal{O}_K$  is generated by two elements. Therefore  $\phi$  is surjective.

If  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  then

$$\phi(A \cdot [x, y]) = (ax + by, cx + dy) \subseteq (x, y)$$

The same argument, using  $A^{-1}$ , gives the opposite inclusion. Therefore  $\phi$  factors through a map  $\bar{\phi} : \Gamma_K \backslash \mathbb{P}^1(K) \rightarrow CL(\mathcal{O}_K)$ .

It remains to prove that  $\bar{\phi}$  is injective. We assume two points of  $\mathbb{P}^1(K)$  have the same image under  $\phi$ . For simplicity, we treat the case where one of the points is  $\infty = [1, 0]$ . Denote the other by  $[x, y]$ . We can assume that both  $x, y \in \mathcal{O}_K$ . Since  $(x, y) = \phi(\infty) = (1)$ , we must have  $ax + by = 1$  for some  $a, b \in \mathcal{O}_K$ . Then

$$A = \begin{pmatrix} a & b \\ -y & x \end{pmatrix}$$

lies  $\Gamma_K$  and it maps  $[x, y]$  to  $\infty$ . Therefore they lie in the same  $\Gamma_K$ -orbit.  $\square$

**Corollary 3.2.3.**  *$\Gamma_K$  has  $h$  cusps. A congruence subgroup  $\Gamma \subset \Gamma_K$  has a finite number of cusps.*

Let  $(\mathbb{H}^2)^* = \mathbb{H}^2 \cup \mathbb{P}^1(K)$ . We put a topology on this, such that

1. It agrees with the usual one on  $\mathbb{H}^2$
2. The sets of form

$$U_C = \{(z_1, z_2) \mid \text{Im}(z_1)\text{Im}(z_2) > C\} \cup \{\infty\}, \quad C \in \mathbb{R}^+$$

forms a fundamental systems of neighbourhoods of  $\infty$ .

3. If  $p = A\infty$ , with  $A \in SL_2(K)$ , then  $AU_C$  forms a fundamental system of neighbourhoods of  $p$ .

Let  $\Gamma \subset \Gamma_K$  be a congruence group. Let  $X(\Gamma) = \Gamma \backslash (\mathbb{H}^2)^* = X^\circ(\Gamma) \cup \{\text{cusps}\}$  with the quotient topology. To analyze the quotient, we let  $\Gamma_p$  be the isotropy group of a cusp  $p$ . The structure of this group is essentially given as follows.

**Lemma 3.2.4.** *There exists a rank 2 group additive subgroup  $M \subset K$ , and finite index multiplicative subgroup  $V \subseteq \{u \in \mathcal{O}_K^* \mid u \text{ totally positive, i.e. } \sigma_i(u) > 0\}$  such that the  $V$  stabilizes  $M$ , and the group*

$$G(M, V) = \left\{ \begin{pmatrix} \epsilon & \mu \\ 0 & \epsilon^{-1} \end{pmatrix} \mid \mu \in M, \epsilon \in V \right\}$$

*has finite index in  $\Gamma_p$ .*

**Lemma 3.2.5.**

1. *For any  $C > 0$ ,  $\Gamma_\infty$  stabilizes  $U_C$  and  $\Gamma_\infty \backslash \bar{U}_C$  is compact.*
2. *For  $C \gg 0$ , the image of  $U_C$  in  $X(\Gamma)$  is homeomorphic to  $\Gamma_\infty \backslash U_C$*
3. *If  $p = A\infty$  is a cusp different from  $\infty$ , i.e. if  $p$  does not lie in the  $\Gamma$  orbit of  $\infty$ , then for  $C \gg 0$ , the images of  $U_C$  and  $AU_C$  in  $X(\Gamma)$  are disjoint.*

*Proof.* See [G, pp 7-9]. □

**Theorem 3.2.6.**  *$X(\Gamma)$  is compact Hausdorff.*

*Proof.* Using the previous lemma, one can build a compact fundamental domain. See [F, p 38] for details. □

### 3.3 Hilbert modular forms

In order to discuss the analytic properties of Hilbert modular surfaces, we need the appropriate category. An analytic space is to a complex manifold what an algebraic variety, or more generally scheme, is to a nonsingular variety. A basic example is to start with an open ball  $B \subset \mathbb{C}^N$ , choose a collection of holomorphic functions  $f_1, f_2, \dots, f_n$  and consider the zero set  $Z = \{x \in B \mid f_i(x) = 0\}$ . We refer to this as a *model*. In general, an analytic space is something which locally looks like a *model*. To make this more precise, we can proceed as with scheme theory (for those familiar with them) by introducing a sheaf. For our model  $Z$ , let  $\mathcal{O}_Z$  denote the sheaf of restrictions of holomorphic functions from  $B$  to  $Z$ . A (reduced) *analytic space* is a pair  $(X, \mathcal{O}_X)$  consisting of a paracompact Hausdorff space, and a sheaf of continuous complex valued functions, such that it is locally isomorphic (as a locally ringed space) to a pair given by a model. See Grauert-Remmert [GR] for further details (where analytic spaces are called complex spaces). An analytic space  $(X, \mathcal{O}_X)$  is called *normal* if all its stalks  $\mathcal{O}_{X,x}$  are integrally closed. Here are a couple of examples.

1. Complex manifolds are normal because the stalks are rings of convergent power series, and these are regular noetherian and therefore integrally closed.
2. If  $G \subset GL_n(\mathbb{C})$ , then  $\mathbb{C}^n/G$  is normal. Away from 0, it's a manifold, The stalk at 0 is the ring of  $G$ -invariant convergent power series, and this is easily seen to be integrally closed.

The importance of this condition for us stems from the following. Suppose that  $X$  is normal, then:

1. The set of singular points (points where it fails to be a manifold) has codimension at least 2 [GR, p 128]
2. Any holomorphic function defined on the nonsingular part of  $X$  extends to a holomorphic function on  $X$  [GR, p 144]

Here is a useful criterion.

**Theorem 3.3.1** (Cartan). *Suppose that  $X - \{x\}$  is a normal analytic space, and there is a system of neighbourhoods  $U$  of  $x$  such that  $U - \{x\}$  is connected and such that holomorphic functions in  $U - \{x\}$  separate points. If we define  $f : U \rightarrow \mathbb{C}$  to be holomorphic if it is continuous and holomorphic away from  $x$ , then  $X$  becomes a normal analytic space.*

*Proof.* [C, exp 11]. □

Let  $K, (\mathbb{H}^2)^*, \Gamma \subseteq \Gamma_K$  and  $X = X(\Gamma)$  be as before. We have projections  $\pi : (\mathbb{H}^2)^* \rightarrow X$  and  $\pi^o : \mathbb{H}^2 \rightarrow X$ . If  $U \subseteq X$  is open, define  $f \in \mathcal{O}_X(U)$  if it is continuous and if  $f \circ \pi^o$  is holomorphic.

**Theorem 3.3.2.**  *$(X, \mathcal{O}_X)$  is a normal analytic space.*

*Proof.* Away from the cusps, this is easy by the above remarks. At a cusp, say  $\infty$ , one checks Cartan's criterion holds. The topological condition for theorem 3.3.1 is clearly satisfied. We just have to check the second. Given a bounded holomorphic function  $f$  on  $U_C$ , the associated Poincaré series, which is the sum

$$\sum_{A \in \Gamma_\infty} f(A \cdot z)$$

can be shown to converge to a  $\Gamma_\infty$ -invariant holomorphic function [F, p 57-58, p 113]. This construction yields sufficiently many holomorphic functions on  $\Gamma_p \backslash U_C$  to separate points. □

The singular points of  $X$  consist of images of the fixed points in  $\mathbb{H}^2$  and the cusps. Since  $X$  is normal, we can define holomorphic functions, and related things, by prescribing them away from these points. So for example, a holomorphic function on  $X$  is given by a  $\Gamma_K$ -invariant function on  $\mathbb{H}^2$ . Such a function is necessarily constant by compactness of  $X$ . To get something more interesting, we have to relax the invariance condition. A Hilbert modular form of weight  $(k, \ell)$ , with respect to  $\Gamma$ , is a holomorphic function on  $\mathbb{H}^2$  satisfying

$$f\left(\frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'}\right) = (cz_1 + b)^k (c'z_2 + b')^\ell f(z_1, z_2) \quad (3.1)$$

for every element of  $\Gamma$ . We say this has weight  $k$ , when  $k = \ell$ , and this what we mostly care about. By a calculation similar to what we did earlier, we can see

that  $f$  is a weight  $2k$  modular form precisely when the tensor  $f(dz_1 \wedge dz_2)^{\otimes k}$  is invariant under  $\Gamma_K$ . Note that, unlike the one dimensional case, we don't have to impose any extra holomorphicity conditions at the cusps, since this comes for free by normality. This fact, which can be checked to directly, goes by name of the "Koecher principle". To be see this, recall that is the cusp  $p \in \mathbb{P}^1(K)$  is stable under translations by

$$G(M, V) \supset \left\{ \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \mid \mu \in M \right\} \cong \mathbb{Z}^2$$

for some choice of  $(M, V)$ . The functional equation (3.1) forces periodicity with respect to this group. Therefore we have a Fourier expansion. The extension property amounts to showing that that the negative degree Fourier coefficients automatically vanish. In fact, a bit more is true.

**Proposition 3.3.3.** *A holomorphic function  $f$  at  $p$  has a Fourier expansion*

$$f(z_1, z_2) = a_0 + \sum_{\nu \in M^\vee} a_\nu \exp(2\pi i(\nu z_1 + \nu' z_2))$$

where  $M^\vee = \{\nu \in K \mid \forall \mu \in M, \text{tr}(\mu\nu) \in \mathbb{Z}\}$ . Furthermore, we have

$$a_{\epsilon\nu} = a_\nu, \forall \epsilon \in V \quad (3.2)$$

and  $a_\nu = 0$  unless  $\nu$  is 0 or totally positive.

*Proof.* The Fourier expansion and (3.2) follows from the invariance under the group  $G(M, V)$ . Suppose that  $a_\nu \neq 0$  where  $\nu \neq 0$  is not totally positive. Then for  $\epsilon \in V$ ,  $\epsilon > 1$ , one finds that the sequence  $\exp(-2\pi \text{tr}(\epsilon^n \nu))$  is bounded away from 0. This, together with (3.2), would force the Fourier series to diverge along the ray  $\{(ir, ir) \mid r \in \mathbb{R}^+\}$ .  $\square$

A modular form is called a *cuspidal form* if it vanishes at all the cusps, or equivalently if the zeroth Fourier coefficients are all zero. We will mainly be concerned with the case  $\ell = k$ , in which case we call this is a modular form of weight  $k$ . We denote the space of these by  $M_k(\Gamma)$ , and subspace of cuspidal forms by  $S_k(\Gamma)$ . We give a basic example. We try to form the series

$$\sum \frac{1}{[(cz_1 + d)(c'z_2 + d')]^k}$$

where  $k$  is even and  $(c, d)$  runs over  $\mathcal{O}_K^2$ . This will have the right formal properties, but it will diverge. The problem is that terms are repeated infinitely often. We can correct the problem by choosing  $(c, d)$  to range over a set of representatives for the orbit space  $\mathcal{O}_K^2 / \mathcal{O}_K^*$  under  $(c, d) \mapsto (\epsilon c, \epsilon d)$ . Then for  $k > 2$ , this will converge to an element of  $M_k(\Gamma_K)$  called an Eisenstein series.

One of the things we can use Hilbert modular forms for is to embed  $X$  into projective space. If  $f_0, \dots, f_N$  is a collection of Hilbert modular forms of the same weight  $k$ , then we get a map  $X \dashrightarrow \mathbb{P}^N$  by  $x \rightarrow [f_0(x), \dots, f_N(x)]$ .

**Theorem 3.3.4** (Baily-Borel). *There are sufficiently many modular forms of some weight to get an embedding  $X \hookrightarrow \mathbb{P}^N$*

Baily and Borel's [BB] theorem holds not just for Hilbert modular surfaces, but more generally for quotients of hermitian symmetric spaces by arithmetic groups. In particular, their theorem will apply to various other the examples considered later on. By using an extension of Chow's theorem due to Serre [GAGA], we obtain

**Corollary 3.3.5.**  *$X$  is a normal projective variety.*

This has a moduli interpretation, but the explanation will have to wait until we get to abelian varieties.

### 3.4 Riemann-Roch for surfaces

Fix  $K, \dots, X = X(\Gamma)$  as above. Since  $X$  is a normal projective surface, we can use methods from algebraic geometry to study it. We also need to appeal to Serre's GAGA theorem [GAGA], to switch from holomorphic to algebraic objects. As a first step, we need to resolve the singularities.

**Theorem 3.4.1.** *There exists a regular map  $\pi : Y \rightarrow X$  such that*

- (a)  *$Y$  is nonsingular.*
- (b)  *$\pi$  is an isomorphism over the nonsingular locus of  $X$ .*
- (c) *If  $Y'$  is a nonsingular surface through which  $\pi$  factors, then  $Y = Y'$*

We fix one such surface  $Y(\Gamma) = Y$ , which is called a *minimal resolution* of  $X$ . We note that is unique under some additional assumptions, and it always exists by the general theory [BPV]. In the present case, it can be constructed quite explicitly [G]. The explicit construction yields more information, which is needed to prove some of the results below. For example, it known that the preimage of a cusp (resp. non-cusp singularity) is a cycle (resp. chain) of rational curves.

We recall some basic facts from algebraic surface theory. Fix a nonsingular projective surface  $S$  over an algebraically closed field  $k$ . We will only need the case where  $k = \mathbb{C}$ , but state the results more generally when possible. A divisor on  $S$  is a finite sum  $D = \sum n_i C_i$ , where  $n_i \in \mathbb{Z}$  and  $C_i \subset X$  are possibly singular irreducible closed curves. Any such curve determines a discrete valuation  $ord_C$  on the field of rational functions  $k(S)$ , which measures the order of zero or pole along it. If  $f$  is a nonzero rational function on  $S$ , we can define the associated principal divisor  $\text{div } f = \sum ord_C(f)C$ . The divisor class group  $Cl(S)$  is defined as for curves by the abelian group of all divisors by the subgroup of principal divisors. If  $\omega$  is a rational 2-form, we can define  $\text{div } \omega = \sum ord_C(\omega)C$  with a suitable definition of  $ord_C(\omega)$ . The divisor class is independent of  $\omega$  and it is called the canonical divisor class  $K_S$ , or simply  $K$  (it is unlikely to be confused

with the field). Given a divisor  $D$ , we define sheaf  $\mathcal{O}_S(D)$  exactly as for curves. In particular, the space of global sections

$$H^0(S, \mathcal{O}_S(D)) = \{f \in k(S)^* \mid \operatorname{div} f + D \geq 0\} \cup \{0\}$$

This is finite dimensional, and we denote its dimension by  $h^0(\mathcal{O}_S(D))$ . The isomorphism class of  $\mathcal{O}_S(D)$  depends only on the class of  $D$  in  $Cl(S)$ . The sheaf  $\mathcal{O}_S(K)$  is isomorphic to the sheaf of regular 2-forms  $\Omega_S^2 = \wedge^2 \Omega_S^1$ . In particular,  $h^0(\mathcal{O}(K))$  is the dimension of the space of regular 2-forms. This is one of the fundamental invariants of  $S$ , called the *geometric genus*  $p_g(S)$ . There is a new phenomenon for surfaces, namely divisor can be intersected. Suppose that  $C$  and  $D$  are distinct irreducible curves. Then  $C \cap D$  is finite. If  $p \in C \cap D$ , define the intersection multiplicity at  $p$  by

$$(C \cdot D)_p = \dim \mathcal{O}_{S,p}/(f, g) = \dim \hat{\mathcal{O}}_{S,p}/(f, g)$$

where  $\mathcal{O}_{S,p}$  is the local ring of the surface at  $p$ , and  $f, g$  are local equations of  $C$  and  $D$  in this ring. For example, this number is 1 if  $C$  and  $D$  are nonsingular and meet transversely at  $p$ , because  $f$  and  $g$  generate the maximal ideal of the completion  $\hat{\mathcal{O}}_{S,p}$ . Define the intersection number

$$C \cdot D = \sum_{p \in C \cap D} (C \cdot D)_p$$

**Theorem 3.4.2.** *There exists a bilinear form  $Cl(S) \times Cl(S) \rightarrow \mathbb{Z}$  which agrees with the above intersection number whenever it is defined.*

*Proof.* See [H, chap 5, sec. 1]. □

We come to the main point. One would like a formula for  $h^0(\mathcal{O}_S(D))$  for any divisor, but what one has is a formula for the Euler characteristic

$$\chi(\mathcal{O}(D)) = h^0(\mathcal{O}(D)) - h^1(\mathcal{O}(D)) + h^2(\mathcal{O}(D))$$

where  $h^i$  represent dimensions of higher cohomology groups. One can view the higher  $h^i$  above as corrections. These will be dealt with shortly.

**Theorem 3.4.3** (Riemann-Roch for surfaces). *For any divisor*

$$\chi(\mathcal{O}(D)) = \frac{1}{2} D \cdot (D - K) + \chi(\mathcal{O}_S)$$

When  $k = \mathbb{C}$

$$\chi(\mathcal{O}_S) = \frac{K^2 + e(S)}{12}$$

where  $e(S)$  is the topological Euler characteristic.

*Proof.* For the first formula, see [H, chap 5, sec. 1]. The second is special case of the Hirzebruch-Riemann-Roch theorem [BPV, p 20] □

In order to get an exact formula for  $h^0(\mathcal{O}(D))$ , we need to combine Riemann-Roch with a so called vanishing theorem. We use the famous Kodaira vanishing theorem. We work over  $\mathbb{C}$  since the result can fail in positive characteristic. A divisor  $D$  is called ample if  $\mathcal{O}(nD)$  has sufficiently many sections to give an embedding of  $S$  into projective space for all  $n \gg 0$ .

**Theorem 3.4.4** (Kodaira Vanishing). *Assume  $k = \mathbb{C}$ . If  $D$  is an ample divisor, then*

$$H^i(X, \mathcal{O}(K + D)) = 0 \quad i > 0$$

*Proof.* See [GH, chap 1] for a proof. We remark that it true in arbitrary dimension. Kodaira's original formulation, which is used in the reference, is that the line bundle  $\mathcal{O}(D)$  carries metric with positive curvature. This is equivalent to ampleness by Kodaira's embedding theorem. There is now a purely algebraic proof, due to Deligne and Illusie, which uses the above formulation  $\square$

Let us return to our Hilbert modular surface  $Y$ . We assume for simplicity that  $\Gamma$  is torsion free. The union of preimages of the cusps in  $X$  forms a divisor  $D$ . The structure of  $D$  can be determined rather explicitly [G, chap 2]. As we saw, an element of  $M_2(\Gamma)$  gives an invariant holomorphic 2-form on  $\mathbb{H}^2$  and therefore a holomorphic form on  $X^\circ(\Gamma)$ . If it vanishes at the cusps, then it would extend to a holomorphic form, and therefore regular form, on  $Y$ . In fact, the converse holds also.

**Theorem 3.4.5.** *There is an isomorphism between  $S_2(\Gamma) \cong H^0(\mathcal{O}_Y(K))$ . Therefore the dimension of this space is the geometric genus  $p_g(Y)$ .*

*Proof.* [G, p 57].  $\square$

For higher weight, we have:

**Theorem 3.4.6.** *The divisor  $F = K + D$  is ample, and for any  $m > 0$ , we have an isomorphism*

$$S_{2m}(\Gamma) \cong H^0(Y, \mathcal{O}_Y(K + (m - 1)F))$$

*Proof.* [G, p 72].  $\square$

From Kodaira vanishing plus Riemann-Roch, we obtain:

**Corollary 3.4.7.**

$$\dim S_{2m}(\Gamma) = \frac{m-1}{2}(K + (m-1)F) \cdot F + \chi(\mathcal{O}_Y)$$

The right side can be evaluated in explicit terms. See [G, chap IV, sect 4] for a more complete discussion.

### 3.5 Picard modular surfaces

Let

$$B = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$$

denote the ball in  $\mathbb{C}^2$ . This can be identified with the subset

$$B' = \{[z_1, z_2, z_3] \in \mathbb{P}^2 \mid |z_1|^2 + |z_2|^2 - |z_3|^2 < 0\}$$

of the complex projective plane, under  $(z_1, z_2) \mapsto [z_1, z_2, 1]$ . If we introduce the nonpositive definite hermitian form  $H$  on  $\mathbb{C}^3$  with matrix

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

then

$$B' \cong \{[v] \in \mathbb{P}^2 \mid H(v, v) < 0\}$$

Let

$$SU(2, 1) = \{A \in GL_3(\mathbb{C}) \mid \bar{A}^T J A = J, \det A = 1\},$$

denote the special unitary group associated to  $H$ . Clearly this group acts on  $B'$ . The following is straightforward.

**Lemma 3.5.1.**  *$SU(2, 1)$  acts transitively on  $B'$ .*

Now we fix a square free integer  $D > 0$  and consider the imaginary quadratic field  $K = \mathbb{Q}(\sqrt{-D})$ . We have embeddings  $\sigma_i : K \rightarrow \mathbb{C}$ , and involution  $x \mapsto x'$ , and the ring of integers  $\mathcal{O}_K$  described exactly as in the real quadratic case. Let  $V = K^3$  with a lattice  $L \subset V$ , i.e. an  $\mathcal{O}_K$ -submodule such that  $K \otimes L \cong V$ . We assume that  $V$  is equipped with a form  $H_0$  which is Hermitian in the sense that  $H_0(ax, y) = aH_0(x, y)$ ,  $H_0(x, y) = H_0(y, x)'$  for  $a \in K$  and  $x, y \in V$ , and  $\mathcal{O}_K$ -valued on  $L$ . We suppose that after extending scalars to  $\mathbb{C}$ , using either  $\sigma_1$  or  $\sigma_2$  (it won't matter),  $H_0$  has signature  $(2, 1)$ . For example,  $H_0(x, y) = x^T J y'$  satisfies these conditions, but it is not the only choice. We can form the special unitary group  $SU(H_0)$ , which is the subgroup of  $SL_3(K)$  of linear transformations which preserve  $H_0$ . The subgroup  $\Gamma_{K, H_0} \subset SU(H_0)$  stabilizing the lattice  $L$  is called a *Picard modular group*. We will fix the above data, and simply denote this group by  $\Gamma$ . By our assumptions,  $H_0$  can be identified with  $H$  after extending scalars, so we can embed  $\Gamma \subset SU(2, 1)$ . This gives an action of  $\Gamma$  on  $B$ . The action is seen to be properly discontinuous. Therefore the quotient  $X^o = X^o(\Gamma, H_0) = \Gamma \backslash B$  inherits a Hausdorff topology.

The space  $X^o$  is not compact. We can compactify it by adding a finite number of points, again called cusps. Using the model  $B'$ , we see that it has a natural boundary  $\partial B'$  consisting of lines generated by vectors which isotropic in the sense that  $H(v, v) = 0$ . Let

$$B^* = B' \cup \{[v] \mid v \in \mathbb{Q}^3, v \text{ isotropic}\}$$

It is not difficult to see that  $\Gamma$  acts on this. Once again we have

**Theorem 3.5.2** (Baily-Borel). *For a suitable topology on  $B^*$ , the quotient  $X = \Gamma \backslash B^*$  is compact. This can be given the structure of a normal algebraic surface.*

As above, we can construct a minimal resolution  $Y$  of  $X$ . This, and related objects, are called *Picard modular surfaces*.