Chapter 4

Abelian varieties

4.1 Abelian varieties

An abelian variety is a higher dimensional version of an elliptic curve. Here is the precise definition. Over a field k, an abelian variety consists of a smooth projective variety X over k, a k-rational point 0 and morphisms $+ : X \times X \to X$ and $- : X \to X$ which make it into a group. We have the following basic facts:

Theorem 4.1.1. *An abelian variety is a commutative group. When $k = \mathbb{C}$, an abelian variety has the structure of a complex torus, i.e. as a complex Lie group, it is isomorphic to $\mathbb{C}^g$ modulo a lattice.*

Proof. See Mumford [M, p 2, p 44]. □

We now focus on the case where $k = \mathbb{C}$. We can ask when is a complex torus $\mathbb{C}^g/L$ an abelian variety? The naive guess is that it is always true, but turns out to be incorrect once $g > 1$. A necessary condition for a compact complex manifold to be projective is that it has at least one nonconstant meromorphic function, but this can fail for higher dimensional tori. To formulate sufficient conditions, we modify what we did before with elliptic curves, but now we replace the element τ in the upper half plane with a $g \times g$ symmetric matrix Ω with positive definite imaginary part. The set of such matrices forms a complex manifold $\mathbb{H}_g$, that we call the Siegel upper half plane. Given such a matrix, we can form the lattice $L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g$; $\Omega\mathbb{Z}^g$ means the group of integer linear combinations of columns of Ω. Define the complex torus $A_\Omega = \mathbb{C}^g/L_\Omega$. Since $\operatorname{Im}\Omega$ is positive definite, the terms in the series

$$\theta(z) = \sum_{n\in\mathbb{Z}^g} \exp(\pi i n^T \Omega n + 2\pi i n^T z)$$

go to zero rapidly as $||n|| \to \infty$. So it converges to a holomorphic function called the Riemann theta function. This is a generalization of the Jacobi function. It satisfies a similar functional equation

$$\begin{aligned}\theta(z+n) &= \theta(z)\\ \theta(z+\Omega n) &= \exp(-\pi i n^t\Omega n - 2\pi i z^t n)\theta(z)\end{aligned} \tag{4.1}$$

for $n \in \mathbb{Z}^g$.

Theorem 4.1.2 (Lefschetz)**.** *A_Ω is an abelian variety.*

Idea. Consider the vector space V of all quasiperiodic functions satisfying (4.1) with Ω replaced by 3Ω. If $f_0 \ldots, f_N$ is a basis of V, we claim that

$$x \mapsto [f_0(x), \ldots, f_N(x)]$$

gives an embedding of $A_\Omega \to \mathbb{P}^N$. By considering products of the form

$$\theta(z+u)\theta(z+v)\theta(z-u-v) \in V, \quad u, v \in \mathbb{C}^g$$

one generates sufficiently many functions to separate points and tangent vectors. See [M, pp 29-33] for full details. □

Let us characterize lattices of the form $L = L_\Omega = \mathbb{Z}^g + \Omega\mathbb{Z}^g$, with $\Omega \in \mathbb{H}_g$, in coordinate free language. Let $e_1, \ldots e_g$, be the standard basis of $\mathbb{Z}^g$. We can extend this to basis of L, by taking e_{g+i} to be the ith column of Ω. The vectors $e_1, \ldots, e_{2g}$ form a *real* basis of $\mathbb{C}^g$. Let $E : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{R}$ be the real bilinear form with matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{4.2}$$

with respect to this basis. By definition, it is skew symmetric $E(v, u) = -E(u, v)$.

Lemma 4.1.3.

(a) $E(u, v) = \operatorname{Im}(u^t(\operatorname{Im}\Omega)^{-1}\bar{v})$

(b) $E(u, v) \in \mathbb{Z}$, *when* $u, v \in L$.

(c) $E(iu, iv) = E(u, v)$

(d) $E(iu, v)$ *is symmetric positive definite.*

(e) There exist a positive definite hermitian form H on $\mathbb{C}^g$, such that $E = \operatorname{Im} H$.

Proof. Item (a) can be checked by calculation, (b) is clear, and (c) and (d) follow from (a). These conditions show that

$$H(x, y) = E(ix, y) + iE(x, y)$$

satisfies (e). It is worth noting that (c) and (d) also follow from (e). □

Given a lattice $L \subset \mathbb{C}^g$, a nondegenerate skew-symmetric form $E : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{R}$ satisfying (b), (c) and (d) (or equivalently (b) and (e)) is called a *polarization* or Riemann form. It is called *principal* if in addition $\det E = 1$. This implies, by standard linear algebra arguments, that L possesses a basis such that E is given by (4.2). Since E and H above determine each other, H is also sometimes referred to as the polarization.

Lemma 4.1.4. *If L has a principal polarization, then after choosing suitable bases for $\mathbb{C}^g$ and L, we have $L = L_\Omega$ for some $\Omega \in \mathbb{H}_g$.*

Proof. We will say a bit more about this later on. □

We can thus rephrase theorem 4.1.2 as saying that $\mathbb{C}^g/L$ is an abelian variety if L possesses a principal polarization. In fact, by allowing arbitrary polarizations, we get an if and only if statement.

Theorem 4.1.5 (Riemann, Lefschetz)**.** *$\mathbb{C}^g/L$ is an abelian variety if and only if L possesses a polarization.*

The "if" direction can be proved using theta functions, as above. Let us briefly explain the converse from the viewpoint of complex algebraic geometry [GH], because it explains what E actually means. From algebraic geometry, we know that an embedding $X \subset \mathbb{P}^N$ is determined by the very ample divisor class $H + X \cap (\text{hyperplane})$ or the very ample line bundle $\mathcal{L} = \mathcal{O}_X(1)$. This has the advantage of giving an object on X which doesn't depend on any "external" data. The divisor H determines a homology class $[H] \in H_{2\dim X - 2}(X, \mathbb{Z})$, and by Poincaré duality a cohomology class $[H] \in H^2(X, \mathbb{Z})$. This coincides with the first Chern class $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$, which is the basic topological invariant of a line bundle. Since X is a torus, we can identify $H^2(X, \mathbb{Z}) = \wedge^2 Hom(L, \mathbb{Z})$. In other words, $c_1(\mathcal{L})$ can be viewed as an alternating integer valued pairing E on the lattice L. This means that E satisfies condition (b) for a polarization. On the other hand, since $c_1(\mathcal{L})$ is the restriction of $c_1(\mathcal{O}_{\mathbb{P}^N}(1))$, it can be represented by the normalized curvature of the Fubini-Study metric. In particular, it can also be represented by a real differential form, called the Kähler form,

$$\omega = \frac{\sqrt{-1}}{2} \sum_{jk} h_{jk} dz_j \wedge d\bar{z}_k$$

with h_{jk} positive definite hermitian. This can be used to show that E satisfies (e) as well.

From this discussion, we obtain.

Corollary 4.1.6. *Under the identification $H^2(X, \mathbb{Z}) = \wedge^2 Hom(L, \mathbb{Z})$, an element is a polarization if and only if is the first Chern class of an ample line bundle.*

Although polarizations do not traditionally appear in the theory of elliptic curves, they exist and are easy to describe. When X is an elliptic curve

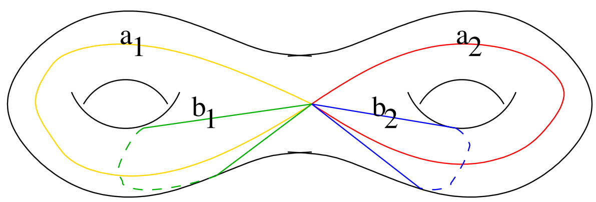
$H^2(X, \mathbb{Z}) \cong \mathbb{Z}$. Under this isomorphism, the Chern class of a divisor $c_1(\mathcal{O}(D))$ is just its degree $\deg D$. It is ample, and therefore corresponds to a polarization, if $\deg D$ is positive. And it is principal, when $\deg D = 1$. So X has a unique principal polarization.

4.2 Jacobians

Let X be a nonsingular projective curve of genus g. Recall that

$$g = \dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \Omega^1_X)$$

Earlier, we took the groups on the right to be sheaf cohomology of the sheaves of regular functions/forms on the Zariski topology, we can also (and will) interpret these as the cohomology groups of the sheaves of holomorphic functions/forms on the classical topology. This is justified by Serre's GAGA theorems. We want to explain that g is the same topological genus, which one half the dimension of the first de Rham cohomology group $H^1(X, \mathbb{C})$ of closed C^∞ complex 1-forms modulo exact forms.



Let say that a 1-form α is *harmonic* if in any system of local analytic coordinates $z = x + iy$, $\alpha = df(x, y)$ where f is harmonic in the usual sense, i.e. it lies in the kernel of $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. (This definition is a bit nonstandard, but people familiar with the the usual condition $(d^*d + dd^*)\alpha = 0$ should be able to check the equivalence.) The key fact, that we state without proof, is the Hodge theorem (which is really due to Weyl in the case of Riemann surfaces).

Theorem 4.2.1 (Hodge theorem)**.** $H^1(X, \mathbb{C})$ *is isomorphic to the space of harmonic* 1*-forms.*

Holomorphic 1-forms are harmonic for example, since any such form is locally df with f holomorphic, and basic complex analysis teaches us that holomorphic functions are harmonic. Conversely, a harmonic $(1, 0)$-form, i.e. a form locally a multiple of dz, is necessarily holomorphic. We can also see that a $(0, 1)$-form (a multiple of $d\bar{z}$) is harmonic if and only if it is a conjugate of a holomorphic 1-form. Thus:

Corollary 4.2.2 (Hodge decomposition)**.** *We have decomposition*

$$H^1(X, \mathbb{C}) = H^{10}(X) \oplus H^{01}(X)$$

where $H^{01}(X) = \overline{H^{10}(X)}$ *and* $H^{10} = H^0(X, \Omega^1_X)$*. In particular,* $\dim H^1(X, \mathbb{C}) = 2g$.

We should explain how to interpret complex conjugation in the above result. To give a conjugation a complex vectors space V is tantamount to finding real vector space $V_{\mathbb{R}}$ and an isomorphism $V_{\mathbb{R}} \otimes \mathbb{C} \cong V$; then $\overline{v \otimes a} = v \otimes \overline{a}$. For $V = H^1(X, \mathbb{C})$, we take $V_{\mathbb{R}} = H^1(X, \mathbb{R})$ to be the de Rham cohomology of real differential forms. Basic facts from topology (the de Rham and universal coefficient theorems), tells us that we can take this further. If $H^1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is singular cohomology with integer coefficients, then

$$\begin{aligned} H^1(X, \mathbb{C}) &\cong H^1(X, \mathbb{Z}) \otimes \mathbb{C} \\ H^1(X, \mathbb{R}) &\cong H^1(X, \mathbb{Z}) \otimes \mathbb{R} \end{aligned} \tag{4.3}$$

To make this more explicit, note that the dual $Hom(H^1(X, \mathbb{Z}), \mathbb{Z})$ can identified with the homology $H_1(X, \mathbb{Z})$. Elements of this are represented by (sums of) closed smooth loops on X. Given a form α representing a class in $H^1(X, \mathbb{C})$, the map (4.3) sends α to the functional $\gamma \mapsto \int_\gamma \alpha$.

We will also consider the transpose of this map (4.3)

$$H_1(X, \mathbb{Z}) \to H^1(X, \mathbb{C})^*,\ \gamma \mapsto \int_\gamma$$

This restricts to

$$H_1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^* \tag{4.4}$$

We define the Jacobian $J(X)$ as quotient

$$J(X) = \frac{H^0(X, \Omega^1_X)^*}{H_1(X, \mathbb{Z})}$$

where we identify $H_1(X, \mathbb{Z})$ with its image.

Proposition 4.2.3. *$J(X)$ is a complex torus.*

Proof. Given $\alpha \in H^1(X, \mathbb{C})$, write $\alpha^{10} \in H^{10}(X)$ and $\alpha^{01} \in H^{01}(X)$ for its components with respect to the Hodge decomposition. Let $p : H^1(X, \mathbb{R}) \to H^{10}(X)$ be defined by $p(\alpha) = \alpha^{10}$. Suppose that $p(\alpha) = 0$. Then $\alpha = \alpha^{10} + \overline{\alpha^{01}} = 0$. Therefore p is injective. It follows that p is an isomorphism, because both space have the same real dimension. Consequently, we can identify the image of (4.4) with the image of $H_1(X, \mathbb{Z})$ in $H^1(X, \mathbb{R})^*$ which is a lattice. $\square$

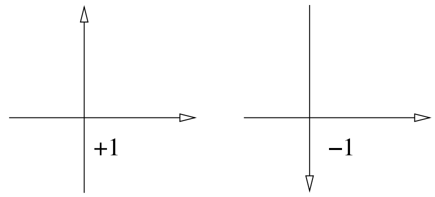
Let $L = H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ be the lattice defining $J(X)$. We have an intersection pairing

$$E : L \times L \to \mathbb{Z}$$

where $E(\gamma, \gamma')$ counts the number of times γ intersects γ', with signs. That is if the curves are transverse

$$E(\gamma, \gamma') = \sum_{p \in \gamma \cap \gamma'} \pm 1$$

according to



There are various ways to construct this rigorously. One way is to construct the dual pairing on $H^1(X,\mathbb{Z})$ using the cup product. In terms of the embedding $H^1(X,\mathbb{Z}) \subset H^1(X,\mathbb{R})$, this given by integration

$$E(\alpha,\beta) = \int_X \alpha \wedge \beta$$

Theorem 4.2.4. *E is a principal polarization. Therefore $J(X)$ is an abelian variety.*

Proof. It is already clear from the above formula that E is skew symmetric. Poincaré duality shows that its determinant is $+1$. If we pullback E to $H^{10}(X)$ under the isomorphism p above, we have

$$E(\alpha,\beta) = \int_X (\alpha+\bar{\alpha}) \wedge (\beta+\bar{\beta}) = \int_X \alpha \wedge \bar{\beta} + \bar{\alpha} \wedge \beta$$

It follows that $E(i\alpha, i\beta) = E(\alpha,\beta)$. Finally suppose $\alpha = f(z)dz$ where f is nonzero holomorphic. Since

$$i\alpha \wedge \bar{\alpha} = 2|f(z)|^2 dx \wedge dy$$

we conclude that

$$E(i\alpha,\alpha) > 0$$

□

Finally, let us explain what information $J(X)$ carries. Choose a base point x_0, and define the Abel-Jacobi map

$$AJ : X \to J(X)$$

by

$$AJ(x) = \int_{x_0}^{x} \in H^{10}(X)^* \mod H_1$$

The integral is only defined after choosing a path from x_0 to x, but its image in $J(X)$ does not depend on it. Given a divisor $D = \sum n_i x_i$, we define $AJ(D) = \sum n_i AJ(x_i)$.

Theorem 4.2.5 (Abel-Jacobi)**.** *We have an isomorphism of abelian groups*

$$Cl^0(X) \cong J(X)$$

induced by AJ, where $Cl^0(X)$ is the degree zero part of the divisor class group.

4.3 Siegel modular varieties

If $A_i = V_i/L_i$ are complex tori, we will say that they are *isomorphic* (respectively *isogenous*) if there is a linear isomorphism $\phi : V_1 \to V_2$ such that $\phi(L_1) = L_2$ (resp. $\phi(L_1) \subseteq L_2$).

Lemma 4.3.1. *Isogeny is an equivalence relation.*

Proof. Transitivity and reflexivity are obvious. We only have to prove that isogeny is symmetric. If $\phi : V_1 \to V_2$ is an isomorphism such that $\phi(L_1) \subseteq L_2$, then L_1 is a sublattice of $\phi^{-1}(L_2)$. Therefore $N\phi^{-1}(L_2) \subset L_1$ for some N. This means that $N\phi^{-1}$ is an isogeny in the opposite direction. □

In case these are abelian varieties, an isomorphism in this sense is automatically an isomorphism of algebraic varieties by [GAGA]. If A_i are equipped with polarizations E_i, we say that ϕ is an isomorphism of polarized abelian varieties if ϕ preserves the forms E_i, i.e.

$$E_1(u, v) = E_2(\phi(u), \phi(v))$$

The problem of describing all abelian varieties up to isomorphism does not have a good solution, but the polarized version does. We now describe it.

Lemma 4.3.2. *An abelian variety is isogenous to a principally polarized abelian variety. Any principally polarized abelian variety of dimension g is isomorphic, as a polarized abelian variety, to an abelian variety of the form $A_\Omega = \mathbb{C}^g/L_\Omega$, $L_\Omega = \Omega\mathbb{Z}^g + \mathbb{Z}^g$ with*

$$E = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

for some $\Omega \in \mathbb{H}_g$.

Proof. The first statement is [BL, 4.1.2]. The second is just a restatement of lemma 4.1.4 □

Of course, the Ω in the previous lemma is not unique. Let us introduce the symplectic group. Given a commutative ring R (e.g. $\mathbb{Z}, \mathbb{R}$) we define

$$Sp_{2g}(R) = \left\{ M \in GL_{2g}(R) \mid M^T E M = E \right\}$$

Lemma 4.3.3. *Given $\Omega \in \mathbb{H}_g$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R})$*

$$M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1} \in \mathbb{H}_g$$

This defines an action of $Sp_{2g}(\mathbb{R})$ on $\mathbb{H}_g$ which is transitive. The isotropy group of iI is

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid AB^T = BA^T, AA^T + BB^T = I \right\} \cong U_n(\mathbb{R})$$

where the isomorphism is given by sending

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + iB$$

Proof. For M as above, one checks the following identities: $A^T C$ and $B^T D$ are symmetric, and $A^T D - C^T B = I$. After expanding, using the above identities, and canceling, we obtain

$$(C\Omega + D)^T (M \cdot \Omega - (M \cdot \Omega)^T)(C\Omega + D) = \Omega - \Omega^T = 0$$

Therefore $M \cdot \Omega$ is symmetric. Similarly

$$(C\Omega + D)^T (\operatorname{Im} M \cdot \Omega)(C\Omega + D) = \operatorname{Im} \Omega > 0$$

which implies that $\operatorname{Im} M \cdot \Omega$ is positive definite.

Let $\Omega = X + iY \in \mathbb{H}_g$. Since Y is symmetric and positive definite, we can find an $A \in GL_g(\mathbb{R})$ so that $Y = AA^T$. Then $M = \begin{pmatrix} A & X(A^T)^{-1} \\ 0 & (A^T)^{-1} \end{pmatrix}$ sends iI to Ω. The formula for the isotropy group can be checked by calculation. □

Corollary 4.3.4. *Thus* $\mathbb{H}_g \cong Sp_{2g}(\mathbb{R})/U_g(\mathbb{R})$.

Let us now explain the idea for the proof of lemma 4.1.4. Given a principally polarized abelian variety $(V/L, E)$. Choose a symplectic basis $\lambda_1, \ldots, \lambda_{2g}$ for L. A basis is symplectic if E is represented by the matrix (4.2). Use the first g vectors $\lambda_1, \ldots, \lambda_g$ as a basis for V. Then if we write the remaining vectors $\lambda_{g+1}, \ldots, \lambda_{2g}$ in the last basis, we get a $g \times g$ matrix Ω. One can see that the conditions for a polarization force $\Omega \in \mathbb{H}_g$ [BL, §4.2]. So once we fix the initial basis λ_i, Ω is determined. If λ'_i is a different symplectic basis, then we get a different $\Omega' \in \mathbb{H}_g$. The relationship is easy to work out. The change of basis matrix $\lambda'_i = \sum m_{ij}\lambda_j$ is necessarily in $Sp_{2g}(\mathbb{Z})$.

Lemma 4.3.5. $\Omega' = M \cdot \Omega$.

We define

$$A_g = Sp_{2g}(\mathbb{Z})\backslash\mathbb{H}_g = Sp_{2g}(\mathbb{Z})\backslash Sp_{2g}(\mathbb{R})/U_g(\mathbb{R})$$

This is called a *Siegel modular variety.* Although at the moment it is just a set.

Corollary 4.3.6. *There is a natural one to one correspondence between elements of* A_g *and isomorphism classes of* g *dimensional principally polarized abelian varieties.*

Next, we study the action of $Sp_{2g}(\mathbb{Z})$ on $\mathbb{H}_g$.

Lemma 4.3.7. *The action of* $Sp_{2g}(\mathbb{Z})$ *is properly discontinuous. Therefore the quotient is a Hausdorff space.*

Proof. Given compact sets $K_1, K_2 \subset \mathbb{H}_g$, we have to show that $S = \{M \in Sp_{2g}(\mathbb{Z}) \mid M(K_1) \cap K_2 \neq \emptyset\}$ is finite. Let us identify $\mathbb{H}_g = Sp_{2g}(\mathbb{R})/U_g(\mathbb{R})$ as above. Note that the group $U_g(\mathbb{R})$ is compact, so that the projection $p : Sp_{2g}(\mathbb{R}) \to \mathbb{H}_g$ is proper. $M \in Sp_{2g}(\mathbb{Z})$ lies in S if and only if $Mp^{-1}K_1 \cap p^{-1}K_2 \neq \emptyset$ if and only if $M \in T = (p^{-1}(K_1))^{-1}p^{-1}(K_2)$. Now T is compact because it is the image of $K_1 \times K_2$ under $(M_1, M_2) \mapsto M_1^{-1}M_2$. Therefore S is the intersection of a compact set with a discrete set, so it's finite. □

The action has fixed points. The solution, as before, is to pass to a congruence subgroup

$$\Gamma(N) = \ker[Sp_{2g}(\mathbb{Z}) \to Sp_{2g}(\mathbb{Z}/N\mathbb{Z})]$$

Proposition 4.3.8. *If $N \geq 3$, then $\Gamma(N)$ is torsion free.*

Proof. We assume that $\gamma \neq I$ is an element of $\Gamma(N)$ of finite order. We can assume that the order is a prime p, by replacing γ a power. Then by assumption, $I - \gamma = N\phi$ where $\phi \in M_{2g\times 2g}(\mathbb{Z})$. Let ζ be a nontrivial eigenvalue of γ, and let η be the corresponding eigenvalue of ϕ. We have a relation

$$N\eta = 1 - \zeta \tag{4.5}$$

This implies $\eta \in \mathbb{Q}(\zeta)$. Furthermore, η is also an algebraic integer because it satisfies the characteristic polynomial of ϕ. Suppose $p = 2$, then $\zeta = -1$. Equation (4.5) implies $N|2$, which is a contradiction because $N \geq 3$. Now suppose $p \geq 3$. Then ζ is a primitive pth root of unity and, as already noted, η is an algebraic integer in the cyclotomic field $\mathbb{Q}(\zeta)$. Taking the norm of (4.5) with respect to $\mathbb{Q}(\zeta)/\mathbb{Q}$ yields an equality of integers

$$N^{p-1}Norm(\eta) = (1-\zeta)(1-\zeta^2)\ldots(1-\zeta^{p-1}) = p$$

But this is impossible because p is prime. □

A consequence of the proposition is that $\Gamma(N)$, with $N \geq 3$, acts freely on $\mathbb{H}_g$. So the quotient

$$A_{g,N} = \Gamma(N)\backslash\mathbb{H}_g$$

can be seen to be a manifold. In more detail, define $\mathcal{O}_{\mathcal{A}_g}(U)$ (and $\mathcal{O}_{\mathcal{A}_{g,N}}(U)$) as to correspond to invariant holomorphic functions on the preimage $\tilde{U} \subset \mathbb{H}_n$. Since the action of $\Gamma(N)$ is free, we find that

Proposition 4.3.9. *When $N \geq 3$, $A_{g,N}$ is a complex manifold.*

A_g is a quotient of $A_{g,N}$ by the finite group $Sp_{2g}(\mathbb{Z}/N\mathbb{Z})$. Therefore

Corollary 4.3.10. *$(A_g, \mathcal{O}_{A_g})$ is a normal analytic space.*

4.4 Siegel modular varieties are moduli spaces

As we did for elliptic curves, we want to upgrade corollary 4.3.6 to a more precise statement. Let us formulate it more generally for $A_{g,N}$. Given $A_\Omega = \mathbb{C}^g/\Omega\mathbb{Z}^g + \mathbb{Z}^g$, the standard basis of the lattice mod N is called a level N-structure. One can see that $M \in \Gamma(N)$ preserves this basis. In general, a level N-structure on a principally polarized abelian variety A is a basis of $H_1(A, \mathbb{Z}/N\mathbb{Z})$ which is symplectic with respect to the form induced by the polarization. In more algebraic terms, it can also be taken as basis of the N-torsion, which is symplectic in the appropriate sense.

Theorem 4.4.1. *$A_{g,N}$ is the coarse moduli space for principally polarized g dimensional abelian varieties with N-structure. When $N \geq 3$, this is a fine moduli space.*

The last statement means that there is a universal family of abelian varieties with the above structure. We will now outline the construction. Given $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_{2g}(\mathbb{R})$ and $(\Omega, z) \in \mathbb{H}_g \times \mathbb{C}^g$, define

$$M \cdot (\Omega, z) = (M \cdot \Omega, ((C\Omega + D)^T)^{-1} z)$$

Lemma 4.4.2. *This defines an action.*

Proof. [MT, p 177]. □

We define a real linear isomorphism

$$i_\Omega : \mathbb{R}^{2g} \to \mathbb{C}^g$$

given by sending $(v_1, v_2) \in \mathbb{R}^g \times \mathbb{R}^g$ to $\Omega v_1 + v_2$. If $\lambda \in \mathbb{Z}^{2g}$, let

$$\lambda \cdot (\Omega, z) = (\Omega, z + i_\Omega(\lambda))$$

Let $\tilde{\Gamma}$ (resp. $\tilde{\Gamma}(N)$) be the subgroup of the group of holomorphic automorphisms of $\mathbb{H}_g \times \mathbb{C}^g$ generated by $Sp_{2g}(\mathbb{Z})$ (resp. $\Gamma(N)$) and $\mathbb{Z}^{2g}$. A calculation shows that $\mathbb{Z}^{2g}$ is a normal subgroup of $\tilde{\Gamma}$. It follows that $\tilde{\Gamma}$ is a so called semidirect product $Sp_{2g}(\mathbb{Z}) \ltimes \mathbb{Z}^{2g}$. This means that we have a split exact sequence

$$1 \to \mathbb{Z}^{2g} \to \tilde{\Gamma} \to Sp_{2g}(\mathbb{Z}) \to 1$$

With the help of this structure, we can see that the action of this group on $\mathbb{H}_g \times \mathbb{C}^g$ is properly continuous, and free when restricted to $\tilde{\Gamma}(N)$, for $N \geq 3$. Consequently the quotient

$$U_{g,N} = \tilde{\Gamma}(N) \backslash \mathbb{H}_g \times \mathbb{R}^{2g}$$

is a complex manifold. Projection on the first factor yields a holomorphic map $\pi : U \to A_{g,N}$. The fibre over a point corresponding to Ω is the abelian variety A_Ω. This is our desired universal family.

An application of Baily-Borel shows that $A_{g,N}$ is a quasiprojective variety. Using a completely different construction, Mumford [GIT] proved that

Theorem 4.4.3 (Mumford)**.** *$A_{g,N}$ is the set of complex points of a quasiprojective scheme over* $\operatorname{Spec}\mathbb{Z}$. *This is a coarse moduli space for all N. It is fine, and smooth over* $\operatorname{Spec}\mathbb{Z}[1/N, \exp(2\pi i/N)]$, *when $N \geq 3$.*