NOTES ON DIFFERENTIAL FORMS ON SINGULAR VARIETIES

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1. Main results

This is a mostly expository account of some recent, and not so recent, work on the topic in the title. A few observations in section 2 are perhaps new. Fix a possibly singular reduced complex analytic space X. Assume that X has pure dimension n, and that it is normal. Normality isn't necessary in what follows, but it does simplify some things. Let me start with the basic, although not very well posed question.

Q1: What is a good notion of a holomorphic p-form on X?

Here are some possible answers:

A. A section of $\Omega^p_X = \wedge^p\Omega_X$.
B. A section of $\Omega^p_X/torsion$.
C. A holomorphic p-form on the smooth locus $U = X_{reg}$.
D. A holomorphic p-form on some (or any) resolution of singularities.

Forms of type A, B, D give rise to forms of type C (for D $\Rightarrow$ C, choose a resolution which is an isomorphism over U). I will mostly focus on relationship between the last two types. Let me refer to forms of type A as Kähler, forms of type C as reflexive, and forms of type D as resolvable. I will focus on the basic question.

Q2: When is a reflexive form resolvable?

It is good to also ask a local version of the question. Let $j : U \to X$ denote the inclusion. Choose a resolution $f : \tilde X \to X$. Reflexive (resp. resolvable) forms are global sections of $j_*\Omega^p_U$ (resp. $f_*\Omega^p_{\tilde X}$). Since X is normal, we can identify $j_*\Omega^p_U = (\Omega^p_X)^{\vee\vee}$, and this is why I chose the word "reflexive" above. There is an inclusion $f_*\Omega^p_{\tilde X} \subseteq j_*\Omega^p_U$.

Q3: When is there equality $f_*\Omega^p_{\tilde X} = j_*\Omega^p_U$?

This is certainly false if X_{sing} has codimension 1 components, and that is part of the reason I imposed normality. Let me simply write "reflexive p-forms are resolvable" if the answer to Q3 is positive, or equivalently Q2 has a positive answer for every open of X. For $p = n$, the answer is given by an old result [KKMS].

Theorem 1.1 (Kempf, 1973). *Reflexive n-forms are resolvable if and only if X has rational singularities.*

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Here is the state of the art from about 30 years ago.

Theorem 1.2 (Flenner, 1988)**.** *If p is less than $codim X_{sing} - 1$, then reflexive p-forms are resolvable.*

Corollary 1.3 (Steenbrink-Van Straten)**.** *If X has isolated singularities, then reflexive p-forms are resolvable when $p \leq n-2$.*

Flenner's theorem was deduced with the help of Steenbrink's vanishing theorem [St]. In a slightly different, although related, direction, when the exceptional divisor E of f has normal crossings and X is proper, Steenbrink's theorem also implies that

$$H^q_E(\tilde{X}, \Omega^q_{\tilde{X}}(\log E)) = 0$$

for $p + q < codim X_{sing}$ [A].

Now I want to talk about a recent advance.

Theorem 1.4 (Kebekus-Schnell, 2018)**.** *If reflexive k-forms are resolvable, then reflexive p-forms are resolvable for all $p \leq k$.*

Corollary 1.5. *If X has rational singularities, then reflexive forms are resolvable in all degrees.*

2. Applications and further remarks

2.1. **Zariski-Lipman conjecture.** The conjecture states that an algebraic variety over a field of characteristic zero is smooth if its tangent sheaf $T_X = (\Omega^1_X)^\vee$ is locally free. Lipman [L] points out that this is false in positive characteristic. The surface $z^p - xy = 0$ is a counterexample.

Theorem 2.2 (Greb, Kovács, Kebekus, Peternell, 2011)**.** *If reflexive 1-forms are resolvable and T_X is locally free, then X is smooth.*

Remark 2.3. *Their result* [GKKP, thm 6.1] *is not stated this way, but this is what they prove.*

Proof. Assume that X is singular. Choose a singular point $p \in X$. By assumption, there exists a basis of sections $\theta_1, \ldots, \theta_n \in T_X(V)$ defined in a neighbourhood V of p. We replace X by V for simplicity. Hironaka has constructed a resolution of singularities $f : \tilde{X} \to X$ which is functorial in a suitable sense. Among the features of this resolution are that both automorphisms and infinitesimal automorphisms of X lift to $\tilde{X}$. In fact, θ_i lift even though they are only infinitesimal automorphisms away from the singularities. We can assume that the exceptional divisor of f is a divisor E with normal crossings. Then a result of Greb, Kovacs, and Kebekus [GKK, cor 4.7] shows that $f_* T_{\tilde{X}}(-\log E)$ is reflexive. Therefore the sections θ_i lift to sections $\widetilde{\theta}_i$ of $T_{\tilde{X}}(-\log E)$. The vector fields $\widetilde{\theta}_i$ form a basis of $T_{\tilde{X}-E}$, although they do not give a basis of $T_{\tilde{X}}$, because they vanish along E. Let $\omega_i \in H^0(\Omega^1_{\tilde{X}-E})$ be the dual basis. By assumption ω_i extend to holomorphic 1-forms, denoted by the same symbols, on $\tilde{X}$. The relation $\omega_i(\widetilde{\theta}_j) = \delta_{ij}$ persists on $\tilde{X}$ (because constant functions extend to constant functions). This forces $\widetilde{\theta}_i$ to be a basis of $T_{\tilde{X}}$ contradicting what was said above. □

Corollary 2.4 (Flenner)**.** *X is smooth if T_X is locally free and the singular set has codimension at least 3.*

Corollary 2.5 (Kebekus-Schnell)**.** *X is smooth if it has rational singularities and T_X is locally free.*

2.6. **Forms on GIT quotients.** Suppose that a reductive group G acts on a smooth affine variety Y, then we can form the GIT quotient $X = Y//G := \operatorname{Spec}\mathcal{O}(Y)^G$. If Y is not necessarily affine then the quotient need not exist in any reasonable sense. Mumford [MF] gave various criteria for existence with varying degrees of niceness. Let me refer to $\pi : Y \to X$ as a GIT quotient if

(1) It is a categorical quotient, i.e. it satisfies the standard universal property [MF, pp 3-4]
(2) It is a uniform quotient, i.e (1) continues to hold after flat base change
(3) π is affine.

Note that the first condition uniquely determine X up to isomorphism, so we write $X = Y//G$. A differential form α on Y is horizontal if $\alpha(v_1, v_2, \ldots) = 0$ for vector fields v_i, when one of them is G-invariant. Let $(\Omega^p_{Y,hor})^G$ be the sheaf of p-forms which are invariant and horizontal. This notion was studied by Brion [Br] and Jamet [J]. The following answers a question of Jamet.

Proposition 2.7. *Suppose Y is a smooth variety with G-action such that the GIT quotient $X = Y//G$ exists. Then there are isomorphisms*

$$f_*\Omega^p_{\tilde{X}} \cong \pi_*(\Omega^p_{Y,hor})^G \cong j_*\Omega^p_U$$

Proof. When Y is affine, Jamet [J] gives inclusions

$$f_*\Omega^p_{\tilde{X}} \subseteq \pi_*(\Omega^p_{Y,hor})^G \subseteq j_*\Omega^p_U$$

Boutot [B] has proved that X has rational singularities. Therefore the above inclusions are equalities by corollary 1.5.

In general, the assumptions imply that if $\{X_i\}$ is an affine open cover of X, then $Y_i = \pi^{-1}X_i$ is an affine cover of Y such that $X_i = Y_i//G$. So we are reduced to the affine case. □

Let me propose a conjecture which would refine the last proposition. Recall that Du Bois [dB] has refined Deligne's construction to define an object $\underline{\Omega}_X$ in the filtered derived category which realizes Deligne's Hodge filtration on cohomology.

Conjecture 2.8. *With $X = Y//G$ as above, $\underline{\Omega}_X$ is isomorphic to $(\Omega^\bullet_{Y,hor})^G$ with its stupid filtration.*

As further evidence, note that it is true when G is finite [dB, §5], or when $G = \mathbb{G}_m^n$ and X is toric [GNPP, chap V, §4].

2.9. **Mixed Hodge structure.** When X is an algebraic variety, Deligne [D] has constructed a canonical mixed Hodge structure on $H^*(X)$. Let F denote the associated Hodge filtration. I will start with a result which is independent of [KS], although it seems to complement those results quite nicely.

Theorem 2.10. *Suppose that X is a proper algebraic variety. Then $F^pH^p(X, \mathbb{C})$ can be identified with a subspace of $H^0(\tilde{X}, \Omega^p_{\tilde{X}})$. Suppose additionally that X is normal and that $R^if_*\mathcal{O}_{\tilde{X}} = 0$ for $0 < i \le k$, then for all $p \le k$, then*

$$H^0(\tilde{X}, \Omega^p_{\tilde{X}}) = F^pH^p(X, \mathbb{C})$$

In other words, $F^pH^p(X)$ is the space of resolvable p-forms.

Proof. Fix $p \leq k$. We use the "blow up sequence"

$$H^{p-1}(E) \to H^p(X) \to H^p(\tilde{X}) \oplus H^p(S) \dots$$

where $S = X_{sing}$. Since $F^p H^{p-1}(E) = 0$, we see that

$$0 \to F^p H^p(X) \to F^p H^p(\tilde{X}) \oplus F^p H^p(S) \to F^p H^p(E)$$

is exact. It also follows that $F^p H^p(X) \cap W_{p-1} H^p(X) \subseteq F^p H^p(S) \cap W_{p-1} H^p(S)$. Therefore by induction on dimension we see that $F^p H^p(X) \cap W_{p-1} H^p(X) = 0$. Thus to prove the theorem, it suffices to prove that the map

$$F^p Gr_p^W H^p(S) \to F^p Gr_p^W H^p(E) \tag{1}$$

is always injective, and an isomorphism under the additional hypothesis. We can construct a commutative diagram

$$\begin{array}{ccc} E & \longleftarrow & \tilde{E} \\ \big\downarrow & & \big\downarrow \\ S & \longleftarrow & \tilde{S} \end{array}$$

where the horizontal maps are desingularizations, and the vertical maps are surjections. By an argument similar to the one used above, or by [D, 8.2.5], $Gr_p^W H^p(S)$ (resp. $Gr_p^W H^p(S)$) embeds into the space of holomorphic p-forms on $\tilde{S}$ (resp. $\tilde{E}$). Injectivity of (1) now follows from the injectivity of pullback of p-forms under $\tilde{E} \to \tilde{S}$.

Now suppose that X satisfies the additional assumptions of normality etc. Choose an analytic neigbourhood $S \subset T \subset X$ such that T deformation retracts to S, and such that the preimage $T' = f^{-1}T$ deformation retracts to E. Since $R^i f_* \mathcal{O}_{\tilde{X}} = 0$ for $0 < i \leq k$, we have $H^p(T, \mathcal{O}) \cong H^p(T', \mathcal{O})$. Let $\underline{\Omega}_S^0 \in D^b_{coh}(\mathcal{O}_X)$ be the zeroth graded piece of the (analytic) du Bois complex [dB]. Here are the key properties. There is a canonical map $\mathbb{C}_S \to \underline{\Omega}_S^0$ factoring through $\mathcal{O}_S$, such that $H^p(S, \mathbb{C}) \to H^p(S, \underline{\Omega}_S^0)$ is the surjective projection $H^p(S) \to Gr_F^0 H^p(S)$. A similar statement holds on E. Since E has normal crossing singularities, it is known that $\mathcal{O}_E \cong \underline{\Omega}_E^0$. Putting these facts together, we get a commutative diagram

$$\begin{array}{ccccccccc} H^p(E,\mathbb{C}) & \xleftarrow{\sim} & H^p(T',\mathbb{C}) & \xrightarrow{p} & H^p(T',\mathcal{O}) & \xrightarrow{r} & H^p(E,\mathcal{O}) & \xrightarrow{\sim} & H^p(E,\underline{\Omega}^0) \\ \big\uparrow & & \big\uparrow & & \cong\big\uparrow & \overset{s}{\nearrow} & \big\uparrow & & \big\uparrow\rho \\ H^p(S,\mathbb{C}) & \xleftarrow{\sim} & H^p(T,\mathbb{C}) & \longrightarrow & H^p(T,\mathcal{O}) & \longrightarrow & H^p(S,\mathcal{O}) & \longrightarrow & H^p(S,\underline{\Omega}^0) \end{array}$$

such that the composite map $H^p(E, \mathbb{C}) \to H^p(E, \underline{\Omega}^0)$ is surjective. It follows that $r \circ p$ is surjective, and therefore also s, and hence also ρ are surjective. Thus we have proved that

$$Gr_F^0 H^p(S) \to Gr_F^0 H^p(E)$$

is surjective. Therefore we have a surjection on the pure weight p parts

$$F^0 Gr_p^W H^p(S) \to F^0 Gr_p^W H^p(E)$$

Taking complex conjugation shows that (1) is also surjective.

□

Corollary 2.11. *If X is proper with rational singularities, then*

$$H^0(\tilde{X}, \Omega^p_{\tilde{X}}) = F^p H^p(X, \mathbb{C})$$

holds for all p

First Proof. This is an immediate corollary of the theorem. □

Second Proof. I will give an alternative argument, which explains how this ties in with the ideas of [KS]. A result of Lee and Huber-Jörder [HJ, thm 7.6] identifies

$$F^p H^p(X) \cong H^0_h(X, \Omega^p_h) \tag{2}$$

where Ω^p_h is the sheafication of Kähler differentials in the Voevodsky's h-topology. One the other hand, [KS, cor 1.11] shows that

$$H^0_h(X, \Omega^p_h) = \{\text{reflexive } p\text{-forms}\} \tag{3}$$

This proves the corollary.

Let me make a few more comments, since I have swept all the subtleties under the rug. Concretely, $H^0_h(X, \Omega^p_h)$ can be identified with the space of $\alpha \in H^0(\tilde{X}, \Omega^p_{\tilde{X}})$, such that

$$\phi^* p_1^* \alpha = \phi^* p_2^* \alpha \tag{4}$$

for some resolution $\phi : \tilde{X}_2 \to \tilde{X} \times_X \tilde{X}$ with projections $p_i : \tilde{X} \times_X \tilde{X} \to \tilde{X}$. But this is exactly the description of $F^p H^p(X)$ one gets by computing the mixed Hodge structure using a smooth hypercover as in [D]. This shows (2). The second equality (3) is more delicate. One needs to check (4) for reflexive forms. The difficulty lies in the fact that $\tilde{X}_2$ may have components mapping to X_{sing}. I refer to section 14 of [KS] for this part of the argument. I should add that their result is local, so it is stronger than what follows from the first proof. □

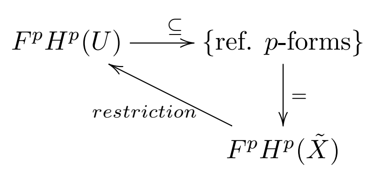
Proposition 2.12. *If X is a proper algebraic variety with rational singularities, then there are isomorphisms*

$$\{\textit{reflexive } p\textit{-forms on } X\} \cong F^p H^p(U, \mathbb{C}) \cong F^p H^p(X, \mathbb{C})$$

Proof. We can assume that $\tilde{X}$ contains U, and that the complement is a divisor E with simple normal crossings. We have by [D]

$$F^p H^p(U, \mathbb{C}) = H^0(\tilde{X}, \Omega^p_{\tilde{X}}(\log E))$$

So we have a diagram



This gives the first isomorphism.

The isomorphism

$$\{\text{ref. } p\text{-forms on } X\} \cong F^p H^p(X, \mathbb{C})$$

follows from the previous corollary. □

2.13. **Log forms.** Kebekus and Schnell have also proved a log version of their theorem. Using this together with a theorem of Kovacs-Schwede-Smith [KSS], it follows that the above isomorphism

$$\{\text{ref. } p\text{-forms on } X\} \cong F^p H^p(U, \mathbb{C})$$

holds when X is proper with normal CM Du Bois singularities.

3. Proof of theorem 1.4

They give two proofs. I will outline the shorter one. This is based on the following criterion.

Theorem 3.1 (Kebekus-Schnell)**.** *A reflexive p-form α on X is resolvable if and only if for every open $V \subset X$ and Kähler form β (resp. γ) of degree $n-p$ (resp. $n-p-1$) on V, $\alpha \wedge \beta$ and $d\alpha \wedge \gamma$ are resolvable, in the sense that they extend to $f^{-1}V$.*

Proof of theorem 1.4. Its enough to prove that a reflexive $k-1$ form α is resolvable. Since the problem is local, we may assume that X embeds into a ball in $\mathbb{C}^{n+c}$. Choosing Kähler forms β, γ of degree $n-k+1$ and $n-k$ on $V \subseteq X$, theorem 3.1 tell us that we need to check that $\alpha \wedge \beta$ and $d\alpha \wedge \gamma$ are resolvable. There is no loss in assuming that $V = X$. Choose coordinates z_i on the ball. Then we can expand

$$\beta = \sum dz_i \wedge \beta_i$$

By assumption $\alpha \wedge dz_i$ and $d\alpha$ are resolvable. It follows that $\sum \alpha \wedge dz_i \wedge \beta_i$ and $d\alpha \wedge \gamma$ are resolvable. □

4. Hodge modules

It remains to prove theorem 3.1. The proof uses Hodge modules [S]. I will summarize the basic facts needed. Let Y be a complex manifold.

(1) A (pure) Hodge module on Y consists of a regular holonomic D_Y-module M with an ascending good filtration F, and a perverse sheaf L of $\mathbb{Q}$-vector space such that $L \otimes \mathbb{C}$ and M correspond under Riemann-Hilbert:

$$DR(M) := (M \stackrel{\nabla}{\to} \Omega^1_Y \otimes M \stackrel{\nabla}{\to} \Omega^2_Y \otimes M \ldots)[\dim X] \cong L \otimes \mathbb{C}$$

Note that the choice of isomorphism is part of the datum. The collection of Hodge modules is subject to some inductive axioms which are too complicated to state here, except for the base axiom: Hodge modules over a point are the same thing as Hodge strucutres. Unless there is a danger of confusion, we will conflate M with the Hodge module $(M, \ldots)$.

(2) Note (for experts), I will assume that Hodge modules are polarizable. It follows that the category $HM(Y)$ of Hodge modules is abelian and semisimple. This can be decomposed into a sum $HM(Y) = \bigoplus HM(Y, w)$ of the category Hodge modules of weight w. $HM(pt, w)$ is the category of pure polarizable Hodge structures of weight w. The perverse sheaf assocated to a simple module is the intersection cohomology complex associated to the local system underlying an irreducible polarized variation of Hodge structure supported on an irreducible subvariety. The converse statement is also true. If $X \subset Y$ is a closed analytic space, then $HM(X)$ can defined as the category of Hodge modules, all of whose simple factor are supported in X.

Up to equivalence $HM(X)$ depends only on X and not Y. (For D-modules, this is Kashiwara's theorem.)

(3) There is a good notion of direct images of Hodge modules $f_+ : D^b(HM(X)) \to D^b(HM(Z))$ under projective morphisms $f : X \to Z$. We have an exact functor to the constructible derived category $rat : D^b(HM(X)) \to D^b_c(X, \mathbb{Q})$, which extends $(M, F, L) \mapsto L$. If $\mathcal{M} = (M, F, L) \in HM(X)$, then the following compatibilities hold

$$\mathbb{R}f_* rat(\mathcal{M}) \cong rat(f_+\mathcal{M})$$
$$\mathbb{R}f_* Gr_p^F M \cong Gr_p^F DR f_+ \mathcal{M}$$

A version of the decomposition theorem of [BBD] holds for Hodge modules. This says that if $M \in HM(X)$, then

$$f_+ M \cong \bigoplus M_i[i]$$

where M_i are Hodge modules on Z.

5. Proof of theorem 3.1

Working locally, we can assume that X embeds into a ball $Y \subset \mathbb{C}^{n+c}$. So c is the codimension. Let $S = X_{sing}$ and let $f : \tilde{X} \to X$ be a resolution of singularities. Note that this map, as well as the composition (also called) $f : \tilde{X} \to Y$, can be assumed to be projective. The sheaf $\mathbb{Q}_{\tilde{X}}$ viewed as a variation of Hodge structure of weight 0 corresponds to Hodge module with D-module $\mathcal{O}_{\tilde{X}}$ and filtration

$$F_p \mathcal{O}_{\tilde{X}} = \begin{cases} 0 & \text{if } p \leq -1 \\ \mathcal{O}_{\tilde{X}} & \text{otherwise} \end{cases}$$

The de Rham complex is

$$DR(\mathcal{O}_{\tilde{X}}) = (\mathcal{O}_{\tilde{X}} \to \Omega^1_{\tilde{X}} \to \dots)[n]$$

We have

$$Gr^F_{-p} \cong \Omega^p_{\tilde{X}}[n-p]$$

Therefore from the discussion in the previous section, we obtain

$$Gr_{-p} DR(f_+\mathcal{O}_{\tilde{X}}) \cong \mathbb{R}f_* Gr^F_{-p} DR(\mathcal{O}_{\tilde{X}}) \cong \mathbb{R}f_* \Omega^p_{\tilde{X}}[n-p] \tag{5}$$

and

$$f_+ \mathcal{O}_{\tilde{X}} = M \oplus M' \tag{6}$$

where M is the sum of simple components supported on all of X, and M' is the sum of components supported in S. The previous discussion also shows that M is the Hodge module corresponding to the intersection cohomology of X.

Proposition 5.1.

$$f_* \Omega^p_{\tilde{X}} \cong \mathcal{H}^{p-n} Gr^F_{-p} DR(M)$$

Proof. By (5) and (6), we have

$$f_* \Omega^p_{\tilde{X}} \cong \mathcal{H}^{p-n} Gr^F_{-p} DR(M) \oplus \mathcal{H}^{p-n} Gr^F_{-p} DR(M')$$

Now using the fact that $f_* \Omega^p_{\tilde{X}}$ is torsion free, we can conclude that both summands are torsion free. But the second summand is supported in S, so it must be zero. □

Another fact needed is the following

Lemma 5.2. *$F_{c-1}M = 0$ and each F_iM is a torsion free $\mathcal{O}_X$-module.*

Proof. The vanishing is a formal consequence of properties of Hodge modules. See [S, §3.2, 5.4]. For the last part, it is enough to observe that M is a torsion free, because it can be realized as a submodule of $\mathcal{O}_X(*D)$ for a Cartier divisor $D \supseteq S$. □

When the previous lemma and proposition are combined, we obtain a key formula

$$f_*\Omega^p_{\tilde{X}} = \ker : \Omega_Y^{p+c} \otimes F_cM_X \xrightarrow{\bar{\nabla}} \Omega_Y^{p+c+1} \otimes Gr^F_{c+1}M \tag{7}$$

Here $\bar{\nabla}$ is map induced by the connection ∇ associated to the D-module structure. As a consequence, we see that there is a bijection which associates to a holomorphic p-form α on Y, a section $\widetilde{\alpha} \in H^0(Y, \Omega_Y^{p+c} \otimes F_cM)$ such that $\nabla\tilde{\alpha} \in H^0(Y, \Omega_Y^{p+c+1} \otimes F_cM)$. A calculation shows

Lemma 5.3. *If α is as above and $\beta \in H^0(Y, \Omega_Y^k)$, then*

$$\widetilde{d\alpha} = \nabla\widetilde{\alpha}$$
$$\widetilde{\alpha \wedge \beta} = \widetilde{\alpha} \wedge \beta$$

Lemma 5.4. *A form $\alpha \in H^0(Y - S, \Omega^p_{Y-S})$ extends to Y if and only if $\widetilde{\alpha}$ and $\nabla\widetilde{\alpha}$ both extend to sections of $\Omega^{p+c} \otimes F_cM$ and $\Omega^{p+c+1} \otimes F_cM$ respectively over Y.*

Proof. One direction is clear. Suppose $\widetilde{\alpha}$ and $\nabla\widetilde{\alpha}$ extend to sections α' and γ' of $\Omega_Y^{p+c} \otimes F_cM$ and $\Omega_Y^{p+c+1} \otimes F_cM$. Then $\nabla\alpha - \gamma'$, which is a section of $\Omega_Y^{p+c+1} \otimes F_{c+1}M$, vanishes on U. So it must vanish on Y because the sheaf is a torsion free $\mathcal{O}_X$-module. Therefore $\alpha' = \widetilde{\alpha''}$ for some form α'' on Y extending α.

□

Lemma 5.5. *A section*

$$\xi \in H^0(Y - S, \Omega_Y^{p+c} \otimes F_cM)]$$

extends to $H^0(Y, \Omega_Y^{p+c} \otimes F_cM)$ if and only if for any $\beta \in H^0(Y, \Omega_Y^{n-p})$, $\xi \wedge \beta$ extends to $H^0(Y, \Omega_Y^{n+c} \otimes F_cM)$.

Proof. Applying (7) when $p = n$, implies that

$$F_cM = (f_*\omega_{\tilde{X}}) \otimes \omega_Y^{-1}$$

is generically a line bundle on X, and in particular a line bundle on $X - S$. We can express

$$\xi = \sum dz_I \otimes \mu_I$$

where

$$dz_{\{i_1, i_2, \ldots\}} = dz_{i_1} \wedge dz_{i_2} \wedge \ldots, \quad i_1 < i_2 < \ldots$$

and the μ_I's are sections $H^0(X - S, F_cM)$. Taking $\beta = dz_J$, with $J = \{1, \ldots i_{n+c}\} - I$, and applying the hypothesis shows that μ_I extends to X. □

Proof of theorem 3.1. Suppose $\alpha \in H^0(U, \Omega^p_U)$ is a regular form satisfying the assumptions of the theorem. By lemma 5.4, we need to check that $\widetilde{\alpha}$ and $\nabla\widetilde{\alpha}$ extend to Y. By lemma 5.5, it is enough to check that $\widetilde{\alpha} \wedge \beta$ and $\nabla\widetilde{\alpha} \wedge \gamma$ extend, for Kähler forms β, γ. These conditions can seen to be equivalent to the original assumptions about α by lemma 5.3.

□

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