FORMULAS

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I want to recall some formulas which are used in the Maple/Sage scripts *hodge.maple* and *hodge.sage* avalable in

http://www.math.purdue.edu/∼dvb/scripts/readme.html

Given a smooth projective variety X, the ith Betti number $b_i(X) = \dim H^i(X, \mathbb{Q})$ if it's defined over $\mathbb{C}$, and $\dim H^i(X_{et}, \mathbb{Q}_\ell)$ in general. The pqth Hodge number $h^{pq}(X) = \dim H^q(X, \Omega^p_X)$. These can be assembled into generating functions called the Poincaré and Hodge-Poincaré polynomials

$$Poincare_X(t) = \sum b_i(X)t^i$$

$$HPoincare_X(x, y) = \sum h^{pq}(X)x^p y^q$$

The Hodge decomposition implies that $Poincare(t) = HPoincare(t, t)$ in characteristic 0. This may fail in positive characteristic, although it continues to hold for the examples considered below. Note that in spite of this relationship, it is worth considering these seperately as the Poincaré polynomial is much easier to calculate.

1. Complete intersections

Let $X \subset \mathbb{P}^{n+k}$ be a smooth complete intersection of multidegree $[d_1, \ldots d_k]$. The Hodge number $h^{pq}(X)$ is trivially computable if $p + q \neq n$; it is given by 1 if $p = q < n$ and zero otherwise. The "middle" case, $p + q = n$ is more interesting. Hirzebruch (c.f.[SGA7, exp XI]) showed that the degree n part of $HPoincare(X)$ coincides with the degree n part of the series

$$\frac{1}{(1+x)(1+y)}\left[\prod_i \frac{(1+x)^{d_i} - (1+y)^{d_i}}{(1+y)^{d_i}x - (1+x)^{d_i}y} - 1\right] + \frac{1}{1-xy}$$

In the Sage script,

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hirzebruch_rat(multideg)
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produces this function, where $multideg = [d_1, \ldots]$;

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hirzebruch_ser(multideg, N)
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expands this as a series to order N;

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hodge_ci(multideg, N)
fast_hodge_ci(multideg, N)
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computes the list of the middle Hodge numbers for $X \subset \mathbb{P}^N$. The second form is faster in general, since it doesn't calculate the rational function as an intermediate step. I've only implemented the last two in the Maple version (and in fact they are identical in this case).

2. Moduli of vector bundles

Let $SU_X(n,d)$ (resp. $U_X(n,d)$) be the moduli space of rank n stable bundles with fixed (resp. variable) determinant of degree d over a smooth projective curve X of genus g. These are smooth projective varieties if $(n,d)=1$. I want to recall formulas of Zagier [Z] and del Baño [B] for computing the Betti and Hodge numbers of these spaces. It is enough to this for $SU_X(n,d)$. The Hodge Poincaré polynomial of $U_X(n,d)$ has an extra factor of $(1+x)^g(1+y)^g$ which comes from the Jacobian $J(X)$.

Let

$$\langle x\rangle = 1+[x]-x$$

following Zagier (del Baño also uses this symbol but with a different convention). Set

$$P_n(t) = \frac{(1+t)^{2g}(1+t^3)^{2g}\dots(1+t^{2n-1})^{2g}}{(1-t^2)^2(1-t^4)^2\dots(1-t^{2n-2})^2(1-t^{2n})}$$

$$HP_n(x,y) = \frac{(1+x)^g(1+y)^g\dots(1+x^ny^{n-1})^g(1+x^{n-1}y^n)^g}{(1-xy)^2(1-x^2y^2)^2\dots(1-x^{n-1}y^{n-1})^2(1-x^ny^n)}$$

These represent (Hodge) Poincaré series of the moduli stack of all rank n bundles with fixed determinant of degree d. Note that $HP_n(t,t)=P_n(t)$.

$$M(n_1,\dots,n_k,\lambda) = \sum_{j=1}^{k-1}(n_j+n_{j+1})\langle(n_1+\dots n_j)\lambda\rangle$$

$$M_g(n_1\dots n_k,\lambda) = M(n_1,\dots,n_k,\lambda)+(g-1)\sum_{i<j}n_in_j$$

Let

$$\Sigma(n_1,\dots n_k) = \frac{(-1)^{k-1}t^{2M_g(n_1,\dots n_k,d/n)}}{(1-t^{2n_1+2n_2})\dots(1-t^{2n_{k-1}+2n_k})}P_{n_1}\dots P_{n_k}$$

$$H\Sigma(n_1,\dots n_k) = \frac{(-1)^{k-1}(xy)^{M_g(n_1,\dots n_k,d/n)}}{(1-(xy)^{n_1+n_2})\dots(1-(xy)^{n_{k-1}+n_k})}HP_{n_1}\dots HP_{n_k}$$

So that setting $x=y=t$ in the second expression yields the first.

Then

$$Poincare(t) = P_1^{-1}\sum_{k=1}^{n}\sum_{n_1+\dots n_k=n}\Sigma(n_1,\dots n_k)$$

$$HPoincare(x,y) = HP_1^{-1}\sum_{k=1}^{n}\sum_{n_1+\dots n_k=n}H\Sigma(n_1,\dots n_k)$$

The commands

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poincare_vb(n,d,g)
hpoincare_vb(n,d,g)
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produce these polynomials, and

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poincare_vb_trunc(n,d,g,N)
hpoincare_vb_trunc(n,d,g,N)
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produce them truncated to degree N. I've only implemented poincare... in Maple.

References

[B] del Baño, On the Chow motive of some moduli spaces, Crelles J (2001)

[SGA7] Deligne et. al, SGA 7 II, Springer LNM 340

[Z] Zagier, Elementary aspects of the Verlinde formula and of the Harder-Narasimhan-Atiyah-Bott formula. Proc. of Hirzebruch's 65th birthday conference (1996)