MA54200 HOMEWORK

ASSIGNMENT 1: SOLUTIONS

1.3. Show that the principal value integral

$$\text{p.v.} \int \frac{\phi(x)}{x}\, dx = \lim_{\epsilon \to 0+} \left(\int_{-\infty}^{-\epsilon} \frac{\phi(x)}{x}\, dx + \int_{\epsilon}^{\infty} \frac{\phi(x)}{x}\, dx \right)$$

exists for all $\phi \in C_c^\infty(\mathbb{R})$, and is a distribution. What is its order?

Solution. Note that we can write

$$\text{p.v.} \int \frac{\phi(x)}{x}\, dx = \int_0^\infty \frac{\phi(x) - \phi(-x)}{x}\, dx.$$

The latter integral exist, since

$$\frac{\phi(x) - \phi(-x)}{x} \le 2 \sup |\phi'|.$$

Moreover, if $\operatorname{supp} \phi \subset [-a, a]$, then

$$\left| \text{p.v.} \int \frac{\phi(x)}{x}\, dx \right| \le 2a \sup |\phi'|,$$

which implies that the p.v. of $1/x$ is a distribution of order at most 1. Finally, the order cannot be 0. Indeed, if $0 \le \phi_\epsilon \le 1$ is such that $\operatorname{supp} \phi_\epsilon \subset [\epsilon, 4\epsilon]$ and $\phi_\epsilon = 1$ on $[2\epsilon, 3\epsilon]$ then

$$\text{p.v.} \int \frac{\phi(x)}{x} \ge \frac{1}{4\epsilon} \sup |\phi_\epsilon|$$

□

1.5. Show that

$$\langle u, \phi \rangle = \sum_{k=1}^{\infty} \partial^k \phi(1/k)$$

is a distribution on $(0, \infty)$, but that there is no $v \in \mathcal{D}'(\mathbb{R})$ whose restriction to $(0, \infty)$ is equal to u.

Solution. Let ϕ be such that $\operatorname{supp} \phi \subset [1/N, N]$. Then

$$\langle u, \phi \rangle = \sum_{k=1}^{N} \partial^k \phi(1/k) \le \sum_{k=1}^{N} \sup_{[1/N, N]} |\partial^k \phi|.$$

Since the compacts $[1/N, N]$ exhaust $(0, \infty)$, it follows that u is a distribution on $(0, \infty)$.

Assume now $u = v\big|_{(0,\infty)}$ for $v \in \mathcal{D}'(\mathbb{R})$. Then there must exist N_0 and C_0 such that

$$|\langle v, \phi \rangle| \le C_0 \sum_{k=1}^{N_0} \sup |\partial^k \phi|, \quad \operatorname{supp} \phi \subset [-1, 1].$$

So, if we take $N > N_0$, we will have

$$|\partial^N \phi(1/N)| = |\langle u, \phi\rangle| = |\langle u, \phi\rangle| \leq C_0 \sum_{k=1}^{N_0} \sup |\partial^k \phi|$$

if $\operatorname{supp}\phi \subset (\frac{1}{N+1}, \frac{1}{N-1})$. This would imply that $\partial^N \delta_{1/N}$ is of order at most $N_0 < N$ and consequently that $\partial^N \delta$ is of order at most N_0 on a small interval $(-\epsilon, \epsilon)$. We claim that this is impossible. Indeed, let $\psi \in C_c^\infty((-\epsilon,\epsilon))$ be such that $\partial^N \psi(0) \neq 0$. Consider then the test functions

$$\psi_\lambda(x) = \lambda^N \psi(x/\lambda)$$

for small $\lambda > 0$. We have $\operatorname{supp}\psi_\lambda \subset (-\epsilon\lambda, \epsilon\lambda)$. Moreover,

$$\partial^N \psi_\lambda(0) = \partial^N \psi(0)$$

and

$$\partial^k \psi_\lambda = \lambda^{N-k} \partial^k \psi.$$

Thus, we would have an estimate

$$|\partial^N \psi(0)| \leq C_0 \sum_{k=1}^{N_0} \lambda^{N-N_0} \sup |\partial^k \psi|$$

for any $\lambda > 0$. This is clearly a contradiction for small λ. □

1.6. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ have the property that $\langle u, \phi\rangle \geq 0$ for all real valued nonnegative $\phi \in C_c^\infty(\mathbb{R}^n)$. Show that u is of order 0.

Proof. Let $K \subset\subset \mathbb{R}^n$ and $\psi_K \in C_c^\infty(\mathbb{R}^n)$ be a nonnegative cut-off function such that $\psi_K = 1$ on K. Then for real-valued test functions ϕ with $\operatorname{supp}\phi \subset K$ we have

$$\big(\sup_K |\phi|\big)\psi_K(x) - \phi(x) \geq 0.$$

Hence

$$\langle u, \big(\sup_K |\phi|\big)\psi_K(x) - \phi(x)\rangle \geq 0.$$

This implies

$$\langle u, \phi(x)\rangle \leq \langle u, \psi_K\rangle (\sup_K |\phi|).$$

For complex valued ϕ we obtain

$$|\langle u, \phi(x)\rangle| \leq 2\langle u, \psi_K\rangle (\sup_K |\phi|)$$

by considering the real and imaginary parts of ϕ. □

1.9. Let $(c_k)_{k\in\mathbb{Z}}$ be complex numbers which satisfy

$$|c_k| \leq C(1+|k|)^m, \quad k \in \mathbb{Z},$$

for some constants $C \geq 0$ and m. Show that

$$u = \sum_{k\in\mathbb{Z}} c_k e^{ikx}$$

converges in $\mathcal{D}'(\mathbb{R})$.

Proof. Let $\phi \in C_c^\infty(\mathbb{R})$. Then

$$\int_{\mathbb{R}} e^{ikx}\phi(x)dx = \frac{(-1)^{m+2}}{(ik)^{m+2}} \int_{\mathbb{R}} e^{ikx}\partial^{m+2}\phi(x)dx, \quad k \neq 0.$$

Hence

$$\left| c_k \int_{\mathbb{R}} e^{ikx}\phi(x)dx \right| \leq \frac{C}{k^2} \sup |\partial^{m+2}\phi|, \quad k \neq 0.$$

thus, the series $\langle u, \phi \rangle$ converges and defines a distribution of order at most $m+2$. $\square$