MA54200 HOMEWORK

ASSIGNMENT 2: SOLUTIONS

2.1. Show that

$$\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2} \to \delta \quad \text{in } \mathcal{D}'(\mathbb{R}) \text{ as } \epsilon \to 0+.$$

Proof. Let $\eta_\epsilon(x) := \frac{1}{\pi}\,\epsilon/(x^2+\epsilon^2)$. Then we have the following properties

- $\eta_\epsilon \geq 0$
- $\displaystyle\int_{\mathbb{R}} \eta_\epsilon(x)dx = 1$
- $\displaystyle\int_{|x|\geq\alpha} \eta_\epsilon(x)dx \to 0$ as $\epsilon \to 0$ for any given $\alpha > 0$.

We claim that any family $\{\eta_\epsilon\}$ of L^1 functions satisfying the properties above converges to δ-function in $\mathcal{D}'(\mathbb{R})$. That is, we need to show that

$$\int_{\mathbb{R}} \phi(x)\eta_\epsilon(x)dx \to \phi(0), \quad \text{for any } \phi \in C_c^\infty(\mathbb{R})$$

Indeed, for a given $\kappa > 0$, let $\alpha > 0$ be such that $|\phi(x)-\phi(0)| < \kappa$ for $|x| \leq \alpha$. Then

$$\begin{aligned}\left|\int_{\mathbb{R}} \phi(x)\eta_\epsilon(x)dx - \phi(0)\right| &= \left|\int_{\mathbb{R}} [\phi(x)-\phi(0)]\eta_\epsilon(x)dx\right| \\ &\leq \int_{|x|\leq\alpha} |\phi(x)-\phi(0)|\eta_\epsilon(x)dx + \int_{|x|\geq\alpha} |\phi(x)-\phi(0)|\eta_\epsilon(x)dx \\ &\leq \kappa + \sup|\phi| \int_{|x|\geq\alpha} \eta_\epsilon(x)dx \leq 2\kappa,\end{aligned}$$

if ϵ is sufficiently small. Since κ was arbitrary, we obtain that $\eta_\epsilon \to \delta$ in $\mathcal{D}'(\mathbb{R})$. □

2.3. Show that, if $u \in \mathcal{D}'(\mathbb{R})$ and $x\partial u + u = 0$, then

$$u = A(1/x) + B\delta,$$

where A and B are complex numbers and $1/x$ is the principal value distribution.

Proof. Using the Leibniz rule, we can write the equation as $\partial(xu) = 0$, which implies (by Theorem 2.4.1) that

$$xu = A$$

for a complex constant A. Now let us show that $A(1/x)$ is a particular solution of the above equation. Namely, we need to verify that

$$\langle x(1/x), \phi\rangle = \langle 1/x, x\phi(x)\rangle = \langle 1, \phi\rangle$$

Indeed, we have

$$\langle x(1/x), \phi\rangle = \lim_{\epsilon\to 0+} \int_{-\infty}^{-\epsilon} \frac{x\phi(x)}{x}dx + \int_{\epsilon}^{\infty} \frac{x\phi(x)}{x}dx = \int_{\mathbb{R}} \phi(x)dx = \langle 1, \phi\rangle.$$

Finally, a general solution to the equation $xu = A$ will be

$$u = A(1/x) + B\delta$$

by Theorem 2.7.1

$\square$

2.6. Let $I \subset \mathbb{R}$ be an interval. Show that, if $u \in \mathcal{D}'(I)$ and $\partial u \in C^\infty(I)$ then $u \in C^\infty(I)$.

Let

$$P(x, \partial) = a_0(x)\partial^m + a_1(x)\partial^{m-1} + \cdots + a_m(x)$$

be a differential operator with C^∞ coefficients, defined on I, and suppose that $a_0 \neq 0$ on I. Show that, if $u \in D'(I)$ and $Pu \in C^\infty(I)$, then $u \in C^\infty(I)$.

Remark. If we don't specify, interval means open interval.

Proof. 1) Suppose $u \in \mathcal{D}'(u)$ and $\partial u = f \in C^\infty(I)$. Fix a point $x_0 \in I$ and define $F(x) = \int_{x_0}^x f(t)dt$ for $x \in I$. Then $F \in C^\infty(I)$ and $\partial F = f$. Hence $\partial(u - F) = 0$, which means $u - F = C$ for a complex number C. Thus, $u \in C^\infty(I)$.

2) Suppose $u \in \mathcal{D}'(I)$ and $Pu = f \in C^\infty(I)$. We need to show that $u \in C^\infty(I)$. We will use induction in m (the order of P).

i) If $m = 0$, we have $a_0 u = f \in C^\infty(I)$ hence $u = f/a_0 \in C^\infty(I)$, as $a_0 \neq 0$ in I.

ii) Suppose we know that the statement holds for differential operators of order up to $m-1$ for some $m \geq 1$. We need to prove the statement for m.

From the localization property, it will be sufficient to show that for any x_0 there exists a small interval $J \ni x_0$ such that $u\big|_J \in C^\infty(J)$.

Now, for any $x_0 \in I$, there exists a C^∞ solution ϕ of the equation $P\phi = 0$ in a possibly small interval $J \ni x_0$ such that $\phi(x_0) = 1$. We may assume also that $\phi \neq 0$ in J, by taking J smaller if needed. Consider then a distribution $v = u/\phi$ in J. Plugging $u = v\phi$ and in the equation

$$a_0(x)\partial^m u + \cdots + a_{m-1}(x)\partial u + a_m(x)u = f(x)$$

and using the Leibniz rule, we obtain that v satisfies a similar equation

$$b_0(x)\partial^m v + \cdots + b_{m-1}(x)\partial v + b_m(x)v = f(x),$$

where $b_0 = a_0\phi \neq 0$ and $b_m = a_m P\phi = 0$ in J. Thus, if $w = \partial v$, it will satisfy $(m-1)$-order equation

$$b_0(x)\partial^{m-1} w + \cdots + b_{m-1}(x)w = f(x),$$

and by the inductive assumption we will obtain that $w \in C^\infty(J)$. Consequently, $\partial v \in C^\infty(J)$, $v \in C^\infty(J)$ and $u \in C^\infty(J)$. $\square$

2.14. Define the function $x \mapsto x_-$ on $\mathbb{R}$ by $x_- = (-x)_+$. Show that for $\lambda \in \mathbb{C} \setminus \{0, -1, \ldots\}$ there is a distribution $x_-^{\lambda-1}$ which is an analytic function of λ and equal to the locally integrable function $x_-^{\lambda-1}$ when $\operatorname{Re}\lambda > 0$. Show that it has simple poles at $\lambda = 0, -1, \ldots$, and calculate the residues at these.

Put, also,

$$|x|^{\lambda-1} = x_+^{\lambda-1} + x_-^{\lambda-1}, \quad (x)^{\lambda-1} \operatorname{sign} x = x_+^{\lambda-1} - x_-^{\lambda-1}.$$

Determine the regions of $\mathbb{C}$ on which these distribution are defined and analytic in λ and compute the residues at the poles.

Solution. We may use the properties of $x_+^{\lambda-1}$ as follows. Note that for $\operatorname{Re}\lambda > 0$ we have that

$$\int_{\mathbb{R}} x_-^{\lambda-1}\phi(x)dx = \int_{\mathbb{R}} x_+^{\lambda-1}\phi(-x)dx$$

Thus, for $\lambda \in \mathbb{C} \setminus \{0, -1, \ldots\}$ we may define

$$\langle x_-^{\lambda-1}, \phi(x)\rangle := \langle x_-^{\lambda-1}, \phi(-x)\rangle.$$

(Clearly, this defines a distribution.) For any test function ϕ we therefore have that

$$f_\phi(\lambda) = \langle x_-^{\lambda-1}, \phi(x)\rangle$$

is an analytic function in $\mathbb{C} \setminus \{0, -1, \ldots\}$, with simple poles at $0, -1, \ldots$, from the properties of $x_+^{\lambda-1}$. Hence, the mapping $\lambda \mapsto x_-^{\lambda-1}$ is a distribution valued analytic function with simple poles. Now, to compute the residues, recall that

$$\operatorname*{res}_{\lambda=-k} \langle x_+^{\lambda-1}, \phi(x)\rangle = \frac{\partial^k\phi(0)}{k!}$$

Hence

$$\operatorname*{res}_{\lambda=-k} \langle x_-^{\lambda-1}, \phi(x)\rangle = \operatorname*{res}_{\lambda=-k} \langle x_+^{\lambda-1}, \phi(-x)\rangle = \frac{(-1)^k\partial^k\phi(0)}{k!}.$$

Therefore,

$$\operatorname*{res}_{\lambda=-k} x_-^{\lambda-1} = \frac{\partial^k\delta}{k!}$$

Recalling also that

$$\operatorname*{res}_{\lambda=-k} x_+^{\lambda-1} = \frac{(-1)^k\partial^k\delta}{k!}$$

we obtain that $|x|^{\lambda-1}$ is analytic in $\mathbb{C} \setminus \{0, -2, -4, \ldots\}$ with simple poles and

$$\operatorname*{res}_{\lambda=-2m} |x|^{\lambda-1} = \frac{2\partial^{2m}\delta}{(2m)!}.$$

Similarly, $(x)^{\lambda-1}\operatorname{sign} x$ is analytic in $\mathbb{C} \setminus \{-1, -3, \ldots\}$ with simple poles and

$$\operatorname*{res}_{\lambda=-(2m-1)} (x)^{\lambda-1}\operatorname{sign} x = -\frac{2\partial^{2m-1}\delta}{(2m-1)!}.$$

□

3.3. Determine all $u \in \mathcal{D}'(\mathbb{R}^2)$ such that

$$(x_1 + ix_2)u = 0.$$

Solution. Let u be as in the statement of the problem. Then we claim that

$$\operatorname{supp} u = \{0\}.$$

Indeed, for any ϕ such that $0 \notin \operatorname{supp}\phi$ we have

$$\langle u, \phi\rangle = \langle u, (x_1 + ix_2)\phi/(x_1 + ix_2)\rangle = \langle (x_1 + ix_2)u, \phi/(x_1 + ix_2)\rangle = 0.$$

Hence, by Theorem 3.2.1 , $u = \sum_{0\le k+j\le N} c_{kj}\partial_1^k\partial_2^j\delta$. For further analysis, it will be more convenient to use complex notations. Introduce

$$z = x_1 + ix_2, \quad \partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar z} = \frac{1}{2}(\partial_1 + i\partial_2)$$

Then u can be written as

$$u = \sum_{0 \le k+j \le N} a_{kj} \partial_z^k \partial_{\bar{z}}^j \delta.$$

Now we claim that $a_{kj} = 0$ for $k \ge 1$. To this end, observe that we have the following identities

$$\partial_z z = 1, \quad \partial_{\bar{z}} z = 0, \quad \partial_{\bar{z}} \bar{z} = 1, \quad \partial_z \bar{z} = 0,$$

which imply that

$$\partial_z^k \partial_{\bar{z}}^j (z^m \bar{z}^n)\big|_{z=0} = \begin{cases} k!j!, & k = m, j = n \\ 0, & \text{otherwise} \end{cases}$$

Thus, if ψ_0 is a cut-off function, identically 1 in a neighborhood of the origin, and $k \ge 1$ then

$$a_{kj} = \frac{(-1)^{k+j}}{k!j!} \langle u, \psi_0 z^k \bar{z}^j \rangle = \frac{(-1)^{k+j}}{k!j!} \langle zu, \psi_0 z^{k-1} \bar{z}^j \rangle = 0$$

Thus,

$$u = \sum_{j=0}^{N} c_j \partial_{\bar{z}}^j \delta.$$

Conversely, every linear combination u of $\partial_{\bar{z}}^j \delta$ solves $zu = 0$. Indeed,

$$\langle z \partial_{\bar{z}}^j \delta, \phi \rangle = (-1)^j \langle \delta, \partial_{\bar{z}}^j (z\phi) \rangle = \langle \delta, z \partial_{\bar{z}}^j (\phi) \rangle = 0,$$

where we have used that $\partial_{\bar{z}}^j (z\phi) = z\, \partial_{\bar{z}}^j \phi$, which is a consequence of $\partial_{\bar{z}} z = 0$. $\square$