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Solution to p. 64, Problem 22

We measure the vertical distance x of the ball from the roof of the building positively upwards, so that $x = 0$ at time $t = 0$. The velocity $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ follows the same sign convention, and we denote the initial velocity $v(0) =: v_0$. The model for this experiment is derived from Newton's law $F = m\frac{\mathrm{d}v}{\mathrm{d}t}$ by writing down the net force. The problem here is that the air resistance changes sign at the highest point, whereas the expression $v^2/1325$ given in the text does not. So we model two different movements, the first upward movement until the ball reaches the highest point, the second downward movement on its way back. The net forces in both cases are

$$F = -mg - kv^2, \tag{1}$$
$$F = -mg + kv^2, \tag{2}$$

where $k = \frac{1}{1325}\frac{kg}{m}$ is the drag constant. Hence the two inital value problems are

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g - \frac{k}{m}v^2, \quad v(0) = v_0, \tag{3}$$
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g + \frac{k}{m}v^2, \quad v(T_1) = 0, \tag{4}$$

where T_1 denotes the time after which the ball reaches its maximum. We first solve equation (3). We transform it into an equation between v and x treated as the independent variable. Note that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}v,$$

hence (3) is equivalent to

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{-g - \frac{k}{m}v^2}{v}. \tag{5}$$

To solve (5), we separate variables,

$$\frac{v\mathrm{d}v}{g + \frac{k}{m}v^2} = -\mathrm{d}x,$$

and integrate:

$$\int \frac{v\mathrm{d}v}{g + \frac{k}{m}v^2} = -\int \mathrm{d}x + c \quad \Longleftrightarrow \quad \frac{m}{2k}\ln(g + \frac{k}{m}v^2) = -x + c$$

with some constant c, or

$$v = \sqrt{\frac{m}{k}(Ce^{-\frac{2k}{m}x} - g)}$$

with some other constant C. To compute C, we use the initial condition $v(0) = v_0$, which implies

$$v_0^2 = \frac{m}{k}(Ce^{-\frac{2k}{m}x} - g) \quad \Rightarrow \quad C = g + \frac{k}{m}v_0^2.$$

Hence the solution of (5) together with the initial condition $v(0) = 0$ is

$$v = \sqrt{(v_0^2 + \frac{m}{k}g)e^{-\frac{2k}{m}x} - \frac{mg}{k}}.$$

(a) Compute the maximal height h: It is characterized by the condition $v(h) = 0$, hence

$$0 = (v_0^2 + \frac{m}{k}g)e^{-\frac{2k}{m}h} - \frac{mg}{k} \quad \Rightarrow \quad h = \frac{m}{2k}\ln(1 + \frac{kv_0^2}{mg}) \approx 18.562\ m.$$

(b) Compute the time T_2, after which the ball hits the ground. For that purpose, we first compute the time T_1, after which the ball reaches its maximal height h. Consider the solution from above:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{(v_0^2 + \frac{m}{k}g)e^{-\frac{2k}{m}x} - \frac{mg}{k}}. \tag{6}$$

To simplify notation, set $K := v_0^2 + \frac{m}{k}g$, $\alpha := \frac{2k}{m}$, and $\beta := \frac{mg}{k}$. Then we have to solve the separable equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{Ke^{-\alpha x} - \beta} \quad \Longleftrightarrow \quad \frac{\mathrm{d}x}{\sqrt{Ke^{-\alpha x} - \beta}} = \mathrm{d}t.$$

The substitution $u := \sqrt{Ke^{-\alpha x} - \beta}$ solves the integral

$$\int \frac{\mathrm{d}x}{\sqrt{Ke^{-\alpha x} - \beta}} = -\frac{2}{\alpha\sqrt{\beta}} \arctan\sqrt{\frac{K}{\beta}e^{-\alpha x} - 1},$$

but $\alpha\sqrt{\beta} = 2\sqrt{\frac{mg}{k}} =: 2\gamma$, hence (6) is equivalent to

$$-\frac{1}{\gamma}\arctan\sqrt{\frac{K}{\beta}e^{-\alpha x} - 1} = t + c, \tag{7}$$

which is the implicit solution of (6). We plug the initial condition $x(0) = 0$ into (7) and get

$$c = -\frac{1}{\gamma}\arctan\sqrt{\frac{K}{\beta} - 1}.$$

At time $t = T_1$, we get from (7)

$$T_1 = \frac{1}{\gamma}\left(\arctan\sqrt{\frac{K}{\beta} - 1} - \arctan\sqrt{\frac{K}{\beta}e^{-\alpha h} - 1)}\right).$$

We solve the movement down (4) analogously as the movement up:

$$v = -\sqrt{\frac{m}{k}(Ce^{-\frac{2k}{m}x} + g)}.$$

But now we measure t and x from the maximal height, so the initial conditions are $x(0) = 0$ and $v(0) = 0$. Hence

$$0 = -\sqrt{\frac{m}{k}(C + g)} \quad \Rightarrow \quad C = -g.$$

Thus the solution is

$$v = -\sqrt{\frac{mg}{k}(1 - e^{-\frac{2k}{m}x})} = -\sqrt{\beta(1 - e^{-\alpha x})}.$$

Now solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sqrt{\beta(1 - e^{-\alpha x})},$$

which is separable:

$$\frac{\mathrm{d}x}{\sqrt{\beta(1 - e^{-\alpha x})}} = -\mathrm{d}t.$$

The substitution $u = \sqrt{\beta(1 - e^{-\alpha x})}$ solves the integral

$$\int \frac{\mathrm{d}x}{\sqrt{\beta(1 - e^{-\alpha x})}} = -\frac{1}{2\gamma}\ln\frac{1 + \sqrt{1 - e^{\alpha x}}}{1 - \sqrt{1 - e^{\alpha x}}},$$

hence the implicit solution is

$$-\frac{1}{2\gamma}\ln\frac{1 + \sqrt{1 - e^{\alpha x}}}{1 - \sqrt{1 - e^{\alpha x}}} = -t + c.$$

The initial condition $x(0) = 0$ implies $c = 0$, hence

$$\ln\frac{1 + \sqrt{1 - e^{\alpha x}}}{1 - \sqrt{1 - e^{\alpha x}}} = 2\gamma t.$$

After time $t = T_2$ the ball hits the ground, i.e. it has height $-h - 30$, hence

$$T_2 = \frac{1}{2\gamma}\ln\frac{1 + \sqrt{1 - e^{-\alpha(h+30)}}}{1 - \sqrt{1 - e^{-\alpha(h+30)}}}.$$

The total time, after which the ball hits the ground, is

$$T_g = T_1 + T_2.$$

We leave the numerical evaluation to the reader.