LECTURE 11

11. Classification of Free Boundary points

11.1. Homogeneous Global Solutions. By Theorem 10.3, blowups of solutions of Problems A and B at fixed $x_0 \in \Gamma$ and of Problem C at $x_0 \in \Gamma'$ are *homogeneous* (of degree two) *global solutions*, i.e. a global solution u satisfying

$$u(\lambda x) = \lambda^2 u(x), \quad x \in \mathbb{R}^n,\ \lambda > 0.$$

Another way to express the homogeneity is by the identity

$$\partial^{(2)} u(x) := x \cdot \nabla u(x) - 2u(x) = 0 \quad \text{in } \mathbb{R}^n. \tag{11.1}$$

In this section we give a complete description of such solutions.

Theorem 11.1 (Classification of homogeneous global solution)**.** *Let u be a homogeneous global solution of Problem A, B, or C. Then u is of one of the following forms.*

- In Problems A, B:

- Polynomial solution $u(x) = \frac{1}{2}(x \cdot Ax)$, $x \in \mathbb{R}^n$. *where A is an $n \times n$ symmetric matrix with* $\operatorname{Tr} A = 1$
- Halfplane solutions $u(x) = \frac{1}{2}(x \cdot e)_+^2$, $x \in \mathbb{R}^n$, *where e is a unit vector.*

-In Problem C:

- Polynomial solutions (positive or negative) $u(x) = \frac{\lambda_+}{2}(x \cdot Ax)$ *or* $u(x) = -\frac{\lambda_-}{2}(x \cdot Ax)$, $x \in \mathbb{R}^n$, *where A is an $n \times n$ nonnegative symmetric matrix with* $\operatorname{Tr} A = 1$.
- Halfplane solutions (positive or negative) $u(x) = \frac{\lambda_+}{2}(x \cdot e)_+^2$ *or* $u(x) = -\frac{\lambda_-}{2}(x \cdot e)_-^2$, $x \in \mathbb{R}^n$, *for a unit vector e.*
- Two-plane solution $u(x) = \frac{\lambda_+}{2}(x \cdot e)_+^2 - \frac{\lambda_-}{2}(x \cdot e)_-^2$, $x \in \mathbb{R}^n$, *for a unit vector e.*

Proof.
Problems A, B. Observe that $u \in P_\infty(M)$ and

$$u(0) = |\nabla u(0)| = 0. \tag{11.2}$$

Consider two possibilities. First suppose that $\operatorname{Int} \Omega^c(u) = \emptyset$. Then, since $\partial\Omega(u)$ has zero Lebesgue measure (see Corollary 8.10) the function u satisfies the equation $\Delta u = \mathit{const}$ a.e. in $\mathbb{R}^n$, and $\|D^2 u_0\|_{L^\infty(\mathbb{R}^n)} \le M$. By Liouville's theorem u is a degree two polynomial and homogeneity comes from (11.2).

Next, suppose $\operatorname{Int} \Omega^c(u) \neq \emptyset$. Then we apply the ACF Monotonicity Formula (see Lecture 5) to the positive and the negative parts of $\partial_e u$, for different directions e. Recall that we have shown earlier that $(\partial_e u)^\pm$ are subharmonic, so the ACF Monotonicity Formula is applicable (see Lemma 6.2 applied with $M_1 = M_2 = 0$).

The homogeneity of u readily implies that for any direction e

$$\phi_e(r) := \Phi(r, (\partial_e u)^+, (\partial_e u)^-) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla(\partial_e u)^+|^2}{|x|^{n-2}} \int_{B_r} \frac{|\nabla(\partial_e u)^-|^2}{|x|^{n-2}},$$

is constant for all $r > 0$. Now, wee need to use Theorem 5.3, which treats the case of equality in the ACF Monotonicity Formula. By this theorem we have two possibilities: either

(i) $\partial_e u \geq 0$ in $\mathbb{R}^n$ or $\partial_e u \leq 0$ in $\mathbb{R}^n$, or
(ii) $\operatorname{supp}(\partial_e u)^+$ and $\operatorname{supp}(\partial_e u)^-$ are complementary halfspaces.

The latter possibility is excluded, as it will imply that $\operatorname{Int}\Omega^c(u) = \emptyset$, contrary to our assumption. Thus the former possibility holds and therefore we obtain that $\partial_e u$ does not change the sign in $\mathbb{R}^n$ for any direction e. This is possible if and only if u is one-dimensional, i.e. there exists a direction e and a function f on $\mathbb{R}$ such that $u(x) = f(x \cdot e)$. The rest of the proof is an easy consequence from this.

Problem C. We start with the claim that $\Gamma''(u) = \emptyset$. First, as above, we apply the ACF Monotonicity Formula to the positive and negative parts of a directional derivative $\partial_e u$. From homogeneity of u, we obtain that $\phi_e(r)$ is a constant for $r > 0$. From Theorem 5.3 we obtain therefore that either

(i) $\partial_e u \geq 0$ in $\mathbb{R}^n$ or $\partial_e u \leq 0$ in $\mathbb{R}^n$, or
(ii) $(\partial_e u)^+ \Delta(\partial_e u)^+ = 0$ in $\mathbb{R}^n$ and $(\partial_e u)^- \Delta(\partial_e u)^- = 0$ in $\mathbb{R}^n$ in the sense of measures.

Suppose now that there exists a point $y_0 \in \Gamma''(u)$ and denote by ν_0 the direction of gradient of u at y_0. There is a neighborhood $B_\rho(y_0)$ where $\partial_{\nu_0} u > 0$ and $\{u = 0\} \cap B_\rho(y_0)$ is a $C^{1,\alpha}$-surface. If $e \cdot \nu_0 \neq 0$ then $\partial_e u(y_0) \neq 0$ and for sufficiently small δ we obtain

$$|\Delta \partial_e u|(B_\delta(y_0)) = \frac{|\lambda_+ + \lambda_-|}{2} \int_{\{u=0\}\cap B_\delta(y_0)} |e \cdot \nu| \, dH^{n-1} > 0,$$

where $\nu = \nabla u / |\nabla u|$ is the normal to the surface $u = 0$. Thus the case (ii) cannot hold for directions e nonorthogonal to ν_0. Thus, (i) holds for all such directions and by continuity for all directions. As before, this implies that u is one-dimensional. But it is easy to show that all one-dimensional homogeneous solutions of Problem C are either halfplane or two-plane solutions for which $\Gamma'' = \emptyset$. This contradicts to assumption that $\Gamma''(u)$ is non-empty.

Once we have that $\Gamma''(u) = \emptyset$, renormalized positive and negative parts of u, $\frac{1}{\lambda_\pm} u^\pm$, will solve Problem A. Thus u has one of the forms described in the statement of the theorem. $\square$

11.2. **Classification of Free Boundary Points.** Since the blowups with fixed centers are homogeneous global solution by Theorem 11.1, this leads to a classification of free boundary points according to their blowup. But first we need to show any two blowups at a given point are of the same type.

We start by finding Weiss's energy of homogeneous global solutions described in Theorem 11.1.

Problem A. If u is a homogeneous global solution and we integrate by parts in the expression for W, using that $\Delta u = 1$ in Ω, we arrive at

$$\begin{aligned} W(r, u, 0) \equiv W(1, u, 0) &= \int_{B_1} \left(|\nabla u|^2 + 2u\right) dx - 2 \int_{\partial B_1} u^2 \, dH^{n-1} \\ &= \int_{B_1} (-\Delta u + 2)\, u \, dx - \int_{\partial B_1} \partial^{(2)} u \, u \, dH^{n-1} = \int_{B_1} u \, dx. \end{aligned}$$

Hence, for polynomial solutions $u(x) = \frac{1}{2}(x \cdot Ax)$, we have

$$W(r, u, 0) = \frac{1}{2} \int_{B_1} x \cdot Ax \, dx = \alpha_n$$

and for halfplane solutions $u(x) = \frac{1}{2}(x \cdot e)_+^2$

$$W(r, u, 0) = \frac{1}{2} \int_{B_1} (x \cdot e)_+^2 dx = \frac{\alpha_n}{2},$$

where

$$\alpha_n = \frac{1}{2} \int_{B_1} x_1^2 \, dx.$$

So, the only values taken by W on homogeneous solutions are

$$\alpha_n, \quad \frac{\alpha_n}{2}.$$

Problem B. As it follows from Theorem 11.1, the homogeneous solutions of Problems A and B are identical and so are their Weiss functionals.

Problem C. Arguing similarly, we obtain that the homogeneous solutions in this case have energies W

$$\lambda_\pm \alpha_n, \quad \frac{(\lambda_+ + \lambda_-)\alpha_n}{2}, \quad \frac{\lambda_\pm \alpha_n}{2}.$$

The computation above and Weiss's Monotonicity Formula lead us to the following definition.

Definition 11.2 (Balanced Energy)**.** Let $u \in P_R(x_0, M)$ be a solution of Problem A, B, or C and assume additionally that $x_0 \in \Gamma'(u)$ in the case of Problem C. Then the limit

$$\omega(x_0) := \lim_{r \to 0} W(r, u, x_0), \tag{11.3}$$

which exists by Theorem 10.2, is called the *balanced energy* of u at x_0.

If $u_0 = \lim_{j \to \infty} u_{x_0, \lambda_j}$ for $\lambda_j \to 0$ is a blowup of u at a fixed center x_0 as in Theorem 10.3 then

$$\omega(x_0) = \lim_{j \to \infty} W(\lambda_j, u, x_0) = \lim_{j \to 0} W(1, u_{x_0, \lambda_j}, 0) = W(1, u_0, 0).$$

Thus, the balanced energy at a point coincides with the Weiss energy of any of blowups with fixed center x_0. This has two consequences: first that the balanced energy can take only a limited number of values and second that all blowups at x_0 are of the same type.

Proposition 11.3. Problems A, B. *The balanced energy is an upper semicontinuous function of $x_0 \in \Gamma(u)$ and*

$$\omega(x_0) \in \left\{ \alpha_n, \frac{\alpha_n}{2} \right\}.$$

Problem C. *The balanced energy is an upper semicontinuous function of $x_0 \in \Gamma'(u)$ and*

$$\omega(x_0) \in \left\{ \lambda_\pm \alpha_n, \frac{(\lambda_+ + \lambda_-)\alpha_n}{2}, \frac{\lambda_\pm \alpha_n}{2} \right\}.$$

Proof. The upper semicontinuity follows from the fact that

$$W(r, u, \cdot) =: \omega_r(\cdot) \searrow \omega(\cdot), \qquad \text{as } r \searrow 0,$$

and the functions $\omega_r(\cdot)$ are continuous for any $r > 0$. The values taken by the balanced energy are obtained from the analysis immediately before and after Definition 11.2. □

Proposition 11.4 (Unique type of the blowup)**.** *Let $x_0 \in \Gamma(u)$ in Problems A, B, or $x_0 \in \Gamma'(u)$ in Problem C. Then any two blowups of solution u with fixed center x_0 have the same type.*

We leave the proof to the reader as an exercise.

Definition 11.5 (Classiffication of Free Boundary Points)**.**
– In Problems A, B for $x_0 \in \Gamma(u)$ we will use the following terminology:

- x_0 is a *high-energy point*, if $\omega(x_0) = \alpha_n$
- x_0 is a *low-energy point*, if $\omega(x_0) = \frac{\alpha_n}{2}$

Equivalently, x_0 is high-energy if blowups with fixed center x_0 are polynomial and low-energy if blowups are halfplane solutions.
– In Problem C, for $x_0 \in \Gamma'(u)$, we say

- x_0 is a *two-phase point*, if $x_0 \in \partial\{u > 0\} \cap \partial\{u < 0\}$
- x_0 is an *one-phase point*, otherwise

One can show that the solution u of Problem C does not change sign in a neighborhood of an one-phase point. Similarly to Problems A, B we distinguish *high-enegy* and *low-energy* one-phase points, depending on their balanced energy.