LECTURE 12

12. First Results on the Regularity of the Free Boundary

12.1. **Problem A: C^1-regularity of the free boundary near low-energy points.** In this lecture, we study the free boundary in Problem A near the low-energy points. An immediate corollary from the upper semicontinuity of the balanced energy (see Proposition 11.3) is that the set of low energy points is a relatively open subset of $\Gamma(u)$. Our purpose here is to show that this set is locally a graph of a C^1-function.

12.1.1. *Flatness of the Free Boundary.*

Lemma 12.1. *Let $u \in P_1(M)$ and a halfplane solution $u_0(x) = \frac{1}{2}(x_1)_+^2$ be such that*

$$\|u - u_0\|_{L^\infty(B_1)} \le \epsilon.$$

Then

$$\Gamma(u) \cap B_{1/2} \subset \{|x_1| \le C\sqrt{\epsilon}\}$$

for a dimensional constant C.

Proof. 1) It is immediate that $u > 0$ in $\{x_1 > \sqrt{2\epsilon}\} \cap B_1$.

2) We claim that $u \equiv 0$ in $\{x_1 < -C\sqrt{\epsilon}\} \cap B_{1/2}$, for a dimensional constant C.

Indeed, take any $x_0 \in \Omega(u) \cap B_{1/2} \cap \{x_1 < 0\}$ and let $r := -x_0 \cdot e_1 > 0$. Then by the nondegeneracy we have

$$\sup_{B_r(x_0)} u \ge u(x_0) + c\,r^2.$$

Now note that $B_r(x_0) \subset \{x_1 < 0\}$, implying that $u_0 = 0$ and consequently $|u| \le \epsilon$ in $B_r(x_0)$. Hence, one must have $c\,r^2 \le 2\epsilon$, or $r \ge \sqrt{2\epsilon/c}$. Therefore $\Omega(u) \cap B_{1/2} \subset \cap\{x_1 > -\sqrt{2\epsilon/c}\}$. □

12.1.2. *Lipschitz regularity.* We first show the Lipschitz character of the free boundary.

The following lemma contains a key idea that will yet appear in modified forms for several times and plays the role of the bridge for transferring the properties of global solutions to local ones.

Lemma 12.2. *Let $u \in P_r(x_0, M)$ be a solution of Problem A and suppose that for some constant $C > 0$ and direction e, we have*

$$C\,r\,\partial_e u - u \ge -\epsilon_0 r^2 \quad \text{in } B_r(x_0), \tag{12.1}$$

where $0 < \epsilon_0 < 1/8n$. Then

$$C\,r\,\partial_e u - u \ge 0 \quad \text{in } B_{r/2}(x_0). \tag{12.2}$$

Proof. Suppose the conclusion of the lemma fails. Then for some point $y \in B_{r/2}(x_0) \cap \Omega$ we have $C\, r\, \partial_e u(y) - u(y) < 0$.

Let

$$w(x) = C\, r\, \partial_e u(x) - u(x) + \frac{1}{2n}|x-y|^2.$$

Then w is harmonic in $\Omega \cap B_{r/2}(y)$, $w(y) < 0$, and $w \geq 0$ on $\partial\Omega$. Hence by the minimum principle the negative infimum of w is attained on $\partial B_{r/2}(y)$. This gives

$$\inf_{\partial B_{r/2}(y) \cap \Omega} (C\, r\, \partial_e u - u) \leq -\frac{r^2}{8n},$$

which contradicts (12.1) since $\epsilon_0 < 1/8n$. □

Lemma 12.3. *Let $u \in P_1(M)$ be a solution of Problem A. Suppose that the origin is a low energy point and that the rescalings $u_{r_j}(x) = \frac{1}{r_j^2} u(r_j x)$ converge to a halfspace solution $u_0(x) = \frac{1}{2}(x_1)_+^2$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for some sequence $r_j \to 0$. Then $u \geq 0$ in B_{r_0} for some $r_0 = r_0(u) > 0$.*

Moreover, for any $0 < \delta \leq 1$ there exists $r_\delta > 0$ such that $\partial_e u \geq 0$ in B_{r_δ} for any direction

$$e \in K_\delta := \{e \in S^{n-1} : \ e \cdot e_1 \geq \delta\}.$$

Proof. If $e \in K_\delta$ then

$$\delta^{-1}\partial_e u_0 - u_0 \geq 0 \quad \text{in } B_1,$$

and by convergence of u_j we have the inequality

$$\delta^{-1}\partial_e u_{r_j} - u_{r_j} > -\frac{1}{8n} \quad \text{in } B_1,$$

for sufficiently large $j \geq j_\delta$. Then by Lemma 12.2 we conclude that for such j

$$\delta^{-1}\partial_e u_{r_j} - u_{r_j} \geq 0 \quad \text{in } B_{1/2}. \tag{12.3}$$

Besides, it follows from Lemma 12.1 that for large j

$$u_{r_j} = 0 \quad \text{in } B_{1/2} \cap \{x_1 < -\frac{1}{4}\}. \tag{12.4}$$

Now consider (12.2) with $e = e_1$ ($\delta = 1$). It can be rewritten as

$$\partial_{x_1}(e^{-x_1} u_{r_j}) \geq 0.$$

Taking into account (12.4) and integrating we arrive at

$$u_{r_j} \geq 0 \quad \text{in } B_{1/4} \quad \text{for } j \geq j_1.$$

Together with (12.3) it gives us the inequality

$$\partial_e u_{r_j} \geq 0 \quad \text{in } B_{1/2} \quad \text{for } j \geq j_\delta,$$

which is equivalent to

$$\partial_e u \geq 0 \quad \text{in } B_{r_j},$$

for small enough $r_j \leq r_\delta$. □

12.1.3. C^1*-regularity.* Now we give a rather short proof of C^1-regularity of the free boundary in a neighborhood of a low-energy point. Deeper investigation, including the uniform estimates of C^1 modulus of continuity for a class of solutions $u \in P_1(M)$ will be done in forthcoming lectures.

Theorem 12.4. *Let $u \in P_1(M)$ be a solution of Problem A, and suppose the origin is a low-energy point. Then there exists $\rho > 0$ such that $\Gamma \cap B_\rho$ is a C^1-graph.*

Proof. Consider a blowup sequence with fixed center at the origin. By our assumption any such blowup is a halfplane solution. Then, by Lemma 12.3 there exists a cone of monotonicity $K_{1/2}$ in some neighborhood B_{r_0} of the origin. It means that in B_{r_0} the function u is non-decreasing along any direction $e \in K_{1/2}$, i.e. $\Gamma \cap B_{r_0}$ is a Lipschitz graph. In fact, Lemma 12.3 guarantees much more. For any $\delta > 0$ the cone K_δ is a cone of monotonicity for u in an appropriate r_δ-neighborhood of the origin. This yields the existence of a tangent plane at the origin.

Now, by Lemma 12.3, every free boundary point $z \in \Gamma \cap B_\epsilon$ is a low-energy point and therefore Γ has a tangent plane at all these points. Once again Lemma 12.3 implies that the normal vector ν_z to Γ, at $z \in \Gamma$, is δ-close to ν_0 (the normal to Γ at the origin) if $z \in B_{r\delta}$, and δ maybe chosen arbitrary small. The latter means that Γ is C^1 at the origin. Hence the statement of the theorem follows with $\rho = \min(\epsilon, r_0)$. $\square$