LECTURE 13

13. First Results on the Regularity of the Free Boundary (continued)

13.1. **Problem B: the local structure of the patches.** The main result of this section is the following theorem.

Theorem 13.1. *Let $u \in P_1(x_0, M)$ and suppose that x_0 is a low-energy point. Then, there is a $\rho = \rho(u) > 0$ such that*

$$\Omega^c \cap B_\rho(x_0) \subset \{u = u(x_0)\}\,.$$

Recall that the energy classification of free boundary points is given in Lecture 11.

The proof of the theorem is based on the ideas of directional monotonicity, similar to Problem A that we discussed in Lecture 12.

Lemma 13.2. *Let $u \in P_1(x_0, M)$ and suppose that*

$$C\partial_e u - |\nabla u|^2 \geq -\epsilon_0 \quad \text{in } B_1$$

where $0 \leq \epsilon_0 < 1/4n^2$. Then

$$C\partial_e u - |\nabla u|^2 \geq 0 \quad \text{in } B_{1/2}$$

Proof. We argue similarly as in the proof of Lemma 12.2 and use that

$$\Delta(|\nabla u|^2) = 2|D^2 u|^2 + 2\nabla u \nabla(\Delta u) \geq \frac{2}{n}$$

in $\Omega(u)$ instead of $\Delta u = 1$. We leave the details to the reader. □

Lemma 13.3. *Let $u \in P_1(x_0, M)$ with x_0 a low energy point and suppose that $u_0(x) = \frac{1}{2}(x_1^+)^2$, $x \in \mathbb{R}^n$, be a blowup of u at fixed center x_0. Then, for any δ there is $r_\delta > 0$ such that*

$$\delta^{-1}\partial_e u - |\nabla u|^2 \geq 0 \quad \text{in } B_{r_\delta}(x_0)$$

for any direction

$$e \in K_\delta := \{e \in \partial B_1 : e \cdot e_1 \geq \delta\}$$

In particular, $\partial_e u \geq 0$ in $B_{r_\delta}(x_0)$ for any $e \in K_\delta$.

Proof. We have that

$$\partial_e u_0 = x_1^+(e \cdot e_1) \quad \text{and} \quad |\nabla u_0|^2 = (x_1^+)^2.$$

Hence, for $e \in K_\delta$

$$\delta^{-1}\partial_e u_0 - |\nabla u_0|^2 \geq 0 \quad \text{in } B_1.$$

Fix now a sequence $r_k \to 0$ such that

$$u_0 = \lim_{k\to\infty} u_{r_k}.$$

By uniform convergence, for all $\epsilon > 0$, there is k_0 such that for all $k \geq k_0$, and $e \in K_\delta$ we have

$$\delta^{-1}\partial_e u_{r_k} - |\nabla u_{r_k}|^2 \geq -\epsilon_0 \quad \text{in } B_1,$$

for $\epsilon_0 < 1/4n^2$. Thus, by the previous lemma

$$\delta^{-1}\partial_e u_{r_k} - |\nabla u_{r_k}|^2 \geq 0 \quad \text{in } B_{1/2}$$

and the proof is complete by taking $r_\delta = \frac{r_k}{2}$, the lemma is proved. □

Lemma 13.4. *Under the assumptions and notations of Lemma 13.3 we have*

$$x_0 - K_\delta \cap B_{r_\delta} \subset \{u \leq u(x_0)\}\,.$$

Proof. We omit the obvious proof. □

Lemma 13.5. *Under the assumptions and notations of Lemma 13.3, there is $r'_\delta > 0$ such that*

$$x_0 - K_{2\delta} \cap B_{r'_\delta} \subset \Omega^c.$$

Proof. Suppose there is a sequence $\{x_k\} \subset -K_{2\delta} \cap B_1 \cap \Omega$ such that $\rho_k = |x_k| \to 0$. Fix a constant $\tau > 0$, such that

$$B_{\tau|x|}(x) \subset -K_\delta, \quad \text{for any } x \in -K_{2\delta}.$$

By the nondegeneracy lemma of u in Ω, there exist $y_k \in B_{\tau\rho_k}(x_k)$ such that

$$u_{\rho_k}({\rho_k}^{-1}y_k) - u_{\rho_k}({\rho_k}^{-1}x_k) \geq c_0\tau^2, \tag{13.1}$$

for some constant $c_0 > 0$, independent of k.

Now, since we assume that x_0 is a low energy point, any blow up of u at x_0 is a half space solution. In particular

$$\liminf_{r\to 0} u_r(x) \geq 0\,, \quad \text{for all } x \in \mathbb{R}^n.$$

On the other hand, by Lemma 13.4,

$$\limsup_{r\to 0} u_r(x) \leq 0 \quad \text{for all } x \in -K_\delta.$$

Hence, for all $x \in -K_\delta$,

$$\lim_{r\to 0} u_r(x) = 0.$$

Since $\{{\rho_k}^{-1}y_k\}$ and $\{{\rho_k}^{-1}x_k\}$ are two bounded sequences contained in $-K_\delta \cap B_2$,

$$\lim_{k\to\infty} u_{\rho_k}({\rho_k}^{-1}y_k) - u_{\rho_k}({\rho_k}^{-1}x_k) = 0,$$

which is a contradiction to (13.1). □

Proof of Theorem 13.1. Fix some $0 < \delta < 1/2$ and let $\rho = \frac{2}{3}\min\{r_\delta, r'_\delta\}$, with r_δ as in Lemma 13.3 and r'_δ as in Lemma 13.5.

We claim that any point $x \in \Omega^c \cap B_\rho$ can be joined to $-K_{2\delta} \cap B_\rho$ with a segment parallel to e_1 and contained in Ω^c. In fact, if the claim is not true, then we can find two points, $x_1 \in \Omega \cap B_\rho$ and $x_2 \in \Omega^c \cap B_\rho$ such that

$$x_2 - x_1 = |x_2 - x_1| e_1.$$

Now, take a small ball $B_\epsilon(x_1) \subset \Omega$ and denote by C the cone generated by x_2 and $B_\epsilon(x_1)$. Move $B_\epsilon(x_1)$ from x_1 to x_2 along the axis of C, reducing its radius to fit in C, until we touch for the first time Ω^c. Let ζ_0 be a point of contact. There is $\rho' > 0$ and $\delta' > 0$ such that

$$\zeta_0 - K_{\delta'} \cap B_{\rho'} \subset B_\rho(x_0) \cap \Omega. \tag{13.2}$$

Since, by Lemma 13.3, $\partial_e u \geq 0$ in $B_\rho(x_0)$, for all $e \in K_\delta$, then

$$\zeta_0 - K_\delta \cap B_{\rho'} \subset \{u \leq u(\zeta_0)\}$$

and

$$\zeta_0 + K_\delta \cap B_{\rho'} \subset \{u \geq u(\zeta_0)\}\,.$$

Let $\hat{u}_0$ be a blow up of u at ζ_0. Then $\hat{u}_0$ can not be a polynomial solution since $\hat{u}_0 \leq 0$ in $-K_\delta$ and $\hat{u}_0 \geq 0$ in K_δ and homogeneous polynomials of degree two are even. Then, $\hat{u}_0$ should be a half space solution. In that case, the same argument as the one in Lemma 13.5 shows that there is $\rho'' > 0$ and $\delta'' > 0$ such that

$$\zeta_0 - K_{\delta''} \cap B_{\rho''} \subset \Omega^c.$$

This contradiction with (13.2) proves the theorem. $\square$