LECTURE 14

14. First Results on the Regularity of the Free Boundary (continued)

14.1. **Problem C: the free boundary near the branching points.** In this lecture we study the structure of the free boundary near two-phase free boundary points in Problem C. Because of the Implicit Function Theorem, we may restrict ourselves to the case of two-phase points where gradient vanishes. Namely we assume that we are given a $C^{1,1}$-solution of

$$\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} \quad \text{in } D$$

with $\lambda_\pm > 0$ and a free-boundary point $x_0 \in \partial\{u>0\} \cap \partial\{u<0\} \cap D$ such that $|\nabla u(x_0)| = 0$. We will call such point *branching points*.

Theorem 14.1. *Let $u \in P_1(x_0, M)$ be a solution of Problem C such that x_0 is a branching point , i.e. $x_0 \in \partial\{u>0\} \cap \partial\{u<0\}$ and $|\nabla u(x_0)| = 0$. Then there exist $r = r(x_0, u) > 0$ such that $\partial\{u>0\} \cap B_r(x_0)$ and $\partial\{u<0\} \cap B_r(x_0)$ are graphs of C^1 functions that touch at x_0.*

The proof is based on the general idea of directional monotonicity that we exploited earlier.

Lemma 14.2. *Let $u \in P_1(x_0, M)$ and suppose that*

$$C\partial_e u - |u| \geq -\epsilon_0 \quad \text{in } B_1$$

where $0 \leq \epsilon_0 < (1/8n)\min\{\lambda_+, \lambda_-\}$. Then

$$C\partial_e u - |u| \geq 0 \quad \text{in } B_{1/2}.$$

The proof is similar to those of Lemmas 12.2 and 13.2, but is more subtle.

Proof. Let $v = C\partial_e u - |u|$ and suppose that the set $\{v<0\}$ is nonempty. Now, note that $\nabla u \neq 0$ on $\gamma = \{v<0\} \cap \{u=0\}$ and therefore this set is locally a $C^{1,\alpha}$ surface. Further, note that

$$\Delta|u| \geq \lambda_+ \chi_{\{u>0\}} + \lambda_- \chi_{\{u<0\}} \quad \text{in } B_1$$

in the sense of distributions and

$$\Delta(\partial_e u) = (\lambda_+ + \lambda_-)(e \cdot \nu) H^{n-1} \lfloor \gamma \quad \text{in } \{v<0\},$$

where $\nu = \nabla u / |\nabla u|$ is the unit normal on γ. Since on γ we have $u = 0$ and $v = C\partial_e u - |u| < 0$, we also have that $\partial_e u < 0$ and therefore $e \cdot \nu < 0$. In particular, this implies that

$$\Delta(\partial_e u) \leq 0 \quad \text{in } \{v<0\}$$

and consequently

$$\Delta v \leq -\lambda_{\min} \quad \text{in } \{v<0\},$$

where $\lambda_{\min} := \min\{\lambda_+, \lambda_-\}$.

Suppose now $\{v<0\}\cap B_{1/2}$ contains a point x_0. Consider then the auxiliary function

$$w(x)=v(x)+\frac{\lambda_{\min}}{2n}|x-x_0|^2.$$

Then $\Delta w\le 0$ in $\{v<0\}$ and $w(x_0)=v(x_0)<0$. Hence, by the minimum principle

$$\inf_{\partial(B_{1/2}(x_0)\cap\{v<0\})} w\le w(x_0)<0.$$

Since $w(x)=\lambda_{\min}/2n|x-x_0|^2\ge 0$ on $\partial\{v>0\}\cap B_{1/2}(x_0)$, we must have

$$\inf_{(\partial B_{1/2}(x_0))\cap\{v<0\}} w<0.$$

Finally, since $w(x)=v(x)+\lambda_{\min}/8n$ on $(\partial B_{1/2}(x_0))\cap\{v<0\}$, we obtain that

$$\inf_{\partial B_{1/2}(x_0)} v<-\frac{\lambda_{\min}}{8n},$$

contrary to our assumption. □

Lemma 14.3. *Let $u\in P_1(x_0,M)$ with x_0 a branching point and suppose that $u_0(x)=\frac{\lambda_+}{2}(x_1^+)^2-\frac{\lambda_-}{2}(x_1^-)^2$ be a blowup of u with fixed center at x_0. Then for any $\delta>0$ there exists $r_\delta>0$ such that*

$$\delta^{-1}2r_\delta\partial_e u-|u|\ge 0\quad\text{in } B_{r_\delta}(x_0)$$

for any unit vector $e\in K_\delta:=\{x:x_1\ge\delta|x|\}$. In particular, $\partial_e u\ge 0$ in $B_{r_\delta}(x_0)$.

Proof. We start by observing that for u_0 we have

$$\partial_e u_0=(e\cdot e_1)(\lambda_+x_1^++\lambda_-x_1^-),$$

hence

$$\delta^{-1}\partial_e u_0-|u_0|\ge 0\quad\text{in } B_1.$$

Consider now a sequence $r_j\to 0$ such that $u_{x_0,r_j}\to u_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$. Then

$$\delta^{-1}\partial_e u_{x_0,r_j}-|u_{x_0,r_j}|\ge-\epsilon_0>-\frac{\lambda_{\min}}{8n}\quad\text{in } B_1$$

for large j. But then by the previous lemma

$$\delta^{-1}\partial_e u_{x_0,r_j}-|u_{x_0,r_j}|\ge 0\quad\text{in } B_{1/2}.$$

Then, letting $r_\delta=r_j/2$ and scaling back we obtain

$$\delta^{-1}2r_\delta\partial_e u-|u|\ge 0\quad\text{in } B_{r_\delta}(x_0).$$

□

We are now ready to prove the theorem that we stated in the beginning.

Proof of Theorem 14.1. 1) Without loss of generality we will assume $x_0=0$. Fix a small $\delta>0$ and let $r_\delta>0$ be as in Lemma 14.3, so that $\partial_e u\ge 0$ for any $e\in K_\delta$. This immediately implies that $\partial\{u>0\}\cap B_{r_\delta/2}$ and $\partial\{u<0\}\cap B_{r_\delta/2}$ are graphs $x_1=f_\pm(x_2,\dots,x_n)$ of Lipschitz functions $f_\pm$.

2) Now, to show the differentiability of $f_\pm$, observe that by letting $\delta\to 0$ we easily obtain that $Df_\pm(0)=0$. Similarly, we obtain that $f_\pm$ are differentiable at any other branching point in $B_{r_\delta/2}$. The rest of the free boundary points $\hat{x}\in\Gamma\cap B_{r_\delta/2}$ are either: (i) two-phase with $|\nabla u(\hat{x})|>0$ or (ii) one-phase points. While the case

(i) is clear, in the case (ii) we claim that $\hat{x}$ is a low-energy point. Indeed, if $\epsilon > 0$ is such that $u \geq 0$ in $B_\epsilon(\hat{x})$ then

$$\hat{x} - K_\delta \cap B_\epsilon \subset \{u = 0\}$$

and

$$\hat{x} + \mathrm{Int}(K_\delta) \cap B_\epsilon \subset \{u > 0\}.$$

Thus, for a blowup $\hat{u}$ at $\hat{x}$ we must have

$$-K_\delta \subset \{\hat{u} = 0\},$$

which is possible only if $\hat{u}$ is a halfplane solution. Then we apply Theorem 12.4 to establish the differentiability of f_+ at $\hat{x}$.

Similarly, we treat the case when $u \leq 0$ near $\hat{x}$.

3) Finally, let us show that $f_\pm$ are C^1 functions. We first show that

$$\lim_{x' \to 0} |\nabla f_\pm(x')| = 0,$$

where $x' = (x_2, \dots, x_n)$. This follows from the fact that we can apply Lemma 14.3 with $\delta \to 0$. Indeed, the directional monotonicity $\partial_e u \geq 0$ in B_{r_δ} for $e \in K_\delta$ readily implies that

$$|\nabla f_\pm(x')| \leq \delta/\sqrt{1-\delta^2}, \quad \text{for } |x'| \leq r_\delta.$$

Thus, the C^1 regularity at the origin follows.

Next, C^1 regularity at other branching points in B_{r_δ} follows similarly. C^1 regularity of $f_\pm$ at the remaining free boundary points has been actually established when we proved the differentiability at those points. This completes the proof of the theorem.

□