LECTURE 16

16. Approximation by Global Solutions

Once we classified the global solutions, we want to show that the solutions in large domains can be approximated by global solutions and that during this process certain properties of global solutions are retained.

16.1. **Problems A and B.**

Lemma 16.1 (Approximation by Global Solutions)**.** *For any $\epsilon > 0$ there exists a radius $R_\epsilon = R_\epsilon(M, n)$ such that if $u \in P_R(M)$ with $R \geq R_\epsilon$, then one can find a global solution $u_0 \in P_\infty(M)$ such that*

$$\|u - u_0\|_{C^1(B_1)} \leq \epsilon.$$

Proof. Fix $\epsilon > 0$ and argue by contradiction. If no R_ϵ exists as above then one can find a sequence $R_j \to \infty$ and solutions $u_j \in P_{R_j}(M)$ such that

$$\|u_j - u_0\|_{C^1(B_1)} > \epsilon \quad \text{for any } u_0 \in P_\infty(M).$$

Note that in the case of Problem B we may additionally ask $u_j(0) = 0$. Then we will have uniform estimates

$$|u_j(x)| \leq \frac{M}{2}|x|^2, \quad |x| \leq R_j \tag{16.1}$$

we may assume that u_j converges is $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a global solution $\hat{u}_0 \in P_\infty(M)$. But then

$$\|u_j - \hat{u}_0\|_{C^1(B_1)} < \epsilon, \quad \text{for } j \geq j_\epsilon,$$

contradicting (16.1). □

Next, we want to find conditions on local solutions in Problems A, B that will guarantee approximation by convex global solutions rather than polynomial ones.

Definition 16.2 (Minimal Diameter)**.** The *minimal diameter* of a bounded set $E \subset \mathbb{R}^n$, denoted $\min\operatorname{diam}(D)$, is the infimum of distances between pairs of parallel planes such that D is contained in the strip determined by the planes.

Definition 16.3 (Thickness Function)**.** For $u \in P_R(x_0, M)$, the *thickness function* of Ω^c at x_0 is defined by

$$\delta_\rho(u, x_0) = \frac{\min\operatorname{diam}(\Omega^c \cap B_\rho(x))}{\rho}, \quad 0 < \rho < R.$$

As usual, we may drop x_0 from the notation in the case $x_0 = 0$.

Lemma 16.4 (Approximation by Convex Global Solutions)**.** *Fix $\sigma > 0$ and $\epsilon > 0$. Then there exists $R_{\epsilon,\sigma} = R_{\epsilon,\sigma}(M, n)$ such that if*

- *$u \in P_R(M)$ for $R \geq R_{\epsilon,\sigma}$,*
- *$\delta_1(u) \geq \sigma$,*

then we can find a convex global solution $u_0 \in P_\infty(M)$ with the properties

(i) *$\|u - u_0\|_{C^1(B_1)} \leq \epsilon$;*

(ii) *there exists a ball* $B = B_\rho(x) \subset B_1$ *of radius* $\rho = \sigma/2n$ *such that* $B \subset \Omega^c(u_0)$.
(iii) $B_{\rho/2}(x) \subset \Omega^c(u)$.

Proof. As before, we argue by contradiction. If no $R_{\epsilon,\sigma}$ exists as above, then we can find a sequence $R_j \to \infty$ and solutions $u_j \in P_{r_j}(M)$ such that

$$\delta_1(u_j) \geq \sigma$$

and that for any global solution $u_0 \in P_\infty(M)$ at least one of the conditions (i)–(iii) in the lemma fails.

Now, we again have uniform estimates

$$|u_j(x)| \leq \frac{M}{2}|x|^2, \quad |x| \leq R_j$$

and therefore without loss of generality we may assume that u_j converges in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a global solution $\hat{u}_0 \in P_\infty(M)$. Thus,

$$\|u_j - \hat{u}_0\|_{C^1(B_1)} < \epsilon \quad \text{for } j \geq j_\epsilon,$$

i.e. the condition (i) in the lemma is satisfied.

To show that in fact (ii) and (iii) are also satisfied with this choice of global solution, we note that

$$\delta_1(\hat{u}_0) \geq \sigma,$$

otherwise, we would have $\delta_1(u_j) < \sigma$ for large j. This implies that $\hat{u}_0$ cannot be a polynomial solution. Thus, by the classification of global solutions $\hat{u}_0$ is a convex function and $\Omega^c(\hat{u}_0)$ is a convex set.

To proceed, we will need the following lemma of F. John on convex sets.

Lemma 16.5 (F. John). *If* C *is a convex body and* E *is the ellipsoid of largest volume contained in* C, *then* $C \subset nE$ *(after we choose the origin at the center of* E*).*

Applying this lemma to the convex set $C = \Omega^c(\hat{u}_0) \cap B_1$, we obtain that C contains a ball B of radius $\rho = \sigma/2n$. Indeed, if E has one of its diameters smaller than 2ρ then nE has one of its diameters smaller than $2n\rho = 1$ and C is contained in a strip of width smaller than 1, a contradiction.

Thus, (ii) is satisfied. Finally, (iii) follows from nondegeneracy for large j.

We arrived at a contradiction since all conditions are satisfied by $\hat{u}_0$. Hence the lemma follows. □

16.2. **Problem C.** In the case of the two-phase obstacle problem, we specify a condition that guarantees the approximation by two-plane solutions.

Lemma 16.6 (Approximation by Two-Plane Solutions). *For any* $\epsilon > 0$ *there exist* $R_\epsilon = R_\epsilon(M, n)$ *and* $\sigma_\epsilon = \sigma_\epsilon(M, n)$ *such that if*

- $u \in P_R(M)$ *with* $R \geq R_\epsilon$
- $|\nabla u(0)| \leq \sigma_\epsilon$,
- $\mathrm{dist}(0, \partial\{u > 0\}) \leq \sigma_\epsilon$, $\mathrm{dist}(0, \partial\{u < 0\}) \leq \sigma_\epsilon$

then there exits a two-plane global solution $u_0 \in P_\infty(M)$ *such that*

$$\|u - u_0\|_{C^1(B_1)} \leq \epsilon.$$

Proof. As before, fix $\epsilon > 0$ and assume the statement is false. Then we can find sequences $R_j \to \infty$, $\sigma_j \to 0$, and solutions $u_j \in P_{R_j}(M)$ such that

$$|\nabla u_j| \le \sigma_j, \quad \mathrm{dist}(0, \partial\{u_j > 0\}) \le \sigma_j, \mathrm{dist}(0, \partial\{u_j < 0\}) \le \sigma_j$$

but so that

$$\|u_j - \hat{u}_0\|_{C^1(B_1)} > \epsilon \quad \text{for any } u_0 \in P_\infty(M).$$

Noticing that we have uniform estimates

$$|\nabla u_j(x)| \le (1+M)|x|, \quad |x| < R_j$$

for j large enough, we may assume that $u_j \to \hat{u}_0 \in P_\infty(M)$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$.

We claim that 0 is a branching point for $\hat{u}^0$. Indeed,

$$|\nabla \hat{u}_0(0)| = \lim_{j\to\infty} |\nabla u_j(0)| = 0.$$

Also, it is not hard to see that $0 \in \partial\{u_0 > 0\} \cap \partial\{u_0 < 0\}$ by using the nondegeneracy property. Hence 0 is a branching point. But then Theorem 15.7 implies that $\hat{u}_0$ is a two-plane solution and we arrive at a contradiction, as

$$\|u_j - \hat{u}_0\|_{C^1(B_1)} < \epsilon \quad \text{for } j \ge j_\epsilon.$$

□