LECTURE 17

17. Lipschitz Regularity of the Free Boundary

In the next few lectures we will prove the Lipschitz regularity of the free boundary for a class of solutions. The main difference from the results discussed earlier is that the estimates are going to depend on the class of solutions rather than individual solutions.

17.1. **Problem A.** We have shown in the previous lecture that local solutions $u \in P_R(M)$ for large R can be approximated with convex global solutions u_0 if $\Omega^c(u) \cap B_1$ is thick. Moreover, the approximating u_0 will contain a ball in $\Omega^c(u_0)$. Since the latter set is convex this will immediately imply the Lipschitz regularity of $\Gamma(u_0)$. Below is a more accurate version of this statement.

Lemma 17.1. *Let $u \in P_\infty(M)$ be a convex global solution such that $\Omega^c(u) \cap B_1$ contains a ball $B = B_\rho(-se_n)$ for some $0 < \rho < s \leq 1$. Set $K(\delta, s, h) = \{|x'| < \delta, -s \leq x_n \leq h\}$ for any $\delta, h > 0$. Then*

(i) *For any unit vector $e \in \mathcal{C}_{4/\rho} := \{x = (x', x_n) : x_n \geq (4/\rho)|x'|\}$ we have*
$$\partial_e u \geq 0 \quad \text{in } K(\rho/2, s, 1);$$
(ii) *The free boundary $\Gamma \cap (K(\rho/8, s, 1/2)$ is a Lipschitz graph*
$$x_n = f(x'),$$
where f is concave in x' and
$$|\nabla_{x'} f| \leq \frac{C}{\rho}.$$
for a dimensional constant C.

(iii) *There exists a constant $C_0 = C_0(\rho, M, n) > 0$ such that*
$$C_0 \partial_e u - u \geq 0 \quad \text{in } K(\rho/8, s, 1/2)$$
for any $e \in \mathcal{C}_{4/\rho}$.

Proof. Let $e \in \mathcal{C}_{4/\rho}$. Then observe the following geometric property: every ray originating at a point in $K(\rho/2, s, 1)$ in the direction $-e$ intersects the ball $B = B_\rho(-se_n)$. Since $\partial_e u = 0$ on B and $\partial_{ee} u \geq 0$ in $\mathbb{R}^n$ (from convexity), we readily obtain (i).

Further, it is easy to see that

$$(x_0 + \mathcal{C}^\circ_{4/\rho}) \cap K(\rho/2, s, 1) \subset \{u > 0\} \tag{17.1}$$

Then, using (i) we find the representation $x_n = f(x', t)$ in $K(\rho/8, s, 1/2)$, with the Lipschitz estimate $|\nabla_{x'} f| \leq C/\rho$.

Finally, to show (iii), assume the contrary. Then there exists a sequence of functions u_k satisfying the assumptions of the theorem and points $x_k \in \Omega(u_k) \cap K(\rho/8, s, 1/2)$ such that

$$k\partial_e u_k(x_k) - u_k(x_k) \leq 0, \quad e = e(k). \tag{17.2}$$

Let now

$$\tilde{x}_k = (x'_k, f_k(x_k)) \in \Gamma(u_k), \quad h_k = (x_k)_n - f_k(x'_k)$$

and consider the rescalings

$$v_k(x) = \frac{u(\tilde{x}_k + h_k x)}{h_k^2}.$$

Then (17.2) can be rewritten as

$$\partial_e v_k(e_n) \leq \frac{h_k}{k} v_k(e_n).$$

Since v_k are locally uniformly bounded in $\mathbb{R}^n$, we may assume that $v_k \to v \in P_\infty(M)$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. If we also assume $e(k) \to e$, we will have

$$\partial_e v(e_n) = 0$$

Now note that by (17.1)

$$\mathcal{C}^\circ_{4/\rho} \cap K(\rho/8, 0, 2) \subset \Omega(v_k)$$

and therefore $e_n \in \Omega(v)$. Since also $\partial_e v \geq 0$ there, from the minimum principle applied to the harmonic function $\partial_e v$ in $\Omega(v)$ we obtain that $\partial_e v = 0$. This would imply that u vanishes in a neighborhood of the origin, a contradiction. □

Now, we prove the Lipschitz regularity of the free boundary for solutions in large balls.

Theorem 17.2. *For every $\sigma > 0$ there exists $R_\sigma = R_\sigma(M, n)$ such that if $u \in P_R(M)$ and $\delta_1(u) \geq \sigma$ then $\Gamma \cap B_{c_n\sigma}$ is Lipschitz regular with Lipschitz constant $L = L(\sigma, n, M)$.*

Proof. Fix a small $\epsilon > 0$, to be specified later, and apply Lemma 16.4. So, if $R > R_{\epsilon,\sigma}$ we can find a global solution u_0 which satisfies conditions (i)–(iii) in Lemma 16.4. In particular, $\Omega^c(u_0) \cap B_1$ must contain a ball B of radius $\rho = \sigma/2n$. Without loss of generality we may assume $B = B_\rho(-se_n)$ for some $0 \leq s \leq 1$. Then applying Lemma 17.1 we will have that

$$C_0 \partial_e u_0 - u_0 \geq 0 \quad \text{in } K(\rho/8, s, 1/2)$$

for any $e \in \mathcal{C}_{4/\rho}$.

Now, if $\epsilon = \epsilon(\sigma, M, n)$, the approximation $\|u - u_0\|_{C^1(B_1)} \leq \epsilon$ implies that

$$C_0 \partial_e u - u \geq -(C_0 + 1)\epsilon > -(\rho/8)^2/8n \quad \text{in } K(\rho/8, s, 1/2)$$

and recalling Lemma 12.2 we will obtain that

$$C_0 \partial_e u - u \geq 0 \quad \text{in } K(\rho/16, s, 1/4).$$

(To be more accurate, one needs to apply Lemma 12.2 in every ball $B_{\rho/8}(te_n)$ for $t \in [-s, 1/4]$). The latter inequality can be rewritten as

$$\partial_e(e^{-C_0(x\cdot e)} u) \geq 0 \quad \text{in } K(\rho/16, s, 1/4).$$

Taking $e = e_n$ and noting that $u = 0$ on $B'_{\rho/16} \times \{-s\}$ we obtain after integration that

$$u \geq 0 \quad \text{in } K(\rho/16, s, 1/4).$$

Combining with the previous inequality this gives

$$\partial_e u \geq 0 \quad \text{in } K(\rho/16, s, 1/4)$$

for any $e \in \mathcal{C}_{4/\rho}$.

The rest of the proof is now left to the reader as an exercise. $\square$

Next, we give a reformulation of Theorem 17.2.

Theorem 17.3. *There exists a modulus of continuity $\sigma(r) = \sigma_{M,n}(r)$ such that if $u \in P_1(M)$ and $\delta_r(u) \geq \sigma(r)$ for some value $r = r_0 \in (0,1)$ then $\Gamma \cap B_{c_n r_0 \sigma(r_0)}$ is a Lipschitz graph with a Lipschitz constant $L \leq L(n, M, r_0)$.*

Proof. This is basically a rescaled version of Theorem 17.2.

Note that in the latter theorem one can take function $\sigma \mapsto R_\sigma$ to be monotone and continuous in σ and such that $\lim_{\sigma\to 0+} R_\sigma = \infty$. Then let $r \mapsto \sigma(r)$ be the inverse of the mapping $\sigma \mapsto 1/R_\sigma$ so that we have

$$R_{\sigma(r)} = 1/r.$$

Now, if $\delta_{r_0}(u) \geq \sigma(r_0)$ then the rescaling

$$u_{r_0}(x) = \frac{u(r_0 x)}{r_0^2} \in P_{1/r_0}(M)$$

satisfies

$$\delta_1(u_{r_0}) \geq \sigma(r_0).$$

And because of the identity $R_{\sigma(r_0)} = 1/r_0$ we can apply Theorem 17.2. Then scaling back to u we obtained the corresponding statement for the free boundary of u. $\square$

17.2. **Problem B.** We are going to show here that Theorems 17.2 and 17.3 hold also for solutions of Problem B.

Lemma 17.4. *Let u be as in Lemma 17.1. Then we also have*

$$C_0 \partial_e u - |\nabla u|^2 \geq 0 \quad \textit{in } K(\rho/8, s, 1/2)$$

for any $e \in \mathcal{C}_{4/\rho}$, where $C_0 = C_0(\rho, M, n)$.

Proof. The proof is completely analogous to that of (iii) in Lemma 17.1. $\square$

Theorem 17.5. *Theorem 17.2 holds also for solutions of Problem B.*

Proof. Arguing similarly to the case of Problem A, but using Lemma 13.2 instead of Lemma 12.2 and Lemma 17.4 instead of Lemma 17.1 (iii), we can show that

$$C_0 \partial_e u - |\nabla u|^2 \geq 0 \quad \text{in } K(\rho/16, s, 1/4)$$

for any $e \in \mathcal{C}_{4/\rho}$, which immediately implies that

$$\partial_e u \geq 0 \quad \text{in } K(\rho/16, s, 1/4).$$

We next claim that u is constant on $\Omega^c(u) \cap K(\rho/16, s, 14)$. The argument is similar to the one in the proof of Theorem 13.1. It will suffice to show that for that every point $x = (x', x_n) \in \Omega^c$ the segment joining $(x', -s)$ and x is completely contained in Ω^c. If the latter statement is false, we can find in $K(\rho/16, s, 1/4)$ two points $x = (x', x_n) \in \Omega^c$ and $\tilde{x} = (x', \tilde{x}_n) \in \Omega$ such that $\tilde{x}_n < x_n$. Without loss of generality we may assume $x \in \Gamma$. Now, let us take a small ball $B_\epsilon(\tilde{x})$ and start moving this ball from $\tilde{x}$ to x along the x_n axis, reducing its radius proportionally to the distance from x. Stop moving if the ball touches the free boundary Γ at some point. Call this point ζ_0. If the moving ball does not touch Γ, then let $\zeta_0 = x$. In either case there will exist a cone $\mathcal{C}$ with axis e_n such that

$$\zeta_0 - \mathcal{C} \cap B_\epsilon \subset \Omega(u)$$

Considering now a blowup $\hat{u}_0$ of u at ζ_0, we realize that it must necessarily be a halfplane solution $\hat{u}_0(x) = \frac{1}{2}(x \cdot e_0)_+^2$ with e_0 satisfying $e \cdot e_0 \geq 0$ for all $e \in \mathcal{C}_{4/\rho}$. On the other hand, one must also have

$$-\mathcal{C} \subset \Omega(\hat{u}_0),$$

which implies that $e_0 \cdot e_n < 0$, a contradiction. Hence, we obtain that u is constant in $\Omega^c(u) \cap K(\rho/16, s, 1/4)$ and therefore deducting a constant we reduce problem to the case of Problem A. □

Theorem 17.6. *Theorem 17.3 holds also for solutions of Problem B.* □

17.3. **Problem C.**

Theorem 17.7. *There exist constants $\sigma_0 > 0$ and $r_0 > 0$ depending only on $\lambda_\pm$, M, n and a given $L > 0$ such that if $u \in P_1(M)$ is a solution of Problem C and*

$$|\nabla u(0)| \leq \sigma_0, \quad \{\pm u > 0\} \cap B_{\sigma_0} \neq \emptyset, \tag{17.3}$$

then $\partial\{\pm u > 0\} \cap B_{r_0}$ are Lipschitz graphs with Lipschitz constant L.

Proof. Fix small $\epsilon > 0$, to be specified later, and let σ_ϵ and $R_\epsilon > 1$ be as in Lemma 16.6. Put $\sigma_0 = \sigma_\epsilon / R_\epsilon$. Now, if u satisfies (17.3), the rescaling

$$u_{1/R_\epsilon}(x) = R_\epsilon^2 u(x/R_\epsilon)$$

satisfies conditions of the approximation Lemma 16.6. Hence, there exists a two-plane solution u_0 such that

$$\|u_{1/R_\epsilon} - u_0\|_{C^1(B_1)} \leq \epsilon.$$

Now, without loss of generality we may assume that $u_0(x) = \frac{\lambda_+}{2}(x \cdot e)_+^2 - \frac{\lambda_-}{2}(x \cdot e)_-^2$. Then we have

$$C_L \partial_e u_0 - |u_0| \geq 0 \quad \text{in } B_1$$

for any $e \in \mathcal{C}_L$. From the approximation we have

$$C_L \partial_e u_{1/R_\epsilon} - |u_{1/R_\epsilon}| \geq -(C_L + 1)\epsilon > -\lambda_{\min}/8n \quad \text{in } B_1,$$

provided $\epsilon = \epsilon(\lambda_\pm, M, m, L) > 0$ is small enough. Then Lemma 14.2 implies

$$C_L \partial_e u_{1/R_\epsilon} - |u_{1/R_\epsilon}| \geq 0 \quad \text{in } B_{1/2},$$

and consequently

$$\partial_e u_{1/R_\epsilon} \geq 0 \quad \text{in } B_{1/2}.$$

for any $e \in \mathcal{C}_L$. From here, arguing as in Lecture 14, we conclude that $\Gamma_\pm(u_{1/R-\epsilon}) \cap B_{1/2}$ are Lipschitz graphs with constant L and scaling back we obtain the Lipschitz regularity of $\Gamma_\pm(u) \cap B_{1/2R_\epsilon}$. Thus we can take $r_0 = 1/2R_\epsilon$. □