LECTURE 18

18. $C^{1,\alpha}$ Regularity of the Free Boundary: Problems A and B

In this lecture we give a proof $C^{1,\alpha}$ regularity of the free boundary. We follow the original idea of Athanasopoulos and Caffarelli based on the Boundary Harnack Principle (also known as the local comparison theorem) for the class of NTA (nontangentially accessible) domains.

18.1. **Boundary Harnack Principle.** Given a domain $\Omega \subset \mathbb{R}^n$ and a constant $M > 0$, we say that a ball $B_r(x) \subset \Omega$ is *M-nontangential*, if

$$M^{-1}r \leq \operatorname{dist}(B_r(x), \partial\Omega) < Mr.$$

An *M-Harnack chain* joining two given points $x_1, x_2 \in \Omega$, is a finite sequence of M-nontangential balls such that the first one contains x_1, the last one contains x_2, and such that consecutive balls have non-empty intersection. The *length* of the chain is the number of balls in the chain.

Definition 18.1 (Corkscrew condition)**.** We say that the domain $\Omega \subset \mathbb{R}^n$ satisfies the (M, r_0)-*corkscrew condition* at $x_0 \in \partial\Omega$ for constants M, $r_0 > 0$ if:

(1) (Interior corkscrew) For any $0 < r \leq r_0$ there exists $A_r(x_0) \in \Omega$ such that $B_{r/M}(A_r(x_0)) \subset \Omega$, and
$$M^{-1}r < d(A_r(x_0), x_0) < Mr;$$
(2) (Exterior corkscrew) $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$ satisfies the previous condition.

Definition 18.2 (NTA property)**.** We say that a domain $\Omega \subset \mathbb{R}^n$ is *non-tangentially accessible* (NTA) if there exist M, $r_0 > 0$ such that

(1) Ω satisfies (M, r_0)-corkscrew condition at every point $x_0 \in \partial\Omega$.
(2) (Harnack chain condition) For every $0 < r \leq r_0$ and $x_1, x_2 \in \Omega$ with $\operatorname{dist}(x_i, \partial\Omega) \geq \epsilon$, $i = 1, 2$, and $|x_1 - x_2| < C\epsilon$, there exists an M-Harnack chain joining x_1 and x_2 with a length depending only on C, M and r_0 (but independent ϵ).

A geometric localization theorem by Peter Jones shows that being NTA is essentially a local property. This theorem says that for an NTA domain Ω in $\mathbb{R}^n$ and $x_0 \in \partial\Omega$, for any $0 < r \leq r_0$ one can find an NTA domain $\Omega_r(x_0)$ (with constants independent of r) such that

$$B_r(x_0) \cap \Omega \subset \Omega_r(x_0) \subset B_{Mr}(x_0) \cap \Omega.$$

Example 18.3. Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a Lipschitz function. Then $\Omega = \{(x', x_n) : x_n > f(x')\}$ is NTA with a constants depending only on the Lipschitz constant of f and the dimension n.

Theorem 18.4 (Boundary Harnack Principle)**.** *Let Ω be NTA with constants M and r_0. Let u and v be two nonnegative harmonic functions in $B_r(x_0) \cap \Omega$, $r < r_0$, continuously vanishing on $B_r(x_0) \cap \partial\Omega$. Then*

$$\sup_{B_{r/K}(x_0)\cap\Omega} \frac{u}{v} \leq C \inf_{B_{r/K}(x_0)\cap\Omega} \frac{u}{v},$$

where $C, K > 1$ depend only on M and n.

Moreover, there exists constants $C > 0$ and $0 < \alpha < 1$, depending only on M and n, such that

$$\operatorname{osc}_{B_\rho(x_0)\cap\Omega} \frac{u}{v} \leq C \left(\frac{\rho}{r}\right)^\alpha \sup_{B_r(x_0)\cap\Omega} \frac{u}{v}. \tag{18.1}$$

18.2. $C^{1,\alpha}$ **Regularity of the Free Boundary: Problems A and B.**

Theorem 18.5. *For every $\sigma > 0$ there exists $R_\sigma = R_\sigma(M, n)$ such that if $u \in P_R(M)$ is a solution of Problem A or B with $\delta_1(u) \geq \sigma$ then $\Gamma \cap B_{c_n\sigma}$ is a $C^{1,\alpha}$ graph with $\alpha = \alpha_{\sigma,M,n} \in (0, 1)$ and the $C^{1,\alpha}$-norm $C \leq C(\sigma, n, M)$.*

Proof. Let R_σ be as in Theorems 17.3 and that the ball $B = B_\rho(-se_n)$ is as in Lemma 17.1 (with $\rho = \sigma/2n$) so that u vanishes on $B_{\rho/2}(-se_n)$. Then, as it follows from the proof of Theorem 17.3, we have

$$\partial_e u \geq 0 \quad \text{in } K(\rho/16, s, 1/4)$$

for any unit vector $e \in \mathcal{C}_{4/\rho}$. Consider now two functions of the type above

$$\begin{aligned} u_1 &= \partial_n u \\ u_2 &= \partial_e u \end{aligned}$$

with e sufficiently close to e_n. Since we already know that $\Omega \cap K(\rho/16, s, 1/4)$ is given as an epigraph $x_n > f(x')$ of a Lipschitz function. Then the Boundary Harnack Principle implies that the ratio

$$\frac{u_2}{u_1}$$

is C^α regular in $\Omega \cap K(\rho/32, s, 1/8)$ up to $\partial\Omega$, with $0 < \alpha < 1$ and C^α norm depending on ρ, n, M, the Lipschitz norm of $\partial\Omega$, as well as on the bound from below on

$$m_i = u_i(A), \quad A = ((3/16)\, e_n).$$

We claim that

$$m_i \geq c_0(\rho, n, M) > 0.$$

It is enough to prove the bound only for m_1, since m_2 can be made as close to m_1 as we wish. Thus, we have to show that

$$\partial_n u \geq c_0 > 0$$

at A. Indeed, if it weren't so, by compactness we would easily construct a function u as above with $\partial_n u = 0$ at A. Then by the minimum principle $\partial_n u$ and consequently u would vanish in $K(\rho/32, s, 1/8)$, a contradiction.

Hence,

$$\frac{\partial_e u}{\partial_n u}$$

is C^α up to $\partial\Omega$ in $\Omega \cap B_{\rho/32}$. Then varying e and ϵ we obtain that the ratios

$$\frac{\partial_i u}{\partial_n u}, \quad i = 1, \dots, n-1,$$

are C^α. This implies that $\partial\Omega \cap K(\rho/32, s, 1/8)$ is the graph $x_n = f(x')$ with

$$\|f\|_{C^{1,\alpha}} \leq C(\rho, n, M)$$

since

$$\partial_i f = \frac{\partial_i u}{\partial_n u}, \quad i = 1, \dots, n-1.$$

□

The rescaled version of the above theorem is as follows.

Theorem 18.6. *There exists a modulus of continuity $\sigma(r) = \sigma_{M,n}(r)$ such that if $\delta_r(u) \geq \sigma(r)$ for some $r = r_0 \in (0,1)$ then $\Gamma \cap B_{c_n r_0 \sigma(r_0)}$ is a $C^{1,\alpha}$-graph with $\alpha = \alpha_{r_0,M,n} \in (0,1)$ and the $C^{1,\alpha}$-norm $C \leq C(r_0, M, n)$.*