LECTURE 2

2. Mathematical formulation

Here we give an exact setting for three free boundary problems that will be investigated in next sections. We consider solutions of equations of the type

$$\Delta u = f(x, u, \nabla u) \quad \text{in } D, \tag{2.1}$$

where D is an open set in $\mathbb{R}^n$ and the right hand side term is supposed to be piecewise continuous, having jumps at some values of the arguments u and ∇u. We also suppose that there is a certain apriori unknown subset $\Omega = \Omega(u)$ of D where the corresponding equation (2.1) is "good" and we are interested in the regularity of the free boundary $\Gamma(u) = \partial\Omega \cap D$. More specifically, we consider the following three problems.

Problem A: *No-sign obstacle problem*

$$f = \chi_\Omega, \qquad \Omega(u) = D \setminus \{u = |\nabla u| = 0\}.$$

When the solution u of is nonnegative, then $\Omega = \{u > 0\}$ and u becomes a solution of the classical obstacle problem.

Problem B: *Superconductivity problem*

$$f = \chi_\Omega, \qquad \Omega(u) = \{|\nabla u| > 0\}.$$

Alternatively, one can take here $\Omega = \operatorname{Int} \overline{\{|\nabla u| > 0\}}$, which will lead to the same equation, however, the notion of the free boundary will be different. The latter definition of Ω will eliminate certain non-physical singular free boundary points.

Problem C: *Two-phase obstacle problem*

$$f = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}}, \qquad \Omega(u) = \{u \neq 0\}$$

where $\lambda_\pm$ are given positive constants and the free boundary $\Gamma(u) = \partial\Omega(u) \cap D$ consists of two parts: $\Gamma'(u) = \Gamma(u) \cap \partial\{|\nabla u| = 0\}$ and $\Gamma''(u) = \Gamma(u) \cap \partial\{|\nabla u| \neq 0\}$. By the implicit function theorem Γ'' is locally $C^{1,\alpha}$ graph for all $0 < \alpha < 1$. Therefore we are mostly interested in the properties of $\Gamma'(u)$.

We will assume that $u \in L^\infty_{\mathrm{loc}}(D)$ and $f \in L^\infty(D \times \mathbb{R} \times \mathbb{R}^n)$ and that the equation (2.1) is satisfied in the sense of distributions, i.e.

$$\int_D u\, \Delta\eta\, dx = \int_D f(x, u, \nabla u)\, \eta\, dx,$$

for all C^∞ test functions η with compact support in D. The standard L^p-theory of elliptic equations will immediately imply the higher regularity of u.

Theorem 2.1. *Let $u \in L^p(D)$, $g \in L^p(D)$, $1 < p < \infty$, satisfy $\Delta u = g$ in D in the sense of distributions. Then $u \in W^{2,p}_{\mathrm{loc}}(D)$ and*

$$\|u\|_{W^{2,p}(K)} \leq C \left(\|u\|_{L^p(D)} + \|g\|_{L^p(D)} \right)$$

for any open $K \subset\subset D$ with $C = C(p, n, K, D)$. □

Thus, for solutions of (2.1) we obtain

$$u \in W^{2,p}_{\mathrm{loc}}(D), \quad \text{for all} \quad 1 < p < \infty, \tag{2.2}$$

Consequently, we also have

$$u \in C^{1,\alpha}_{\mathrm{loc}}(D), \quad \text{for all} \quad 0 < \alpha < 1, \tag{2.3}$$

by the Sobolev embedding $W^{2,p} \hookrightarrow C^{1,\alpha}$ with $\alpha = 1 - n/p$ for $p > n$. An easy counterexample shows that in general we cannot have $p = \infty$ in (2.2) and $\alpha = 1$ in (2.3). Instead we have the following

Theorem 2.2. *Let $u \in L^\infty(D)$, $g \in L^\infty(D)$ satisfy $\Delta u = g$ in the sense of distributions. Then $u \in W^{2,p}_{\mathrm{loc}}(D) \cap C^{1,\alpha}_{\mathrm{loc}}(D)$ for all $1 < p < \infty$, $0 < \alpha < 1$ and*

$$|\nabla u(x) - \nabla u(y)| \le C \|g\|_{L^\infty(D)} |x - y| \log \frac{1}{|x - y|},$$

for any $x, y \in K \subset\subset D$ with $|x - y| \le 1/e$ and $C = C(n, K, D)$. □

As we will see later, the logarithmic term in this theorem can be dropped if u is a solution of Problems A-C. That would give us a starting point for the analysis of the free boundary in those problems.

2.1. **Viscosity solutions.** (*This subsection requires some familiarity with the notion of viscosity solutions of Crandall-Ishii-Lions.*) Here we give a definition of viscosity solutions of Problems A and B and even though these problems are governed by linear operators, this approach allows to extend some of the results to the fully nonlinear case.

Problem A. We say that an upper semicontinuous (u.s.c.) function $u < \infty$ is a *viscosity subsolution* of Problem A, if

$$\Delta P \ge 1$$

for any paraboloid P touching u from above at a point x, provided either $P(x) \ne 0$ or $|\nabla P(x)| \ne 0$. A lower semicontinuous (l.s.c.) function $u > -\infty$ is a *viscosity subsolution*, if

$$\Delta P \le 1$$

for any paraboloid P touching u from below at a point x with the same extra conditions ($P(x) \ne 0$ or $|\nabla P(x)| \ne 0$). A *viscosity solution* is simultaneously a sub- and supersolution.

Problem B. Viscosity solutions in this case are defined similarly to Problem A, with the difference that we as only the condition $|\nabla P(x)| \ne 0$ at the touching point x.