LECTURE 20

DRAFT

20. Higher regularity: Problems A and B

In this lecture we show how using a special technique developed by Kinderlehrer and Nirenberg, called partial hodograph-Legendre transform, one can bootstrap the $C^{1,\alpha}$ regularity of the free boundary in Problems A and B to C^∞ and even C^ω regularity.

20.1. **Partial hodograph-Legendre transformation.** Let Ω be a certain bounded open set in $\mathbb{R}^n$, Γ a relatively open subset of $\partial\Omega$ and the function u satisfy

$$\Delta u = f \quad \text{in } \Omega, \qquad u = |\nabla u| = 0 \quad \text{on } \Gamma.$$

We will assume that

- Γ is C^1 (i.e. locally a graph of a C^1 function)
- $u \in C^2(\Omega \cup \Gamma)$, (i.e. $u \in C^2(\Omega)$ and every second order derivative $\partial_{x_i x_j} u$ can be extended continuously up to Γ.
- $f \in C^k(\Omega \cup \Gamma)$ for some $k = 0, 1, \dots$, and $0 < a \le f \le b < \infty$ in Ω.

Suppose now $0 \in \Gamma$ and that $e_1 = (1, 0, \dots, 0)$ is the inward normal of Ω at 0. Then we have

$$\partial_{x_i x_j} u(0) = 0 \quad \text{for } i = 1, \dots, n,\ j = 2, \dots, n.$$

This implies that

$$\partial_{x_1 x_1} u(0) = f(0) > 0.$$

Now consider the so-called partial hodograph transformation

$$y_1 = -u_{x_1}, \quad y_j = x_j, \quad j = 2, \dots, n. \tag{20.1}$$

Since $\det(D_x y) = -\partial_{x_1 x_1} u < 0$ at the origin, there is a small $r > 0$ such that this transformation is invertible in $\overline{\Omega} \cap B_{2r}$ and maps it onto an open neighborhood of the origin in $\{y_1 \le 0\}$, while $\Gamma \cap B_{2r}$ maps onto a relatively open subset of $\{y_1 = 0\}$. One then can introduce a function

$$v(y) = x_1 y_1 + u(x), \tag{20.2}$$

known as the partial Legendre transform of u. Direct computations show that

$$v_{y_1} = x_1, \quad v_{y_j} = u_{x_j}, \quad j = 2, \dots, n$$

$$u_{x_1 x_1} = -\frac{1}{v_{y_1 y_1}}, \quad u_{x_1 x_j} = \frac{v_{y_1 y_j}}{v_{y_1 y_1}}, \quad j = 2, \dots, n$$

$$u_{x_j x_k} = u_{y_k y_k} - \frac{v_{y_1 y_j} v_{y_1 y_k}}{v_{y_1 y_1}}, \quad j, k = 2, \dots, n$$

Thus, v satisfies

$$Lv := -\frac{1}{\partial_{y_1 y_1} v} - \frac{1}{\partial_{y_1 y_1} v} \sum_{i=2}^{n} (\partial_{y_i y_1} v)^2 + \sum_{i=2}^{n} \partial_{y_i y_i} v = f(x(y)).$$

As can be shown, Lv is a uniformly elliptic equation on the image of $\Omega \cap B_{2r}$ under the hodograph transform. Moreover v vanishes on the image of $\Gamma \cap B_{2r}$, which is a subset of $\{y_1 = 0\}$. Now, if we assume that slightly more regularity on u, Γ and f than we asked initially, namely $\Gamma \in C^{1,\alpha}$, $u \in C^{2,\alpha}(\Omega \cup \Gamma)$ and $f \in C^{k,\alpha}(\Omega \cup \Gamma)$, then using up to the boundary regualarity for solutions of the equation $Lv = f$, we will have that v is $C^{k+2,\alpha}$ regular on the image of $\overline{\Omega} \cap B_r$ and considering the inverse transformation

$$y \mapsto x = (\partial_{x_1} v, y_2, \dots, y_n)$$

we find that u as well as $\Gamma \cap B_r$ are $C^{k+1,\alpha}$ regular. Moreover, if $f \in C^\omega$, then $\Gamma \cap B_r \in C^\omega$.