DRAFT

21. The singular set

21.1. The characterization of the singular set. For a solution u of the obstacletype problem we say that $x_0 \in \Gamma(u)$ is a *regular* free boundary point if for some small r > 0 we can represent

$$\Omega(u) \cap B_r(x_0) = \{ (x', x_n) \in \mathbb{R}^n : x_n > f(x') \} \cap B_r(x_0)$$

after a suitable rotation of the coordinate axes, where f is a C^1 function on \mathbb{R}^{n-1} . The point $x_0 \in \Gamma(u)$ is called *singular* otherwise. In this chapter we study the set $\Sigma(u)$ of singular free boundary points.

Theorem 21.1. Let u be a solution of the obstacle-type problem in D. Then the following statements are equivalent

- (i) $x_0 \in \Sigma(u)$
- (ii) $\lim_{r \to 0} \delta_r(x_0, u) = 0$
- (iii) $\omega(x_0) = \alpha_n$
- (iv) all blowups of u at x_0 , i.e. the limits of rescalings

$$u_{r,x_0}(x) = \frac{u(x_0 + rx)}{r^2}$$

over subsequences $r = r_k \rightarrow 0$ are homogeneous quadratic polynomials Qwith $\Delta Q = f(x_0)$.

Proof. The implication (ii) \Rightarrow (i) is clear; the converse implication follows from Theorem 18.6. The equivalence of (iii) and (iv) is also clear. In fact, recall that all blowups at a point are either halfspace or polynomial solutions, see Proposition 5.21. To show (iv) \Rightarrow (i), simply notice that if x_0 is a regular free boundary point then any blowup at x_0 will vanish at least on a halfspace, thus cannot be polynomial. Finally, let us show (ii) \Rightarrow (iv). Let u_0 be a blowup of u at x_0 over a subsequence $r = r_k \rightarrow 0$. Since $\Delta u_{x_0,r} = 1$ in B_1 except a strip of width $\delta_r(x_0, u) \rightarrow 0$, we will obtain that

$$\Delta u_0 = 1$$
 a.e. in B_1

which contradicts the assumption that u_0 is a halfspace solution.

21.2. Examples of singularities.

21.2.1. A trivial example. Let $u(x) = Q(x) = \frac{1}{2}(x \cdot Ax)$ be a homogeneous quadratic polynomial in \mathbb{R}^n for a symmetric $n \times n$ matrix A with $\operatorname{Tr} A = 1$. Then the free boundary

$$\Pi = \ker A$$

consists completely of singular points. Note that the dimension k of Π can be anything from 0 to n-1. The latter case deserves more attention. For definiteness,



FIGURE 1. The shaded regions correspond to the coincidence set Λ in Schaeffer's examples with ϕ_1 (left) and ϕ_2 (right) respectively. The outer curves correspond to the boundary of D.

let $u(x) = Q_1(x) = \frac{1}{2}x_1^2$. Then the free boundary is a hyperplane $\Pi_1 = \{x_1 = 0\}$ of codimension one which is as smooth as it can be. Nevertheless, all points on Π_1 are singular, since $\Lambda(Q_1) = \Pi_1$ is "thin". In contrast, if $u(x) = h_1(x) = \frac{1}{2}(x_1^+)^2$, the free boundary is still Π_1 , however, this time it consists of regular points, since $\Lambda(h_1) = \{x_1 \leq 0\}$ is "thick". This means that the singular set can be as large as the set of regular points.

21.2.2. Schaeffer's examples. The following two very illustrative examples of singular points in two dimensions are due to Schaeffer. We will use both real and complex notations for their description.

Let D be a simply connected domain in $\mathbb{C} = \mathbb{R}^2$ and Λ a closed subset of D with a piecewise C^1 boundary. Let $\Omega = D \setminus \Lambda$ and suppose that we are given a conformal mapping $\phi : R_{1,2} \to \Omega$ continuous up to the boundary. Here by $R_{a,b}$ we denote the ring $\{a < |z| < b\}$ for 0 < a < b. We will assume that ϕ maps ∂B_2 to ∂D and ∂B_1 to $\partial \Lambda$. Additionally, we will assume that D and Λ are symmetric with respect to the real axis and that

$$\phi(\overline{z}) = \overline{\phi(z)}, \quad z \in \overline{R_{1,2}}$$

If now ϕ is holomorphically extensible to a mapping $R_{1/2,2} \to \mathbb{C}$, we can define

$$f(z) = -\phi(1/\phi^{-1}(z)), \quad z \in \Omega.$$

Note that f(z) defined as above will be continuous up to $\partial \Lambda$ if we put

$$f(z) = -\overline{z}, \quad z \in \partial \Lambda.$$

Next consider

$$v(z) = \operatorname{Re} \int f(z) dz, \quad z \in \Omega$$

the real part of the indefinite integral of f. Even though $\int f(z)dz$ can be multivalued due to Ω not being simply connected, we claim that v is well defined. Indeed, note that

$$\operatorname{Re} \int_{\partial \Lambda} f(z) dz = \int_{\partial \Lambda} \operatorname{Re}(\overline{z} dz) = \int_{\partial \Lambda} d(|z|^2/2) = 0$$

and therefore $\operatorname{Re} \int_{\gamma} f(z) dz = 0$ for any closed curve γ in Ω . Consequently v is well defined in Ω . Of course v is harmonic in Ω and

$$\partial_x v - i \partial_y v = f(z), \quad z \in \Omega.$$

Therefore $\nabla v(z) = -(x, y)$ continuously for $z = x + iy \in \partial \Lambda$. By choosing the constant of integration, we can arrange also that $v(z) = -|z|^2/2$ on $\partial \Lambda$. Define now

$$u(z) = \begin{cases} \frac{1}{4} |z|^2 + \frac{1}{2}v(z), & z \in \Omega \\ 0, & z \in \Lambda \end{cases}$$

Then $u \in C^1(D) \cap C^2(D \setminus \gamma_1)$ and it is easy to see that

$$\Delta u = \chi_{\Omega}$$
 in D , $u = |\nabla u| = 0$ on Λ ,

where the first equality is in the sense of distributions.

More specifically, consider two mappings given by

$$\phi(z) = \phi_i(z) = (z + 1/z)/2 + \epsilon P_i(z)(z - 1/z)/2, \quad i = 1, 2$$
$$P_1(z) = z^2 + 2 + 1/z^2$$
$$P_2(z) = (z - 2 + 1/z)^2$$

for $\epsilon > 0$. They have the properties discussed above for ϵ sufficiently small. The corresponding boundaries of Λ have the parametrizations

$$\phi_1(e^{i\theta}) = \cos\theta + i4\epsilon\cos^2\theta\sin\theta$$
$$\phi_2(e^{i\theta}) = \cos\theta + i32\epsilon\cos(\theta/2)\sin^5(\theta/2)$$

and have the shapes similar to those in Fig. 1 with singularities at $z_0 = 0$ (self-touching) and $z_0 = 1$ (cusp).

It can be shown that the solutions constructed above are also nonnegative, if $\epsilon > 0$ is sufficiently small.

We refer to the original paper of Schaeffer for the proof.

21.3. Structure of the singular set: nonnegative case.

Theorem 21.2. Let $u \ge 0$ be a solution of the obstacle problem in a domain D in \mathbb{R}^n .

(i) For any $x_0 \in \Sigma(u)$ there exists a unique quadratic polynomial

$$Q_{x_0}(x) = \frac{1}{2}(x - x_0) \cdot A_{x_0}(x - x_0)$$

such that $\Delta Q_{x_0} = \operatorname{Tr} A_{x_0} = 1$ with the property that

$$||u - Q_{x_0}||_{L^{\infty}(B_r(x_0))} \le \sigma(r)r^2$$

for a modulus of continuity σ depending only on n and dist $(x_0, \partial D)$.

- (ii) The matrix A_{x_0} depends continuously on $x_0 \in \Sigma(u)$.
- (iii) Let $\Sigma_k(u) = \{x_0 \in \Sigma(u) : \dim \ker A_{x_0} = k\}$ for $k = 0, \ldots, n-1$. Then $\Sigma_k(u)$ is contained in the union of countably many k-dimensional C^1 manifolds.

We start with a uniform version of Theorem 21.1 (ii) and (iv) for normalized solutions of the obstacle problem.

Lemma 21.3. Let $u \in P_1(M)$ and assume that $0 \in \Sigma(u)$. Then there exists a modulus of continuity σ depending only on M and n such that for any 0 < r < 1/2

(i)
$$\delta_r(x_0, u) \leq \sigma(r)$$

(ii) there exists a homogeneous quadratic polynomial Q^r with $\Delta Q^r = 1$ such that

$$||u - Q^r||_{L^{\infty}(B_{2r})} \le \sigma(r)r^2.$$

Proof. The statement of (i) follows from Theorem 18.6. The statement of (ii) is equivalent to the following one: for any $\epsilon > 0$ there exists $r_{\epsilon} > 0$ such that for any $0 < r \leq r_{\epsilon}$ there exists a homogeneous quadratic polynomial Q^r such that $|u - Q^r| \leq \epsilon r^2$ in B_{2r} . Assuming that the latter fails for some $\epsilon > 0$, we obtain a sequence $r_j \to 0$ and $u_j \in P_1(M)$ such that

$$\|u_j - Q\|_{L^{\infty}(B_{2r_i})} \ge \epsilon r_j^2$$

for any homogeneous quadratic polynomial Q with $\Delta Q = 1$. Consider then the rescalings

$$v_j(x) = \frac{u_j(r_j x)}{r_j^2}, \quad x \in B_{1/r_j}$$

Then $v_j \in P_R(CMR^2)$ if $r_j < 1/R$. Hence, over a subsequence $v_j \to v_0 \in P_{\infty}(CM)$ in $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^n)$.

For the global solution v_0 we have two possibilities:

- (1) v_0 is a polynomial solution
- (2) $\Lambda(v_0)$ is a convex set with a nonempty interior.

In the case (1), taking $Q = v_0$, we obtain that for sufficiently large j,

$$||u_j - Q||_{L^{\infty}(B_{2r_j})} = r_j^2 ||v_j - v_0||_{L^{\infty}(B_2)} < \epsilon r_j^2,$$

which is a contradiction with our assumption on u_j . In the case (2), there exist a ball $B_{\delta}(y_0) \subset \Lambda(v_0) \cap B_1$ and from the nondegeneracy $B_{\delta/2}(y_0) \subset \Lambda(v_j) \cap B_1$ for large enough j. Then

$$\delta(r_i, u_i) = \delta(1, v_i) \ge \delta$$

and we arrive at a contradiction as soon as $\sigma(r_j) < \delta$ for σ as in part (i) of this lemma.

Lemma 21.4. Let Q^r be as in Lemma 21.3, 0 < r < 1/2. Then for any unit vector e

$$|\Phi(r, (\partial_e u)^+, (\partial_e u)^-) - \Phi(r, (\partial_e Q^r)^+, (\partial_e Q^r)^-)| \le C\sigma(r)^{1/3},$$

where C = C(M, n).

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Proof. Consider the rescaling $v(x) = \frac{1}{r^2}u(rx)$ and denote $\epsilon = \sigma(r)$ and $Q = Q^r$ for convenience. Then we will have

$$\Lambda(v) \cap B_1 \subset S_{\epsilon}, \qquad \|v - Q\|_{L^{\infty}(B_2)} \le \epsilon,$$

where S_h is a strip of width h. We want to estimate

$$I(r, (\partial_e u)^+) = I(1, (\partial_e v)^+) = \int_{\{\partial_e v > 0\} \cap B_1} \frac{|\nabla \partial_e v|^2 dx}{|x|^{n-2}}$$

in terms of $I(r, (\partial_e Q)^+) = I(1, (\partial_e Q)^+)$. First, by the interior $C^{1,\alpha}$ estimates we will have

(21.1)
$$\|\nabla v - \nabla Q\|_{L^{\infty}(B_1)} \le C \|v - Q\|_{L^{\infty}(B_2)} \le C\epsilon.$$

To use $C^{1,1}$ estimates, we need to stay away from the free boundary by distance, say, $\epsilon^{1/3}$:

(21.2)
$$\|D^2 v - D^2 Q\|_{L^{\infty}(B_1 \setminus S_{\epsilon^{1/3}})} \leq \frac{C}{\epsilon^{2/3}} \|v - Q\|_{L^{\infty}(B_2)} \leq C \epsilon^{1/3}.$$

To proceed, assume $Q(x) = \frac{1}{2}(x \cdot Ax)$ for a symmetric matrix A. Note that $D^2Q = A$ and that by $C^{1,1}$ estimates for v, combined with (21.2), we have $|A| \leq C(M, n)$. Let now

$$V = \{\partial_e v > 0\} \cap B_1, \quad U = \{\partial_e Q = x \cdot Ae > 0\} \cap B_1.$$

Then by (21.1)

$$U_{\pm} \subset V \subset U_{-}, \text{ where } U_{\pm} = \left\{ x \cdot Ae > \pm C\epsilon^{2/3} \right\} \cap B_1$$

and therefore

(21.3)
$$\int_{U_{+}} \frac{|\nabla \partial_{e} v|^{2} dx}{|x|^{n-2}} \leq \int_{V} \frac{|\nabla \partial_{e} v|^{2} dx}{|x|^{n-2}} \leq \int_{U_{-}} \frac{|\nabla \partial_{e} v|^{2} dx}{|x|^{n-2}}$$

Next, using (21.2), we can estimate

(21.4)
$$\left| \int_{U_{\pm}} \frac{|\nabla \partial_e v|^2 dx}{|x|^{n-2}} - \int_{U_{\pm}} \frac{|\nabla \partial_e Q|^2 dx}{|x|^{n-2}} \right|$$
$$\leq C \int_{S_{\epsilon^{1/3}} \cap B_1} \frac{dx}{|x|^{n-2}} + C\epsilon^{1/3} \int_{B_1 \setminus S_{\epsilon^{1/3}}} \frac{dx}{|x|^{n-2}} \leq C\epsilon^{1/3}$$

where C = C(M, n). Furthermore,

(21.5)
$$\left| \int_{U_{\pm}} \frac{|\nabla \partial_e Q|^2 dx}{|x|^{n-2}} - \int_U \frac{|\nabla \partial_e Q|^2 dx}{|x|^{n-2}} \right| \\ \leq C \min\{\epsilon^{2/3}/|Ae|, 1\} |Ae|^2 \leq C\epsilon^{2/3}.$$

Note that both in (21.4) and (21.5) we have used that

$$\int_{S_h \cap B_1} \frac{dx}{|x|^{n-2}} \le \int_{-h}^h \int_{\{|x'| \le 1\}} \frac{dx'}{|x'|^{n-2}} dx_n \le Ch, \quad 0 < h < 1.$$

Now, putting (21.3)–(21.5) together, we obtain

$$|I(1, (\partial_e v)^+) - I(1, (\partial_e Q)^+)| \le C\epsilon^{1/3}.$$

We have an analogous estimate for $I(1, (\partial_e v)^-)$ and taking the product we obtain

$$|\Phi(1, (\partial_e v)^+, (\partial_e v)^-) - \Phi(1, (\partial_e Q)^+, (\partial_e Q)^-)| \le C\epsilon^{1/3},$$

which implies the statement of the lemma.

Lemma 21.5. Let Q^r be as Lemma 21.3 and $Q^r(x) = \frac{1}{2}(x \cdot A^r x)$ for a symmetric matrix A^r with $\operatorname{Tr} A^r = 1$. Then for $0 < r_1 \le r_2 < 1/2$ and any unit vector e

$$|A^{r_1}e|^2 \le |A^{r_2}e|^2 + C\sigma(r_2)^{1/3}$$

Proof. Recall that $\Phi(r, (\partial_e u)^+, (\partial_e u)^-)$ is monotone increasing in r. Then the statement follows from Lemma 21.4 and the observation that

$$\Phi(r, (\partial_e Q^r)^+, (\partial_e Q^r)^-) = C_n |A^r e|^2. \quad \Box$$

Lemma 21.6. Let A^{r_1} and A^{r_2} be as in Lemma 21.5 and additionally assume that they are nonnegative. Then

$$||A^{r_1} - A^{r_2}|| \le C\sigma(r_2)^{1/3}.$$

Proof. Let $B = A^{r_1} - A^{r_2}$. Then $\operatorname{Tr} B = 0$ and let $\lambda = \lambda_{\min} \leq 0$ be the smallest eigenvalue of B and e a corresponding unit eigenvector. We have

$$|A^{r_1}e| = |Be + A^{r_2}e|^2 \le |A^{r_2}e|^2 + C\sigma(r_2)^{1/3}$$

and since $Be = \lambda e$, this gives

$$\lambda^2 + 2\lambda(e \cdot A^{r_2}e) \le C\sigma(r_2)^{1/3}$$

The second term on the right hand side is nonpositive, since A^{r_2} is a nonnegative matrix. This implies that

$$-\lambda_{\min} \leq C\sigma(r_2)^{1/6}$$

Since Tr B = 0, the positive and negative eigenvalues are balanced and therefore if $\lambda_{\max} \ge 0$ is the largest eigenvalue of B

$$\lambda_{\max} \le -(n-1)\lambda_{\min}.$$

This implies the statement of the lemma, since $||B|| = \max\{-\lambda_{\min}, \lambda_{\max}\}$.

We are now ready to prove the main result of this section.

(i) With no loss of generality we may assume that $x_0 = 0$ and $u \in P_1(M)$ and let $Q^r = \frac{1}{2}(x \cdot A^r x)$ be as above. Then by Lemma 21.6, A^r converges to a unique limit A^0 as $r \to 0$ and

$$A^r - A^0 \| \le C\sigma(r)^{1/6}$$

Consequently, $Q^0(x) = \frac{1}{2}(x \cdot A^0 x)$ satisfies

$$\begin{aligned} \|u - Q^0\|_{L^{\infty}(B_r)} &\leq \|u - Q^0\|_{L^{\infty}(B_r)} + \|Q^0 - Q^r\|_{L^{\infty}(B_r)} \\ &\leq C\sigma(r)r^2 + \|A^r - A^0\|r^2 < C\sigma(r)^{1/6}r^2. \end{aligned}$$

This proves part (i).

(ii) Let now $Q_{x_0}(x) = \frac{1}{2}(x-x_0) \cdot A_{x_0}(x-x_0)$ be the polynomials corresponding to $x_0 \in \Sigma(u)$ as in (i) and let $x_1 \in \Sigma(u)$ be another singular point. Then we have

$$\begin{aligned} \|Q_{x_0} - Q_{x_1}\|_{L^{\infty}(B_r(x_0) \cap B_r(x_1))} \\ &\leq \|Q_{x_0} - u\|_{L^{\infty}(B_r(x_0))} + \|u - Q_{x_1}\|_{L^{\infty}(B_r(x_1))} \\ &\leq 2\sigma(r)r^2, \end{aligned}$$

if $|x_1 - x_0| \leq \delta r$ for small $\delta > 0$ and $r < \frac{1}{4} \operatorname{dist}(x_0, \partial D)$. Now, if we take $\delta < \sigma(r)^{1/2}$, then we will have

$$\|A_{x_0} - A_{x_1}\| \le C\sigma(r).$$

This proves part (ii).

(iii) This part is a direct application of Whitney's extension theorem.

Lemma 21.7 (Whitney's Extension Theorem). Let E be a compact set in \mathbb{R}^n and $f: E \to \mathbb{R}$ arbitrary. Suppose that for any $x \in E$ there exists a polynomial P_x of degree m such that

(1)
$$P_x(x) = f(x) \text{ for } x \in E;$$

(2) $|D^k(P_x - P_y)(x)| = o(|x - y|^{m-k}) \text{ for } x, y \in E \text{ and } k = 0, \dots m.$

Then f extends to a C^m function on \mathbb{R}^n such that

$$f(y) = P_x(y) + o(|x - y|^m)$$

for all $x \in K$.

Fix a number a > 0 and consider the subset $\Sigma_{k,a}(u)$ of points $x_0 \in \Sigma_k(u)$ for which the smallest nonzero eigenvalue of A_{x_0} is at least a. Let also K be a compact subset of D.

Take $E = \Sigma_{k,a}(u) \cap K$, f = 0 and $P_{x_0} = Q_{x_0}$. Then the conditions of the Whitney's extension theorem are verified by part (ii) and therefore we can extend f to a C^2 function on \mathbb{R}^n . To complete the proof, now observe

$$E \subset \{\nabla f = 0\} = \bigcap_{i=1}^{n} \{\partial_{x_i} f = 0\}.$$

For $x_0 \in E$, we can arrange the coordinate axes so that the vectors e_1, \ldots, e_{n-k} are eigenvalues of $D^2 f(x_0) = A_{x_0}$. Moreover by our assumption $\partial_{x_i x_i} f(x_0) \ge a$, $i = 1, \ldots, n-k$. Then the implicit function theorem implies that

$$\bigcap_{i=1}^{n-k} \{\partial_{x_i} f = 0\}$$

is a k-dimensional C^1 manifold in a neighborhood of x_0 . This completes the proof of part (iii) and thereby that of the theorem.

21.4. Structure of the singular set: "no-sign" case.

Theorem 21.8. Let u be a solution of the "no-sign" obstacle problem in a domain D in \mathbb{R}^n with $f \in C^{0,1}(D)$.

(i) For any $x_0 \in \Sigma(u)$ there exists a unique linear subspace Π_{x_0} of \mathbb{R}^n such that for any blowup $Q(x) = \frac{1}{2}(x \cdot Ax)$ of u at x_0 as in Theorem 21.1(ii) have

 $\Pi_{x_0}(x) = \ker A.$

- (ii) Let $\Sigma_k(u) = \{x_0 \in \Sigma(u) : \dim \Pi_{x_0} = k\}$ for k = 0, ..., n-1. Then the mapping $x_0 \mapsto \Pi_{x_0}$ is continuous in $x_0 \in \Sigma_k(u)$ (as the mapping to a Grassmanian of k-dimensional subspaces of \mathbb{R}^n).
- (iii) $\Sigma_k(u)$ is contained in the union of countably many k-dimensional C^1 manifolds, k = 0, ..., n - 1.

Lemma 21.9. Let $u \in P_1(M)$ with $f \in C^{0,1}(B_1)$. Then Lemmas 21.3 and 21.4, with σ and C depending additionally on M and $||f||_{L^{\infty}(B_1)}$.

Lemma 21.10. Let $u \in P_1(M)$ with $f \in C^{0,1}(B_1)$, Q^r be as Lemma 21.3 and $Q^r(x) = \frac{1}{2}(x \cdot A^r x)$ for a symmetric matrix A^r with $\operatorname{Tr} A^r = 1$. Then for $0 < r_1 \leq r_2 < 1/2$ and any unit vector e

$$|A^{r_1}e|^2 \le (1+r_2^\beta)|A^{r_2}e|^2 + C\sigma(r_2)^{1/3}$$

Proof. The only essential difference is that with the proof of Lemma 21.5 we utilize the Almost Monotonicity Formula [CJK] in the following form: If $\phi(r) = \Phi(r, (\partial_e u)^+, (\partial_e u)^-)$ and $0 \le r_1 \le r_2 \le 1/2$, then

$$\phi(r_1) \le (1 + r_2^\beta)\phi(r_2) + Cr_2^\beta.$$

We leave the details to the readers.

One complication that we face allowing u to change sign is that A^r is not necessarily nonnegative now and the proof of Lemma 21.6 does not apply. As a result we don't know if A^r converge to a unique matrix A^0 . Instead, we get only that $(A^0)^2$ is unique.

Lemma 21.11. There exists a unique matrix B^0 such that if $A^r \to A$ for a subsequence $r = r_k \to 0$ then

$$A^2 = B^0.$$

Consequently, $\Pi^0 = \ker A = \ker B^0$ is also unique

Proof. Let A' and A'' be the limits of A^r $r = r'_k \to 0$ and $r = r''_k \to 0$ respectively. Then by Lemma 21.10

$$|A'e|^2 = |A''e|^2$$

for any unit vector e. On the other hand, if A is a symmetric matrix, $Ae \cdot Ae = e \cdot A^2 e$ and therefore the above equality implies that $A'^2 = A''^2$.

Proof of Theorem 21.8. Part (i) follows from Lemma 21.11. Consider now the subset $\Sigma_k(u)$ of singular points x_0 with dim $\Pi_{x_0} = k$. We want to show that Π_{x_0} depends continuously on $x_0 \in \Sigma_k(u)$. To this end let $x_j \in \Sigma_k(u)$ and $x_j \to x_0$. Consider now the corresponding matrices B_{x_j} and B_{x_0} . We claim that

(21.6)
$$\limsup_{i \to \infty} e \cdot B_{x_j} e \le e \cdot B_{x_0} e$$

for any unit vector *e*. Indeed, note that we can take $A_{x_0}^{2r}$ as $A_{x_j}^r$ for *j* large enough so that $|x_j - x_0| < r$ (by modifying $\sigma(r)$, if necessary). Then

$$e \cdot B_{x_j} e \le |A_{x_j}^r e|^2 + C\sigma(r)^{1/3} = |A_{x_0}^{2r} e|^2 + C\sigma(r)^{1/3}$$

Passing to the limit as $j \to \infty$ and then as $r \to 0$ we establish (21.6). This implies that

$$\Pi_{x_0} \subset \limsup_{j \to \infty} \Pi_{x_j}.$$

Since all Π_{x_i} and Π_{x_0} are k-dimensional linear subspaces, we must necessarily have

$$\Pi_{x_0} = \lim_{j \to \infty} \Pi_{x_j}$$

This proves part (ii) of Theorem 21.8.

Let now π_{x_0} denote the orthogonal projection to the orthogonal complement of Π_{x_0} so that ker $\pi_{x_0} = \Pi_{x_0}$. Note that π_{x_0} depends continuously on $x_0 \in \Sigma_k(u)$. This not yet enough to apply Whitney's Extension Theorem, we need something stronger. For $k = 1, \ldots, n-1$ and a > 0 let $\Sigma_{k,a}(u)$ be the subset of points $x_0 \in \Sigma_k(u)$ for which the smallest nonzero eignevalue of B_{x_0} is at least a^2 . Note that we have

$$e \cdot B_{x_0} e \ge a^2 |\pi_{x_0} e|^2$$
 for any $x_0 \in \Sigma_{k,a}(u)$.

Lemma 21.12. For any a > 0, k = 1, ..., n - 1 there exists constant C such that

$$|\pi_{x_0}(x-x_0)| \le C\sigma^{1/3}(|x-x_0|)|x-x_0| \quad \text{for any } x, x_0 \in \Sigma_{k,a}(u)$$

Proof. Let $x_0, x \in \Sigma_{k,a}(u)$ with $|x - x_0| = r$ small. Using that $u(x) = |\nabla u(x)| = 0$, we obtain

$$|A_{x_0}^r(x-x_0)| = |\nabla Q_{x_0}^r(x)| = |\nabla u(x) - \nabla Q_{x_0}^r(x)| \le C\sigma(r)r.$$

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On the other hand, using that $a^2 |\pi_{x_0} e|^2 \le e \cdot B_{x_0} e$ for $x_0 \in \Sigma_{k,a}(u)$, we estimate

$$\begin{aligned} a^{2} |\pi_{x_{0}}(x-x_{0})|^{2} &\leq (x-x_{0}) \cdot B_{x_{0}}(x-x_{0}) \\ &\leq (1+r^{\beta}) |A^{r}_{x_{0}}(x-x_{0})|^{2} + C\sigma(r)^{1/3}r^{2} \\ &\leq C\sigma^{2}(r)r^{2}(1+r^{\beta}) + C\sigma(r)^{1/3}r^{2} \leq C\sigma(r)^{1/3}r^{2}. \end{aligned}$$

Having this lemma in mind, define

$$P_{x_0}(x) = (x - x_0) \cdot \pi_{x_0}(x - x_0).$$

It is easy to see that P_{x_0} satisfies the conditions of the Whitney's Extension Theorem on $E = \sum_{k,a}(u) \cap K$ for any $K \subset D$. Then we complete the proof of part (iii) similarly to the nonnegative case.