LECTURE 3

3. Existence

3.1. **Regularization.**

3.1.1. *Classical Obstacle Problem.* We have shown in Lecture 1 that the classical obstacle problem can be reduced to the obstacle problem with zero obstacle, i.e. the problem of minimizing

$$J(u) = \int_D (|\nabla u|^2 + 2fu)dx$$

over the convex subset $K_{g,0} = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D), u \geq 0 \text{ a.e. in } D\}$. Here we assume $g \in W^{1,2}(D) \cap L^\infty(D)$, $g \geq 0$ on ∂D in the sense $g_- \in W_0^{1,2}(D)$ and $f \in L^\infty(D)$.

Because of the strict convexity of J on $K_{g,0}$, it is clear that the functional has one minimizer. We show in this section that this minimizer is in $W^{2,p}_{\text{loc}}(D) \cap C^{1,\alpha}_{\text{loc}}(D)$ for any $p < \infty$ and $0 < \alpha < 1$ and will consequently solve

$$\Delta u = f\chi_{\{u>0\}} \quad \text{in } D.$$

The first step is getting rid of the obstacle at the expense of losing the regularity of the functional J.

Proposition 3.1. *A function $u \in W^{1,2}(D)$ is a minimizer of J over $K_{g,0}$ iff u is a minimizer of the functional*

$$\tilde{J}(u) = \int_D (|\nabla u|^2 + 2fu^+)dx$$

over $K_g = \{u \in W^{1,2}(D) : u - g \in W_0^{1,2}(D)\}$.

Proof. 1) Let $u \in K_{g,0}$ be a minimizer of J over $K_{g,0}$. Take any $v \in K_g$. Then $v_+ \in K_{g,0}$. One has

$$D_i v^+ = D_i v \chi_{\{v>0\}}$$

and therefore

$$\tilde{J}(v) = \int_D |\nabla v|^2 + 2fv_+ \geq \int_D |\nabla v|^2 \chi_{\{v>0\}} + 2fv_+ = J(v_+) \geq J(u) = \tilde{J}(u).$$

This implies that u is a minimizer of $\tilde{J}$ over K_g.

2) Conversely, let $u \in K_g$ be a minimizer of $\tilde{J}$ over K_g. Then $u^+ \in K_{g,0} \subset K_g$. Moreover, evidently

$$J(u_+) \leq J(u)$$

with equality iff

$$\int_D |\nabla u|^2 \chi_{\{u \leq 0\}} dx = 0.$$

The latter implies $D_i u^- = 0$ a.e. in D, hence u^- must be a locally constant in D. Since also $u^- \in W_0^{1,2}(D)$, $u^- = 0$ in D. This implies $u \geq 0$ a.e. in D and consequently $u \in K_{g,0}$. Finally, $\tilde{J}$ coincides with J on $K_{g,0}$ and therefore u is a minimizer of J over $K_{g,0}$. $\square$

Thus, we reduced the problem to studying the minimizers of $\tilde{J}$ with given boundary values g on ∂D. To write the corresponding Euler-Lagrange equations, we consider a family of regularized problems

$$\begin{aligned} \Delta u &= f\chi_\epsilon(u) \quad \text{in } D \\ u &= g \qquad\quad\ \text{on } \partial D, \end{aligned}$$

where $\chi_\epsilon(s)$ is a smooth approximation of the Heaviside function $\chi_{\{s>0\}}$ such that

$$\chi_\epsilon' \geq 0, \quad \chi_\epsilon(s) = 0 \quad \text{for } s \leq -\epsilon, \quad \chi_\epsilon(s) = 1 \quad \text{for } s \geq \epsilon.$$

A solution u_ϵ to this problem can be obtained by minimizing the functional

$$J_\epsilon(u) = \int_D (|\nabla u|^2 + 2f(x)\Phi_\epsilon(u))dx$$

over K_g, where

$$\Phi_\epsilon(s) = \int_{-\infty}^{s} \chi_\epsilon(t)dt.$$

Now, recall that we assume that g is uniformly bounded in D. This will imply by the maximum principle of Alexandrov (see [Gilbarg-Trudinger, Theorem 9.1]

$$\|u_\epsilon\|_{L^\infty(D)} \leq \|g\|_{L^\infty(D)} + C(n, D)\|f\|_{L^\infty(D)}.$$

Consequently, applying the interior L^p-estimates, we will have that

$$\|u_\epsilon\|_{W^{2,p}(K)} \leq C(p, K, D, f, g)$$

for any open $K \subset\subset D$ and $1 < p < \infty$. Thus, the family $\{u_\epsilon\}$ of minimizers of J_ϵ is uniformly bounded in $W^{2,p}(K)$ and therefore we can find a subsequence $\epsilon_k \to 0$ and a function u, such that over $\epsilon = \epsilon_k \to 0$

$$u_\epsilon \to u \quad \text{weakly in } W^{2,p}_{\text{loc}}(D)$$

for any $1 < p < \infty$. Clearly, $u \in W^{2,p}_{\text{loc}}(D)$ for any $1 < p < \infty$.

Now, It is an easy exercise to show that $\{u_\epsilon\}$ is uniformly bounded in $W^{1,2}(D)$. Thus, we may assume $u_\epsilon \to u$ weakly in $W^{1,2}(D)$. Moreover, since $u_\epsilon - g \in W^{1,2}_0(D)$ and $W^{1,2}_0(D)$ is a closed subset of a Hilbert space $W^{1,2}(D)$, we obtain that $u \in K_g$. Applying Fatou's lemma and the dominated convergence theorem we see that

$$J(u) \leq \liminf_{\epsilon=\epsilon_k \to 0} J_\epsilon(u_\epsilon) \leq \liminf_{\epsilon=\epsilon_k\to 0} J_\epsilon(v) = J(v)$$

for any $v \in K_g$. Thus, u is the solution of the obstacle problem. Finally, we verify that u satisfies

$$\Delta u = f\chi_{\{u>0\}} \quad \text{a.e. in } D.$$

To this end, note that we can assume that over $\epsilon = \epsilon_k \to 0$

$$u_\epsilon \to u \quad \text{in } C^{1,\alpha}_{\text{loc}}(D),$$

by the Sobolev embedding theorem. Then the locally uniform convergence implies that $\Delta u = 0$ in $\{u < 0\}$ and $\Delta u = f$ in $\{u > 0\}$. Using also that $u \in W^{2,p}_{\text{loc}}(D)$, we also have $\Delta u = 0$ a.e. on $\{u = 0\}$. Thus, we obtain that $\Delta u = f(x)\chi_{\{u>0\}}$ a.e. in D.

3.1.2. *Two-Phase Obstacle Problem.* Given any two constants λ_+ and λ_- we show here that we can find a locally bounded solution to the two-phase obstacle problem

$$\begin{aligned} \Delta u &= \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} && \text{in } D \\ u &= g && \text{on } \partial D \end{aligned}$$

in a bounded domain D and $g \in W^{1,2}(D) \cap L^\infty(D)$, by applying the regularization technique from the previous subsection.

Namely, we want to find a minimizer of the energy functional

$$J(v) = \int_D (|\nabla v|^2 + 2\lambda_+ u^+ + 2\lambda_- u^-) dx$$

on the set K_g. To this end consider the approximating problems

$$\begin{aligned} \Delta u &= \lambda_+ \chi_\epsilon(u) - \lambda_- \chi_\epsilon(-u) && \text{in } D \\ u &= g && \text{on } \partial D, \end{aligned}$$

and the solutions u_ϵ obtained by minimizing the functional

$$J(u) = \int_D (|\nabla u|^2 + 2\lambda_+ \Phi_\epsilon(u) + 2\lambda_- \Phi_\epsilon(-u)) dx,$$

where the approximations χ_ϵ and Φ_ϵ are as in the previous subsection. Then, following the arguments as before one can establish that for a subsequence $\epsilon = \epsilon_k$ the minimizers u_ϵ converge weakly in $W^{2,p}_{\mathrm{loc}}(D)$ for any $1 < p < \infty$ to a solution of the desired problem. We leave to the reader to fill in the details.

We conclude with the remark that if $\lambda_\pm \geq 0$ or more generally $\lambda_+ + \lambda_- \geq 0$, then the solution of the problem is unique, as the corresponding functional is convex. However, in the non-convex case, i.e. when $\lambda_+ + \lambda_- < 0$ there is a class of non-variational solutions, that may exhibit certain peculiar properties (e.g. they may not be $C^{1,1}$ regular.)

3.2. **Viscosity solutions: Perron-Wiener's Method.** We have chosen here to give a proof of the existence for the case of superconductivity problem, which can be applied to the obstacle problem directly. We follow Caffarelli and Salazar.

So let us recall the equation that appears in the problem of superconductivity

$$\Delta u = f(x, u) \chi_{\{|\nabla u| > 0\}}, \qquad \text{in } B_1, \tag{3.1}$$

with $f > 0$, and $f(x, s) \in C^\alpha_x \cap C^0_s$.

3.2.1. *Sub- and Supersolution.* A *subsolution* to equation (3.1) is an upper semi-continuous (u.s.c.) function u, bounded from above, such that the inequality

$$\Delta P(x) \geq f(x, P(x))$$

holds for any paraboloid

$$P(x) = c_0 + \bar{b} \cdot x + \sum_{i=1}^n a_j x_j^2, \qquad \bar{b} = (b_1, \cdots, b_n)$$

touching u from above at x, with $|\nabla P(x)| \neq 0$. A *supersolution* of (3.1) is a lower semicontinuous (l.s.c.) function u, bounded below, such that

$$\Delta P(x) \leq f(x, P(x))$$

for any paraboloid P with $|\nabla P(x)| \neq 0$, and touching the graph of u at x, from below . A solution is both a sub- and a supersolution.

We denote by u^*, and u_* the upper semicontinuous, and the lower semicontinuous envelopes, respectively, of a given function u, i.e.,

$$u^*(x) = \limsup_{z\to x} u(z), \qquad u_*(x) = \liminf_{z\to x} u(z)\,.$$

3.2.2. *Properties of sub- and supersolutions.*

Lemma 3.2. *Let $\{u_\iota\}$ be a nonempty family of subsolutions of* (3.1) *and set*

$$u = \sup_\iota u_\iota\,.$$

Then, u^* is a subsolution, provided $u^* < \infty$.

Proof. Fix x^0, and let P be any paraboloid, with $|\nabla P(x^0)| \neq 0$, and touching u^* from above at x^0. Let further $\epsilon > 0$ be a fixed constant, and define

$$Q(x) = P(x) + \frac{\epsilon}{2n}|x - x^0|^2\,.$$

By continuity, there is $\delta > 0$ such that for all $x \in B_\delta(x^0)$, we have

a) $|\nabla P(x)| \neq 0, \quad |\nabla Q(x)| \neq 0$;
b) $\Delta P(x) \le \Delta P(x^0) + \epsilon$;
c) $f(x,r) \ge f(x^0, u^*(x^0)) - \epsilon$, for all r such that $|r - u^*(x^0)| \le \epsilon\frac{\delta^2}{n}$.

Choose $\eta < \delta/2$ such that

$$|P(x) - P(x^0)| < \frac{\epsilon\delta^2}{16n}, \quad \forall x \in B_\eta(x^0)\,.$$

There exists an index ι and a point $x' \in B_\eta(x^0)$, (this point doesn't necessarily coincide with x^0, since we might well have $u^*(x^0) > u(x^0)$) such that

$$u_\iota(x') > u^*(x^0) - \frac{\epsilon\delta^2}{16n}\,,$$

and hence

$$\begin{aligned}Q(x') - u_\iota(x') &< Q(x') - u^*(x^0) + \frac{\epsilon\delta^2}{16n}\\ &= P(x') - P(x^0) + \frac{\epsilon}{2n}|x' - x^0|^2 + \frac{\epsilon\delta^2}{16n} < \frac{\epsilon\delta^2}{4n}\,.\end{aligned}$$

Since also $P(x) \ge u^*(x) \ge u_\iota(x)$ in $B_\delta(x^0)$, we arrive at

$$\frac{\epsilon}{2n}|x - x^0|^2 \le Q(x) - u_\iota(x) \qquad B_\delta(x^0)\,.$$

From the last two inequalities it follows that the infimum of $Q - u_\iota$ is attained at a point $x^1 \in B_{\delta/\sqrt{2}}$ (interior of the ball). It is crucial to note that this point is in the interior of the ball $B_{\delta/\sqrt{2}}$, and not on the boundary.

At this point, $Q - Q(x^1) + u_\iota(x^1)$ is a touching paraboloid for u_ι, from above. Since u_ι is a sub-solution, we have

$$\Delta Q(x^1) \ge f(x^1, u_\iota(x^1)).$$

Moreover,

$$\begin{aligned}|u^*(x^0) - u_\iota(x^1)| &\le |Q(x^0) - Q(x^1)| + Q(x^1) - u_\iota(x^1)\\ &\le \frac{\epsilon\delta^2}{16n} + \frac{\epsilon}{2n}|x^1 - x^0|^2 + \frac{\epsilon\delta^2}{4n} \le \frac{13}{16}\frac{\epsilon\delta^2}{n}.\end{aligned}$$

Putting all these inequalities together, we obtain

$$\Delta P(x^0) + 3\epsilon \geq \Delta P(x^1) + 2\epsilon \geq \Delta Q(x^1) + \epsilon \geq f(x^1, u_\iota(x^1)) + \epsilon \geq f(x^0, u^*(x^0)).$$

Since ϵ was arbitrary we arrive at the subsolution property $\Delta P(x^0) \geq f(x^0, u^*(x^0))$. $\square$

Lemma 3.3. *Let $\underline{v} \leq \overline{v}$ be given continuous sub- and supersolutions, respectively. Then there exist a function u, $\underline{v} \leq u \leq \overline{v}$, such that u_* is a supersolution and u^* is a subsolution.*

Proof. Let

$$u = \sup\{w : \ w \leq \overline{v} : \ w \in C^0, \ \Delta w(x) \geq f(x, w(x))\}.$$

By Lemma (3.2), u^* is a subsolution. We also have $u = u_*$ (supremum of continuous functions are lower semicontinuous). To prove that u is a supersolution let P be a paraboloid touching u from below at a point x^0, such that $|\nabla P(x^0)| \neq 0$.

If $u(x^0) = \overline{v}(x^0)$, then by supersolution property of $\overline{v}$, we have $\Delta P(x^0) \leq f(x^0, u(x^0))$, and we will be done.

Let us investigate the case

$$u(x^0) < \overline{v}(x^0), \qquad a := \Delta P(x^0) - f(x^0, u(x^0)) > 0.$$

By continuity we may choose $\delta_1 > 0$, and $\nu > 0$ such that for all $x \in B_{\delta_1}(x^0)$ and $|r - u^*(x^0)| < \nu$, we have

$$f(x, r) \leq f(x^0, u(x^0)) + \frac{a}{3}, \qquad \Delta P(x) \geq \Delta P(x^0) - a/3.$$

Let $0 < \delta_2 \leq \delta_1$ be such that for all $x \in B_{\delta_2}(x^0)$

$$-\frac{\nu}{2} \leq P(x) - P(x^0) - \frac{a}{6n}|x - x^0|^2 \leq \frac{\nu}{2}.$$

Then, for $|\beta| < \nu/2$, the paraboloid

$$Q(x) = P(x) - \frac{a}{6n}|x - x^0|^2 + \beta$$

is a subsolution to (3.1) in B_{δ_2},

$$\Delta Q(x) = \Delta P(x) - \frac{a}{3} \geq \Delta P(x^0) - \frac{2a}{3} \geq f(x^0, u(x^0)) + \frac{a}{3} \geq f(x, Q(x))\,.$$

The last inequality holds because

$$|Q(x) - u(x^0)| \leq \left| P(x) - P(x^0) - \frac{a}{6n}|x - x^0|^2 \right| + |\beta| \leq \nu\,.$$

To reach a contradiction, we shall construct a continuous subsolution less than or equal to $\overline{v}$ and strictly greater than u at x^0.

First we choose $\gamma > 0$ and $0 < \delta < \delta_2$ such that

$$\overline{v} - P \geq \gamma \ \text{ on } \ B_\delta(x^0)\,.$$

By the axiom of choice and the compactness of ∂B_δ, there is a continuous subsolution $v \leq \overline{v}$, such that

$$v - P \geq -\frac{a\delta^2}{12n} \ \text{ on } \ \partial B_\delta\,.$$

Taking $\beta < \min\{\nu/2, \gamma, \frac{a\delta^2}{12n}\}$, $\beta > 0$, we see that the function

$$w(x) = \begin{cases} \max\left(v(x), Q(x)\right), & x \in B_\delta, \\ v(x), & x \notin B_\delta, \end{cases}$$

is a continuous subsolution less than or equal to $\overline{v}$, and $w(x^0) > u(x^0)$. This is a contradiction to the maximality of u. □

3.2.3. *Existence.* A most natural question is whether the function u above is actually continuous. We answer this question under suitable additional hypotheses.

Theorem 3.4. *Let $\underline{v}$ and $\overline{v}$ be as in Lemma 3.3. Assume that there is $c \geq 0$ such that for all $x \in B_1$, all $r \in \mathbb{R}$, and all $h \geq 0$,*

$$f(x, r+h) \geq f(x, r) + ch\,.$$

Then, there is a viscosity solution u such that $\underline{v} \leq u \leq \overline{v}$.

Remark 3.5. *By Lemma 3.3, taking u equal to the supremum of all continuous subsolutions less than or equal to $\overline{v}$, only the continuity of u remains to be proved.*

Before going into the proof of this proposition, we need some notation and properties of Jensen's approximate solutions.

Suppose there is a point $x^0 \in \Omega$ such that

$$u^*(x^0) > u(x^0);$$

otherwise, u is continuous and there is nothing to prove.

Following Jensen's idea, define

$$u^\epsilon(x) = \sup_{y \in \Omega_\alpha} \left\{ u^*(y) + \epsilon - \frac{1}{\epsilon}|y - x|^2 \right\}, \qquad x \in \Omega_\alpha := \{\overline{v} \geq \underline{v} + \alpha\}$$

where α is a positive constant, whose precise value will be fixed later.

Jensen's approximation of a continuous solution enjoys many nice properties; a list of them can be found in the book by Caffarelli and Cabré [Caff-Cab], p. 43, Theorem 5.1. Suitable versions of those properties, adapted to our case, are listed below. We omit the proofs since those given in [Caff-Cab] work with minor changes.

a) u^ϵ is a decreasing family of continuous functions.

b) Let f be a continuous function such that $f \geq u^*$. For each $\beta > 0$ there is an $\epsilon_0 > 0$ such that

$$u^\epsilon \leq f + \beta \quad \text{on} \quad \Omega_{2\alpha}\,, \qquad \forall \epsilon < \epsilon_0\,.$$

c) For each $x \in \Omega_\alpha$, there corresponds a point $x' \in \Omega_\alpha$ such that

$$u^\epsilon(x) = u^*(x') + \epsilon - \frac{1}{\epsilon}|x - x'|^2\,.$$

d) The point x' in c) satisfies

$$|x - x'|^2 \leq \epsilon \sup_{\Omega_\alpha} |\overline{v} - \underline{v}|\,.$$

We state a key lemma for the proof of continuity of the above constructed function.

Lemma 3.6. *Under the hypothesis of Theorem 3.4, for each $\delta > 0$, there exists an $\epsilon_1 > 0$ such that for all $\epsilon < \epsilon_1$, the function $u^\epsilon(x) - \delta$ is a viscosity subsolution of (3.1) in $\Omega_{2\alpha}$.*

Proof. Let P be a paraboloid touching $u^\epsilon - \delta$ from above at a point $x^0 \in \Omega_{2\alpha}$. Assume $|\nabla P(x^0)| \neq 0$ and define

$$Q(x) = P(x + x^0 - x') + \delta + \frac{1}{\epsilon}|x^0 - x'|^2 - \epsilon\,,$$

where x' is the corresponding point for x^0 in c) above. Then, one readily verifies that

$$u^*(x) \leq u^\epsilon(x + x^0 - x') + \frac{1}{\epsilon}|x^0 - x'|^2 - \epsilon \leq Q(x)\,,$$
$$u^*(x') = Q(x')\,, \qquad \nabla Q(x') = \nabla P(x^0) \neq 0\,, \qquad \Delta P(x^0) = \Delta Q(x').$$

Hence

$$\begin{aligned}\Delta Q(x') &\geq f\big(x', u^*(x')\big) = f\big(x', u^\epsilon(x^0) + \frac{1}{\epsilon}|x^0 - x'|^2 - \epsilon\big)\\ &\geq f\big(x', u^\epsilon(x^0) - \delta\big) + c\big(\delta + \frac{1}{\epsilon}|x^0 - x'|^2 - \epsilon\big)\,,\end{aligned}$$

provided $\delta + \frac{1}{\epsilon}|x^0 - x'|^2 - \epsilon \geq 0$.

By d), since $\Omega_\alpha \times I$ (where $I = \{r \in \mathbb{R};\ \inf_{\Omega_\alpha} \underline{v} - \delta \leq r \leq \sup_{\Omega_\alpha} \overline{v} + 1\}$) is compact and f is continuous, we can find $\epsilon_1 > 0$ such that for all $\epsilon \leq \epsilon_1$, $|x^0 - x'|$ is small enough and we have

$$f\big(x', u^\epsilon(x^0) - \delta\big) \geq f\big(x^0, u^\epsilon(x^0) - \delta\big) - c\,\frac{\delta}{2}\,.$$

Consequently, for $\epsilon_1 \leq \delta/2$, we arrive at

$$\Delta P(x^0) \geq f\big(x^0, u^\epsilon(x^0) - \delta\big)\,.$$

□

Proof of Theorem 3.4. Let $\delta = u^*(x^0) - u(x^0)$ and fix $\epsilon_0 > 0$ such that

$$u^\epsilon \leq \underline{v} + \delta/3 \ \text{ on } \ \Omega_{2\delta/3}\,, \qquad \forall \epsilon < \epsilon_0;$$

see property b) above. In addition, by Lemma 3.6, let $\epsilon_1 > 0$ be such that the function $u^\epsilon - \delta$ is a continuous viscosity subsolution of (1) in $\Omega_{2\delta/3}$.

Then, for $\epsilon \leq \epsilon_0 \wedge \epsilon_1$, we have

i) $u^\epsilon(x^0) - \delta \geq u(x^0) + \epsilon$,
ii) $u^\epsilon - \delta \leq \overline{v}$ in $\Omega_{2\delta/3}$,
iii) $u^\epsilon - \delta \leq \overline{v} - 2\delta/3 = \underline{v}$ on $\partial\Omega_{2\delta/3}$.

This in particular implies that the function

$$w(x) = \begin{cases} \max\left((u^\epsilon(x) - \delta), \underline{v}(x)\right)\,, & x \in \Omega_{2\delta/3},\\ \underline{v}(x)\,, & x \notin \Omega_{2\delta/3},\end{cases}$$

is a continuous subsolution less than or equal to $\overline{v}$ and $w(x^0) > u(x^0)$. This leads to a contradiction. □