LECTURE 4

4. Optimal Regularity

As we have seen at the end of Lecture 2, locally bounded solutions of the equation $\Delta u = f$ in an open subset $D \subset \mathbb{R}^n$ with uniformly bounded f are of class $W^{2,p}_{\mathrm{loc}}$ for any $1 < p < \infty$. Hence, working on a compactly supported open subset of D, if necessary, we may assume without loss of generality that

$$u \in W^{2,p}(D) \cap C^{1,\alpha}(D), \quad \text{for some } p > n \text{ and } 0 < \alpha < 1. \tag{4.1}$$

Our aim in this lecture is to show that the solution of Problems A-C are in fact of class $C^{1,1}$. The latter is the optimal regularity for these solutions, as the Laplacian may have discontinuities in these problems.

4.1. **Classical Obstacle Problem.** We start with considering nonnegative solutions of the obstacle problem, since the proofs are much simpler in that case. Throughout this section we consider nonnegative distributional solutions $u \in L^\infty_{\mathrm{loc}}(D)$ of

$$\Delta u = f(x)\chi_{\{u>0\}} \quad \text{in } D, \tag{4.2}$$

for $f \in L^\infty(D)$.

We start with the following result on the growth of u away from the free boundary $\partial\{u > 0\}$.

Theorem 4.1 (Quadratic growth)**.** *Let $u \in L^\infty_{\mathrm{loc}}(D)$, $u \geq 0$ satisfy (4.2), $\Omega = \{u > 0\}$, $x_0 \in \partial\Omega$, and $B_{2R}(x_0) \subset\subset D$. Then*

$$\sup_{B_R(x_0)} u \leq C\|f\|_{L^\infty(D)} R^2,$$

where $C = C(n)$.

Proof. Decompose u into the sum $u_1 + u_2$ in $B_{2R}(x_0)$, where

$$\Delta u_1 = \Delta u, \quad \Delta u_2 = 0 \quad \text{in } B_{2R}(x_0)$$
$$u_1 = 0, \quad u_2 = u \quad \text{on } \partial B_{2R}(x_0).$$

To estimate u_1, we consider the auxiliary function

$$\phi(x) = \frac{1}{2n}(4R^2 - |x - x_0|^2),$$

which is the solution of

$$\Delta\phi = -1 \quad \text{in } B_{2R}(x_0), \qquad \phi = 0 \quad \text{on } \partial B_{2R}(x_0).$$

Then we have

$$-M\,\phi(x) \leq u_1(x) \leq M\phi(x), \quad x \in B_{2R}(x_0),$$

where $M = \|f\|_{L^\infty(D)}$. This follows from the comparison principle, since

$$-M \leq \Delta u_1 \leq M \quad \text{in} \quad B_{2R}(x_0),$$

and that both u_1 and ϕ vanish on $\partial B_{2R}(x_0)$. In particular, this implies that

$$|u_1(x)| \le C(n) M R^2, \quad x \in B_{2R}(x_0).$$

To estimate u_2, observe that $u_2 \ge 0$ in $B_{2R}(x_0)$, since $u_2 \ge u \ge 0$ on $\partial B_{2R}(x_0)$. Also note that $u_1(x_0) + u_2(x_0) = u(x_0) = 0$ and the estimate of u_1 gives

$$u_2(x_0) \le C(n) M R^2.$$

Applying now the Harnack inequality, we obtain

$$u_2(x) \le C(n) u_2(x_0) \le C(n) M\, R^2, \quad x \in B_R(x_0).$$

Combining the estimates for u_1 and u_2, we obtain the desired estimate for u. □

Corollary 4.2. *Let u be as in Theorem 4.1 and $\Lambda = D \setminus \Omega$. Then*

$$u(x) \le C(n) \|f\|_{L^\infty(D)} \operatorname{dist}(x, \Lambda)^2,$$

as long as $2 \operatorname{dist}(x, \Lambda) < \operatorname{dist}(x, \partial D)$. □

In order to obtain $C^{1,1}$ estimates for the solutions of (4.2) that we need to assume a little bit more on the function f in (4.2). Namely, we require f to have a $C^{1,1}$-regular potential, i.e.

$$f = \Delta \psi \quad \text{in } D, \quad \text{with } \psi \in C^{1,1}(D). \tag{4.3}$$

We use the following second order derivative estimates associated with such f: if v is a solution of

$$\Delta v = f \quad \text{in} \quad B_{2R}(x_0) \subset D \tag{4.4}$$

then

$$\|D^2 v\|_{L^\infty(B_R(x_0))} \le C(n) \left(\frac{\|v\|_{L^\infty(B_{2R}(x_0))}}{R^2} + \|D^2 \psi\|_{L^\infty(B_{2R}(x_0))} \right). \tag{4.5}$$

We leave this as an easy exercise to the reader.

Theorem 4.3 ($C^{1,1}$-regularity)**.** *Let $u \ge 0$, f satisfy (4.2)–(4.3). Then $u \in C^{1,1}_{\mathrm{loc}}(D)$ and*

$$\|u\|_{C^{1,1}(K)} \le C(\|u\|_{L^\infty(D)} + \|D^2 \psi\|_{L^\infty(D)}),$$

for any open $K \subset\subset D$, where $C = C(n, K, D)$.

Proof. For $K \subset\subset D$ and $x_0 \in K$, let $\delta = \frac{1}{2} \operatorname{dist}(K, \partial D)$ and $d = \frac{1}{2} \operatorname{dist}(x_0, \Lambda)$. Then we have two possibilities.

1) $d < \delta/4$. In this case, let $y_0 \in \partial B_{2d}(x_0) \cap \partial \Omega$. Then $B_{6d}(y_0) \subset B_{8d}(x_0) \subset\subset D$. Applying Theorem 4.1, we have

$$\|u\|_{L^\infty(B_{3d}(y_0))} \le C(n) \|f\|_{L^\infty(D)}\, d^2.$$

Now note that $B_{2d}(x_0) \subset B_{3d}(y_0)$ and $\Delta u = f$ in $B_{2d}(x_0)$. By the interior estimate (4.5)

$$\|D^2 u\|_{L^\infty(B_d(x_0))} \le C(n)(\|f\|_{L^\infty(D)} + \|D^2 \psi\|_{L^\infty(D)}).$$

In fact, the term $\|f\|_{L^\infty(D)}$ is redundant as $\|f\|_{L^\infty(D)} \le C(n) \|D^2 \psi\|_{L^\infty(D)}$.

2) $d \ge \delta/4$. In this case, the interior derivative estimate for u in $B_d(x_0)$ gives

$$\|D^2 u\|_{L^\infty(B_d(x_0))} \le C(n) \left(\frac{\|u\|_{L^\infty(D)}}{\delta^2} + \|D^2 \psi\|_{L^\infty(D)} \right).$$

Combining cases 1) and 2) above, together with the interpolation inequality, we obtain

$$\|u\|_{C^{1,1}(K)} \le C(n)\left(\frac{\|u\|_{L^\infty(D)}}{\widetilde{\delta}^2} + \|D^2\psi\|_{L^\infty(D)}\right),$$

where $\widetilde{\delta} = \min(\delta, 1)$. $\square$