LECTURE 5

5. ACF-type Monotonicity Formulas

5.1. **Harmonic Functions.** For a continuous $u \in W^{1,2}(B_1)$ define the following quantity:

$$J(r,u) = \frac{1}{r^2}\int_{B_r} \frac{|\nabla u|^2 dx}{|x|^{n-2}}, \quad 0 < r < 1.$$

It is relatively straightforward to show that $J(r,u)$ is monotone in r if u is a harmonic function. Namely, if we represent u as a locally uniformly convergent series

$$u(x) = \sum_{k=0}^{\infty} f_k(x),$$

where $f_k(x)$ are k-th order homogeneous harmonic polynomials, and use the orthogonality of homogeneous harmonic polynomials of different order, we will have

$$\begin{aligned}
J(r,u) &= \frac{1}{r^2}\int_0^r \int_{\partial B_1} |\nabla u(\rho\theta)|^2 \rho\, d\theta d\rho = \\
&= \frac{1}{r^2}\int_0^r \int_{\partial B_1} \rho \sum_{k=1}^{\infty} |\nabla f_k(\rho\theta)|^2 d\theta d\rho \\
&= \frac{1}{r^2}\int_0^r \int_{\partial B_1} \sum_{k=1}^{\infty} \rho^{2k-1}[k^2 f_k(\theta)^2 + |\nabla_\theta f_k(\theta)|^2] d\theta d\rho \\
&= \sum_{k=1}^{\infty} a_k r^{2(k-1)},
\end{aligned}$$

with

$$a_k = \frac{1}{2k}\int_{\partial B_1} [k^2 f_k(\theta)^2 + |\nabla f_k(\theta)|^2] d\theta \geq 0.$$

This implies that $J(r,u)$ is monotone increasing in r.

We next illustrate how this monotonicity formula can be used to obtain interior gradient estimates for harmonic functions.

a) Letting $r \to 0+$, we obtain

$$J(0+,u) \leq J(1/2,u).$$

On the other hand, since u is C^1 (actually real analytic) at the origin, it is easy to see that $J(0+,u) = c_n|\nabla u(0)|^2$, for $c_n > 0$, which implies that

$$c_n|\nabla u(0)|^2 \leq J(1/2,u).$$

b) It turns out that $J(1/2,u)$ is controllable by the L^2-norm of u over B_1 if we assume additionally that $u(0) = 0$. Indeed, consider the function $|x|^{2-n}$ in $B_{1/2}$

and extend it to a function V on B_1 in a smooth nonnegative way, so that $V \equiv 0$ near ∂B_1. Then, using the equality $|\nabla u|^2 = \Delta(u^2/2)$, we have

$$\begin{aligned} J(1/2, u) &= 2^2 \int_{B_{1/2}} \frac{|\nabla u|^2}{|x|^{n-2}} dx \\ &\leq 2^2 \int_{B_1} \Delta\left(\frac{u^2}{2}\right) V dx \\ &= 2^2 \int_{B_1 \setminus B_{1/2}} \left(\frac{u^2}{2}\right) \Delta V dx \end{aligned}$$

which implies

$$J(1/2, u) \leq C_n \|u\|^2_{L^2(B_1)}.$$

Combining the estimates in a) and b) above we arrive at

$$|\nabla u(0)| \leq C_n \|u\|_{L^2(B_1)}$$

Obviously, this is not the best way to establish the inequality above. This method is rather a prelude to the application of the monotonicity formula of Alt-Caffarelli-Friedman for a pair of nonnegative subharmonic functions with "disjoint" support.

5.2. **ACF Monotonicity Formula.**

Theorem 5.1 (Alt-Caffarelli-Friedman (ACF) Monotonicity Formula)**.** *Let $u_\pm \in W^{1,2}(B_1)$ be a pair of nonnegative continuous subharmonic functions in B_1 such that $u_+(0) = u_-(0) = 0$ and $u_+ \cdot u_- = 0$ in B_1. Then the functional*

$$\Phi(r) = \Phi(r, u_+, u_-) := J(r, u_+) J(r, u_-) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u_+|^2 dx}{|x|^{n-2}} \int_{B_r} \frac{|\nabla u_-|^2 dx}{|x|^{n-2}}$$

is finite and nondecreasing in r for $0 < r < 1$.

The standard picture to understand this theorem is to have in B_1 a (relatively nice) surface S, passing through the origin and separating B_1 into two domains D_+ and D_-. The functions u_+ and u_- are harmonic in D_+ and D_- respectively and vanish on S.

Then we have the following series of remarks.

a) Each of the terms $J(r, u_\pm)$ can be understood as a weighted average of $|\nabla u_\pm|^2$. For instance, if $u_\pm = \alpha_\pm x_1^\pm$, then

$$J(r, u_\pm) \equiv c_n \alpha_\pm^2, \quad \Phi(r, u_+, u_-) \equiv c_n^2 \alpha_+^2 \alpha_-^2$$

b) More generally, if S is assumed smooth and $\partial_\nu u_\pm$ exist on S then

$$J(0+, u_\pm) = c_n (\partial_\nu u_\pm)^2.$$

In particular, the monotonicity formula implies

$$c_n^2 (\partial_\nu u_+)^2 (\partial_\nu u_-)^2 \leq \Phi(1/2, u_+, u_-).$$

c) Let Γ be a cone with vertex at the origin, i.e. given a subset $\Sigma_0 \subset \partial B_1$,

$$\Gamma = \{r\,\theta : r > 0, \theta \in \Sigma_0\}.$$

Then consider a homogeneous harmonic function in Γ of the form

$$h(r\,\theta) = r^\alpha f(\theta),$$

vanishing on $\partial\Gamma$. We have

$$\begin{aligned}\Delta h &= \partial_{rr} h + \frac{n-1}{r}\,\partial_r h + \frac{1}{r^2}\Delta_\theta h\\ &= r^{\alpha-2}[(\alpha(\alpha-1)+(n-1)\alpha)f(\theta)+\Delta_\theta f(\theta)].\end{aligned}$$

Thus, we have that h is harmonic in Γ iff f is an eigenfunction for the spherical Laplacian Δ_θ in Σ_0:

$$-\Delta_\theta f(\theta) = \lambda f(\theta) \quad \text{in } \Sigma_0,$$

where

$$\lambda = \alpha(n-2+\alpha).$$

Thus, if we take two disjoint open set $\Sigma_\pm$ on the unit sphere, find there first eigenvalues $\lambda_\pm$ and the corresponding eigenfunctions $f_\pm$, then the homogeneous harmonic functions

$$u_\pm = r^{\alpha_\pm} f_\pm(\theta), \quad \text{in } \Gamma_\pm = \{r\,\theta : \theta \in \Sigma_\pm\}$$

where $\alpha_\pm > 0$ are found from the identity

$$\lambda_\pm = \alpha_\pm(n-2+\alpha_\pm).$$

Then, it is easy to calculate that

$$\Phi(r, u_+, u_-) = C r^{2(\alpha_+ + \alpha_- - 2)}$$

for a $C > 0$ and therefore the monotonicity formula will follow in this case once we know

$$\alpha_+ + \alpha_- \geq 2.$$

This inequality has been established first by Friedland and Hayman. What is interesting is it actually implies the monotonicity formula for all $u_\pm$, not necessarily homogeneous, as we show at the end of this lecture. We refer to the book of [Caffarelli and Salsa, Geometric Approach to Free Boundary Problems], Chapter 12 for a detailed proof of the Friedland-Hayman inequality.

5.3. **Generalizations.** If u is a nonnegative subharmonic function, then $J(r,u)$ can be controlled in terms of L^2-norm of u, precisely as we have done for harmonic functions in §5.1. The only difference is that we have to use the inequality $|\nabla u|^2 \leq \Delta(u^2/2)$ instead of the equality there. Thus, one has

$$J(1/2, u) \leq C_n \|u\|^2_{L^2(B_1)}.$$

Hence, one also has the following variant of the monotonicity formula, which takes the form of an estimate.

Theorem 5.2. *Let $u_\pm$ be as in Theorem 5.1. Then*

$$\Phi(r, u_+, u_-) \leq C_n \|u_+\|^2_{L^2(B_1)} \|u_-\|^2_{L^2(B_1)}$$

for $0 < r \leq 1/2$.

In some applications, this weaker form of the monotonicity formula turns out to be sufficient. However, in other applications, one needs to use Theorem 5.1 at its full strength, moreover, one needs to have information on the case of $\Phi(r)$ being a constant in some interval.

Theorem 5.3. *Let $u_\pm$ be as in Theorem 5.1 and suppose that $\Phi(r_1) = \Phi(r_2)$ for some $0 < r_1 < r_2 < 1$. Then one of the following holds:*

(i) *either* $u_+ = 0$ *in* B_{r_2} *or* $u_- = 0$ *in* B_{r_2},
(ii) *for every* $r_1 < r < r_2$, $\operatorname{supp} u_\pm \cap \partial B_r$ *is a half-spherical cap and* $u_+ \Delta u_+ = u_- \Delta u_- = 0$ *in the sense of measures.*

This follows directly from analyzing the proof of the ACF Monotonicity Formula, in particular, from analyzing the case of inequality in Friedland-Hayman inequality: $\alpha_+ + \alpha_- = 2$ if and only if $\Sigma_\pm$ are complementary half-spherical caps. For more details we refer to the paper by Caffarelli-Karp-Shahgholian [CKS] where this theorem first appeared.

Next, we state a generalization of the ACF Monotonicity Formula, due to Caffarelli, Jerison and Kenig.

Theorem 5.4 (Caffarelli-Jerison-Kenig (CJK) Estimate)**.** *Let* $u_\pm \in W^{1,2}(B_1)$ *be a pair of nonnegative continuous functions satisfying* $\Delta u_\pm \geq -1$ *in* B_1 *the sense of distributions and such that* $u_+ \cdot u_- = 0$ *in* B_1. *Then*

$$\Phi(r, u_+, u_-) \leq C_n(1 + J(1, u_+) + J(1, u_-))^2, \quad 0 < r < 1.$$

This estimate still has some features of the ACF Monotonicity Formula, so sometimes it is referred to as CJK Almost Monotonicity Formula. The proof can be found in the original paper [CJK].

If u is a nonnegative continuous functions such that $\Delta u \geq -1$ in B_1, then using $2|\nabla u|^2 \leq \Delta(u^2) + 2u$, one can show that

$$J(1/2, u) \leq C_n \left(1 + \|u\|^2_{L^2(B_1)}\right).$$

This leads to the following form of the CJK estimate, akin to Theorem 5.2.

Theorem 5.5. *Let* $u_\pm$ *be as in Theorem 5.4. Then*

$$\Phi(r, u_+ . u_-) \leq C_n \left(1 + \|u_+\|^2_{L^2(B_1)} + \|u_-\|^2_{L^2(B_1)}\right)^2$$

for $0 < r \leq 1/2$.

We will also need to use the following form of the CJK estimate, which has more feature of a monotonicity formula. However, one needs to assume more on the growth of function $u_\pm$ near the origin.

Theorem 5.6. *Let* $u_\pm$ *be as in Theorem 5.4 and assume additionally that* $u_\pm(x) \leq C_0|x|^\epsilon$ *in* B_1 *for some* $\epsilon > 0$. *Then there exists* $C_1 = C(C_0, n, \epsilon)$ *such that*

$$\Phi(r_1) \leq (1 + r_2^\epsilon)\Phi(r_2) + C_1 r_2^{2\epsilon}$$

for $0 < r_1 \leq r_2 < 1$. *In particular, the limit* $\Phi(0+)$ *exists.*

Reduction to Friedland-Hayman inequality

Here we follow [Caffarelli, The Obstacle Problem].

We start with a remark that the functional J scales linearly, in the sense that if

$$u_\lambda(x) = \frac{1}{\lambda} u(\lambda x),$$

then

$$J(r/\lambda, u_\lambda) = J(r, u).$$

In particular, this implies that we can assume $u_\pm$ to be defined in B_R for a certain $R > 1$. Then it will suffice to show that $\Phi'(r) \geq 0$ only for $r = 1$.

It will be convenient to introduce

$$I(r,u)=\int_{B_r}\frac{|\nabla u|^2}{|x|^{n-2}}dx$$

Thus, $J(r,u)=\frac{1}{r^2}I(r,u)$ and $\Phi(r,u_+,u_-)=\frac{1}{r^4}I(r,u_+)I(r,u_-)$. We also denote $I_\pm=I(\cdot,u_\pm)$. We have

$$\Phi'(1)=I_+'I_-+I_+I_-'-4I_+I_-.$$

Thus, we want to show

$$\frac{I_+'}{I_+}+\frac{I_-'}{I_-}\geq 4.$$

We now want to rewrite this as an inequality on the unit sphere. To this end, for $u=u_\pm$, let $\Sigma=\{u>0\}\cap\partial B_1$. Then we have

$$I(1,u)=\int_{B_1}\frac{|\nabla u|^2}{|x|^{n-2}}dx\leq\int_{B_1}\frac{\Delta\left(\frac{u^2}{2}\right)}{|x|^{n-2}}dx=\int_\Sigma\left(u\,\partial_r u+\frac{n-2}{2}u^2\right)d\theta,$$

using $\int u\Delta v-v\Delta u=\int uv_\nu-vu_\nu$. On the other hand,

$$I'(1,u)=\int_\Sigma|\nabla u|^2d\theta.$$

Thus,

$$\frac{I'(1,u)}{I(1,u)}\geq\frac{\displaystyle\int_\Sigma[(\partial_r u)^2+|\nabla_\theta u|^2]d\theta}{\displaystyle\int_\Sigma[u\,\partial_r u+\frac{n-2}{2}u^2]d\theta}$$

Note at this point that

$$\frac{\displaystyle\int_\Sigma|\nabla_\theta u|^2}{\displaystyle\int_\Sigma u^2}\geq\lambda,$$

where $\lambda=\lambda(\Sigma)$ is the first eigenfunction of the spherical Laplacian Δ_θ in Σ, so we want to split $u\,\partial u_r$ in an optimal fashion, to spread its control between $\int(\partial_r u)^2$ and $\int|\nabla_\theta u|^2$, i.e.,

$$\int_\Sigma u\,\partial_r u\leq\frac{1}{2}\left[A\int_\Sigma u^2+\frac{1}{A}\int_\Sigma(\partial_r u)^2\right].$$

This will leave us with

$$2\frac{\displaystyle\int_\Sigma(\partial_r u)^2+|\nabla_\theta u|^2}{\displaystyle\frac{1}{A}\int_\Sigma(\partial_r u)^2+(A+n-2)\int_\Sigma u^2}.$$

To perfectly balance both terms, we want

$$\frac{1}{A}=\frac{A+n-2}{\lambda},\quad\text{or}\quad A[A+n-2]=\lambda$$

This choice will give us

$$\frac{I'(1,u)}{I(1,u)}\geq 2A.$$

But now observe that A is precisely the homogeneity of the homogeneous harmonic function, constructed from the first eigenfunction of the spherical Laplacian in Σ. So, if $\Sigma_\pm = \{u_\pm > 0\} \cap \partial B_1$, then these are disjoint open sets on ∂B_1 and if $A_\pm$ are the corresponding homogeneities, then we have

$$\frac{I'_+}{I_+} + \frac{I'_-}{I_-} - 4 \geq 2(A_+ + A_- - 2)$$

and therefore the required inequality will follow from the Friedland-Hayman inequality

$$A_+ + A_- - 2 \geq 0.$$