LECTURE 6

6. Optimal Regularity (Continued)

6.1. **Obstacle-type Problems.** In this section, following the idea of Shahgholian, we use the Caffarelli-Jerison-Kenig estimate from the previous lecture to prove the optimal regularity in "no-sign", superconductivity and two-phase obstacle problem. In fact, we place these equations into a more general framework and establish the $C^{1,1}$ regularity there.

Namely, suppose that we are given a function $u \in W^{2,p}(D)$, $p > n$, which satisfies

$$\Delta u = g \quad \text{in} \quad D, \tag{6.1}$$

in a sense of distribution for some $g \in L^\infty(D)$. Suppose further that there exist an open subset $G \subset D$ such that

$$|\nabla u| = 0 \quad \text{in} \quad D \setminus G \tag{6.2}$$

and in G the right-hand side is given by

$$g(x) = f(x, u(x)), \quad x \in G, \tag{6.3}$$

where $f : G \times \mathbb{R} \to \mathbb{R}$ satisfies the following structural conditions: there exists $M_1, M_2 > 0$ such that

$$|f(x,t) - f(y,t)| \le M_1 |x - y|, \quad x, y \in G,\ t \in \mathbb{R} \tag{6.4}$$

$$f(x,s) - f(x,t) \ge -M_2(s-t), \quad x \in G,\ s,t \in \mathbb{R},\ s \ge t. \tag{6.5}$$

Locally, these conditions are equivalent to

$$|\nabla_x f(x,t)| \le M_1, \quad \partial_t f(x,t) \ge -M_2$$

in the sense of distributions.

Let us now see how Problems A–C fit into this framework.

- Problem A: No-sign obstacle problem
$$\Delta u = f(x)\chi_\Omega, \quad \Omega = D \setminus \{u = |\nabla u| = 0\}$$
with $f \in C^{0,1}(D)$. Here we take $G = \Omega$.
- Problem B: Superconductivity problem
$$\Delta u = f(x)\chi_\Omega, \quad \Omega = \{|\nabla u| > 0\}.$$
with $f(x) \in C^{0,1}(D)$. In this problem we also take $G = \Omega$.
- Problem C: Two-phase obstacle problem
$$\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} \quad \text{with} \quad \lambda_+ + \lambda_- \ge 0.$$
Here me take $G = D$.

Theorem 6.1 ($C^{1,1}$-regularity)**.** *Let $u \in L^\infty(D)$ satisfy (6.1)–(6.5). Then $u \in C^{1,1}_{\mathrm{loc}}(D)$ and*

$$\|u\|_{C^{1,1}(K)} \le CM \left(1 + \|u\|_{L^\infty(D)} + \|g\|_{L^\infty(D)}\right),$$

for any open $K \subset\subset D$, where $C = C(n, K, D)$ and $M = \max\{1, M_1, M_2\}$.

The proof is based on the following lemma which is a direct consequence of the structural assumptions on f and u.

Lemma 6.2. *Let $u \in C^1(D)$ satisfy (6.1)–(6.5). Then for any unit vector e,*

$$\Delta(\partial_e u)^{\pm} \geq -L \quad \text{in } D,$$

where $L = M_1 + M_2\|\nabla u\|_{L^\infty(D)}$.

Proof. Fix a direction e and let $v = \partial_e u$. Let also

$$E := \{v > 0\}.$$

Note that $E \subset G$ because of the assumption (6.2). Then, formally, for $x \in E$,

$$\begin{aligned}\Delta(v^+) = \partial_e \Delta u(x) &= e \cdot \nabla_x f(x,u) + \partial_t f(x,u) D_e u\\ &\geq -M_1 - M_2\|\nabla u\|_{L^\infty(D)} =: -L.\end{aligned}$$

To justify this computation, observe that $\Delta(v^+) \geq -L$ in D is equivalent to the inequality

$$-\int_D \nabla(v^+)\nabla\eta\,dx \geq -L\int_D \eta\,dx \tag{6.6}$$

for any nonnegative $\eta \in C_0^\infty(D)$. Suppose first that $\operatorname{supp}\eta \subset \{v > \delta\}$ with $\delta > 0$. Then writing the equation

$$-\int_D \nabla u \nabla\eta\,dx = \int_D f\eta\,dx$$

with $\eta = \eta(x)$ and $\eta = \eta(x-he)$, we obtain an equation for the incremental quotient

$$v_{(h)}(x) := \frac{u(x+he)-u(x)}{h}.$$

Namely, we obtain

$$-\int_D \nabla v_{(h)}\nabla\eta\,dx = \frac{1}{h}\int_D [f(x+he, u(x+he)) - f(x,u(x))]\eta\,dx \tag{6.7}$$

for small $h > 0$. Note that $u(x+he) > u(x)$ on $\operatorname{supp}\eta \subset \{v > \delta\}$ and from the hypotheses on f we have

$$\begin{aligned}&f(x+he,u(x+he)) - f(x,u(x))\\ &\qquad\geq [f(x+he,u(x+he)) - f(x+he,u(x))] + [f(x+he,u(x)) - f(x,u(x))]\\ &\qquad\geq -M_1 h - M_2[u(x+he)-u(x)]\end{aligned}$$

for small h. Letting in (6.7) $h \to 0$ and then $\delta \to 0$ we arrive at

$$\begin{aligned}-\int_D \nabla v\nabla\eta\,dx &\geq -\int_D (M_1 + M_2 v)\eta dx\\ &\geq -L\int \eta\,dx\end{aligned}$$

for arbitrary $\eta \geq 0$ with $\operatorname{supp}\eta \subset\subset \{v > 0\}$.

Thus, we proved that $\Delta v \geq -L$ in the open set $E = \{v > 0\}$ in the sense of distributions. Then it is a simple exercise to show that (6.6) holds for any nonnegative $\eta \in C_0^\infty(D)$.

To prove the same inequality for v^-, we simply reverse the direction e.

□

The proof of the $C^{1,1}$-regularity theorem that we give below is based on the application of an estimate by Caffarelli, Jerison and Kenig that we stated in Lecture 5.

Proof of Theorem 6.1. We start by observation that u is twice differentiable a.e. in D, since $u \in W^{2,p}_{\mathrm{loc}}(D)$ with $p > n$, see e.g. Theorem 1.72 in [Maly-Ziemer]. Then fix a point $x_0 \in K \subset\subset D$ where u is twice differentiable and define

$$v(x) = \partial_e u(x)$$

for a unit vector e orthogonal to $\nabla u(x_0)$ (if $\nabla u(x_0) = 0$, take arbitrary unit e). Without loss of generality we may assume $x_0 = 0$. Our aim is to obtain a uniform estimate for $\partial_{x_j e} u(0) = \partial_{x_j} v(0)$, $j = 1, \dots, n$. By construction, $v(0) = 0$ and v is differentiable at 0. Hence, we have the Taylor expansion

$$v(x) = \xi \cdot x + o(|x|), \quad \xi = \nabla v(0).$$

Now, if $\xi = 0$ then $\partial_{x_j} v(0) = 0$ for all $j = 1, \dots, n$ and there is nothing to estimate. If $\xi \neq 0$, consider the cone

$$\Gamma = \{x \in \mathbb{R}^n : \xi \cdot x \geq |\xi||x|/2\},$$

which has a property that

$$\Gamma \cap B_r \subset \{v > 0\}, \quad -\Gamma \cap B_r \subset \{v < 0\}$$

for sufficiently small $r > 0$. Consider also the rescalings

$$v_r(x) = \frac{v(rx)}{r}, \quad x \in B_1.$$

Note that $v_r(x) \to v_0(x) := \xi \cdot x$ uniformly in B_1 and consequently $\nabla v_r^\pm \to \nabla v_0^\pm$ weakly in $L^2(B_1)$. Then by Fatou's lemma, we have

$$\begin{aligned} c|\xi|^4 &= \int_{\Gamma \cap B_1} \frac{|\nabla v_0^+(x)|^2 dx}{|x|^{n-2}} \int_{\Gamma \cap B_1} \frac{|\nabla v_0^-(x)|^2 dx}{|x|^{n-2}} \\ &\leq \liminf_{r \to 0} \frac{1}{r^4} \int_{\Gamma \cap B_r} \frac{|\nabla v^+(x)|^2 dx}{|x|^{n-2}} \int_{-\Gamma \cap B_r} \frac{|\nabla v^-(x)|^2 dx}{|x|^{n-2}} \\ &\leq \liminf_{r \to 0} \Phi(r, v^+, v^-), \end{aligned}$$

where Φ is as in ACF Monotonicity Formula (see Lecture 5). In the next step we apply the CJK estimate (see Lecture 5), however we should suitably adjust (scale) $v^\pm$ first. Let now $\delta = \frac{1}{2} \operatorname{dist}(K, \partial D)$ and $K_\delta = \{\operatorname{dist}(\cdot, K) < \delta\}$. By Lemma 6.2, we have $\Delta v^\pm \geq -L_\delta$ in $B_\delta(x_0) \subset K_\delta$, where $L_\delta = M_1 + M_2 \|\nabla u\|_{L^\infty(K_\delta)}$. Then it is easy to check that the rescalings

$$w_\pm(x) = \frac{v^\pm(\delta x)}{L_\delta \delta^2}, \quad x \in B_1$$

satisfy all hypotheses in CJK estimate. Hence, we have

$$\begin{aligned}
c|\xi|^4 &\leq \liminf_{r\to 0}\Phi(r,v^+,v^-) = CL_\delta^4\delta^4 \lim_{r\to 0}\Phi(r,w^+,w^-)\\
&\leq CL_\delta^4\delta^4\left(1+\|w_+\|^2_{L^2(B_1)}+\|w_-\|^2_{L^2(B_1)}\right)^2\\
&\leq CL_\delta^4\delta^4\left(1+\frac{\|\nabla u\|^2_{L^\infty(K_\delta)}}{L_\delta^2\delta^4}\right)^2\\
&\leq C\left(L_\delta^2\delta^2+\frac{\|\nabla u\|^2_{L^\infty(K_\delta)}}{\delta^2}\right)^2 \leq CL^4,
\end{aligned}$$

where $C = C(n, K, D)$ and

$$L = M(1+\|\nabla u\|_{L^\infty(K_\delta)}) \leq C(\|u\|_{L^\infty(D)}+\|g\|_{L^\infty(D)})$$

with $M = \max\{1, M_1, M_2\}$. Recalling now that $\xi = \nabla\partial_e u(x_0)$, we arrive at

$$|\nabla\partial_e u(x_0)| \leq CL.$$

This doesn't give the desired estimate on $|D^2u|$ yet, since e is subject to the condition $e\cdot\nabla u(x_0) = 0$, unless $\nabla u(x_0) = 0$. If $\nabla u(x_0) \neq 0$, choose the coordinate system so that $\nabla u(x_0)$ is parallel to e_1. Then, taking $e = e_2, \dots, e_n$ in the estimate above, we obtain

$$|\partial_{x_ix_j}u(x_0)| \leq CL, \quad i = 2,\dots,n, \quad j = 1,2,\dots,n$$

To obtain the estimate in the missing direction e_1, we use the equation $\Delta u = g$:

$$\begin{aligned}
|\partial_{x_1x_1}u(x_0)| &\leq |\Delta u(x_0)| + |\partial_{x_2x_2}u(x_0)| + \ldots + |\partial_{x_nx_n}u(x_0)|\\
&\leq \|g\|_{L^\infty(D)} + CL.
\end{aligned}$$

This completes the proof of the theorem. $\square$