LECTURE 7

7. Optimal Regularity (Continued)

7.1. **A counterexample.** In this section we follow Andersson and Weiss to describe an example of a non-$C^{1,1}$ two-dimensional solution of the equation

$$\Delta u = -\chi_{\{u>0\}},$$

known as the unstable obstacle problem. This shows in particular the importance of the condition (6.5) in Theorem 6.1.

Let B_1 be a unit ball in $\mathbb{R}^2$ and $0 < \alpha < 1$ and n a positive integer. Consider then a subset $C^\alpha_{*n}(\overline{B}_1)$ of $C^\alpha(\overline{B}_1)$ of functions $u(x_1, x_2)$ that satisfy

$$u(x_1, -x_2) = u(x_1, x_2), \qquad u \circ U_{2\pi/n} = u,$$

where U_θ is a rotation by angle θ in the counterclockwise direction. In other words, functions in $C^\alpha_{*n}(\overline{B}_1)$ are obtained from their restriction on the sector with $0 \leq \theta \leq \pi/n$ (using the polar coordinates) by even reflection with respect to the rays $\theta = k\pi/n$, $k = 0, \ldots, 2n-1$.

Similarly, we define the subspace $C^\alpha_{*n}(\partial B_1)$.

Proposition 7.1. *For any $g \in C^\alpha_{*n}(\partial B_1)$ there exists a constant κ such that the boundary value problem*

$$\begin{aligned} \Delta u &= -\chi_{\{u>0\}} && \text{in} \quad B_1 \\ u &= g - \kappa && \text{on} \quad \partial B_1 \end{aligned}$$

has a solution $u \in C^\alpha_{*n}(B_1) \cap C^{1,\beta}_{\mathrm{loc}}(\overline{B}_1) \cap W^{2,p}_{\mathrm{loc}}(B_1)$, which also satisfies $u(0) = 0$.

Proof. For $\epsilon > 0$ let $f_\epsilon \in C^\infty(\mathbb{R})$ be a mollification of $\chi_{\{s>0\}}$ such that

$$\chi_{\{s>0\}} \leq f_\epsilon \leq \chi_{\{s>-\epsilon\}}$$

Consider now the operator $T_\epsilon : C^\gamma_{*n}(\overline{B}_1) \to C^\gamma_{*n}(\overline{B}_1)$ with $\gamma < \alpha$ given by solving the Poisson problem

$$\begin{aligned} \Delta T_\epsilon(u) &= -f_\epsilon(u - u(0)) && \text{in} \quad B_1 \\ T_\epsilon(u) &= g && \text{on} \quad \partial B_1. \end{aligned}$$

By the theory of strong solutions and the symmetry we have $T_\epsilon(u) \in C^\alpha_{*n}(\overline{B}_1)$. In particular, T_ϵ is a compact operator on $C^\gamma_{*n}(\overline{B}_1)$ into itself. Moreover,

$$\|T_\epsilon u\|_{C^\alpha(\overline{B}_1)} \leq C(n, g)$$

for any $u \in C^\gamma_{*n}(\overline{B}_1)$. Then by the Schauder fixed point theorem there exists a fixed point u_ϵ of the operator T_ϵ,

$$T_\epsilon(u_\epsilon) = u_\epsilon,$$

or in other words, a solution of the semilinear problem

$$\begin{aligned} \Delta u_\epsilon &= -f_\epsilon(u_\epsilon - u_\epsilon(0)) && \text{in} \quad B_1 \\ u_\epsilon &= g && \text{on} \quad \partial B_1. \end{aligned}$$

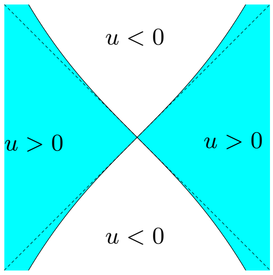

FIGURE 1. Non-$C^{1,1}$ solution of $\Delta u = -\chi_{\{u>0\}}$ with a cross-shaped singularity

The family $\{u_\epsilon\}$ is uniformly bounded in C^α-norm on $\overline{B}_1$ and in $W^{2,p}$-norm on any $B_{1-\delta}$, $\delta > 0$. Therefore for a subsequence $\epsilon = \epsilon_k \to 0$ $u_\epsilon - u_\epsilon(0)$ converges uniformly to a function $u_0 \in C^\alpha(\overline{B}_1) \cap W^{2,p}_{\mathrm{loc}}(B_1)$. Moreover, since

$$\chi_{\{u_0>\delta\}} \leq f^\epsilon(u_\epsilon - u_\epsilon(0)) \leq \chi_{\{u_0>-\delta\}}$$

for $\delta > 0$ and sufficiently small $\epsilon > 0$, in the limit we obtain that

$$-\chi_{\{u_0\geq 0\}} \leq \Delta u_0 \leq -\chi_{\{u_0>0\}}$$

weakly in B_1. On the other hand, $\Delta u_0 = 0$ a.e. on $\{u_0 = 0\}$, since $u_0 \in W^{2,p}_{\mathrm{loc}}(B_1)$. Hence,

$$\Delta u_0 = -\chi_{\{u_0>0\}}$$

weakly in B_1. Moreover, it is immediate that $u_0 = g - \kappa$ on ∂B_1 for a constant $\kappa = \lim_{k\to\infty} u_{\epsilon_k}(0)$. Finally, $|\nabla u_0(0)| = 0$ follows from the symmetry and the $C^{1,\beta}_{\mathrm{loc}} \cap W^{2,p}_{\mathrm{loc}}$ regularity of u_0 follows from the elliptic L^p estimates (see Theorem 2.1). $\square$

Proposition 7.2. *Let u be the solution of $\Delta u = -\chi_{\{u>0\}}$ in B_1 with*

$$u(x_1, x_2) = M(x_1^2 - x_2^2) - \kappa, \qquad (x_1, x_2) \in \partial B_1$$

obtained by Proposition 7.1. Then $u \notin C^{1,1}_{\mathrm{loc}}(B_1)$ if M is sufficiently large. Moreover,

$$\frac{u(rx)}{\|u\|_{L^2(\partial B_r)}} \to \frac{x_1^2 - x_2^2}{\|x_1^2 - x_2^2\|_{L^2(\partial B_1)}} \qquad \text{as} \quad r \to 0$$

The latter convergence implies that the origin is a cross-shaped singularity, see Fig. 1.

At the moment, we postpone the proof of this proposition, as we will need to know more about so-called Weiss monotonicity formulas.