LECTURE 8

8. Preliminary Analysis of the Free Boundary

In this lecture we start the analysis of the free boundary. Our main focus is on the obstacle-type problems that we stated in Lecture 2. Namely, we study the following three problems

$$\begin{aligned}
&\textit{Problem A:}\quad \Delta u = \chi_\Omega \quad \text{in } D, \quad \Omega = D \setminus \{u = |\nabla u| = 0\}\\
&\textit{Problem B:}\quad \Delta u = \chi_\Omega \quad \text{in } D, \quad \Omega = D \setminus \{|\nabla u| = 0\}\\
&\textit{Problem C:}\quad \Delta u = \lambda_+\chi_{\{u>0\}} - \lambda_-\chi_{\{u<0\}} \quad \text{in } D, \quad \lambda_\pm > 0.
\end{aligned}$$

Here D is a domain in $\mathbb{R}^n$ and we will assume that the solution $u \in C^{1,1}(D)$ throughout this lecture and that the equations are satisfied in the a.e. sense.

8.1. **Nondegeneracy.** The first property we discuss is the nondegeneracy property of the solutions, which is in a sense opposite to $C^{1,1}$ estimates. This is going to be important when studying the blowups of the solutions.

In Problems A, B, for any $x_0 \in \Gamma = \Omega(u) \cap D$ we have the estimate

$$\sup_{B_r(x_0)} u \leq u(x_0) + \frac{M}{2}|x - x_0|^2,$$

if $B_r(x_0) \subset\subset D$, where $M = \|D^2 u\|_{L^\infty(D)}$. However, the $C^{1,1}$ regularity doesn't exclude that $u(x) - u(x_0)$ will decay faster than quadratically at x_0. This is taken care of by the following lemma.

Lemma 8.1 (Nondegeneracy: Problem A)**.** *Let u be a solution of Problem A in D. Then we have the inequality*

$$\sup_{\partial B_r(x_0)} u \geq u(x_0) + \frac{r^2}{8n}, \quad \textit{for any } x_0 \in \overline{\Omega(u)} \tag{8.1}$$

provided $B_r(x_0) \subset\subset D$.

Remark 8.2. Since u is subharmonic if it solves Problem A, we can replace sup over $\partial B_r(x_0)$ to one over $B_r(x_0)$, obtaining an equivalent statement.

Before giving the proof, consider a similar nodegeneracy statament for solutions of $\Delta u = 1$.

Lemma 8.3. *Let u safisfy $\Delta u = 1$ in a ball B_R. Then*

$$\sup_{\partial B_r} u \geq u(0) + \frac{r^2}{2n}, \quad 0 < r < R$$

Proof. Consider the auxuliary function

$$w(x) = u(x) - \frac{|x|^2}{2n}, \quad x \in B_R.$$

Then w is harmonic in B_R. Therefore by the maximum principle we obtain that

$$w(0) \le \sup_{\partial B_r} w = \left(\sup_{\partial B_r} u\right) - \frac{r^2}{2n},$$

which implies the required inequality. □

Proof of Lemma 8.1. 1) Assume first that $x_0 \in \Omega(u)$ and moreover $u(x_0) > 0$. Consider then the auxuliary function

$$w(x) = u(x) - u(x_0) - \frac{|x - x_0|^2}{2n}, \tag{8.2}$$

similar to the one in the proof the previous lemma. We have $\Delta w = 0$ in $B_r(x_0) \cap \Omega$. Since $w(x_0) = 0$, by the maximum principle we have that

$$\sup_{\partial(B_r(x_0)\cap\Omega)} w \ge 0.$$

Besides, $w(x) = -u(x_0) - |x - x_0|^2/(2n) < 0$ on $\partial\Omega$. Therefore, we must have

$$\sup_{\partial B_r(x_0)\cap\Omega} w \ge 0.$$

The latter is equivalent to

$$\sup_{\partial B_r(x_0)\cap\Omega} u \ge u(x_0) + \frac{r^2}{2n}$$

and the lemma is proved in this case.

2) Suppose now $x_0 \in \Omega(u)$ and $u(x_0) \le 0$. If $B_{r/2}(x_0)$ contains a point x_1 such that $u(x_1) > 0$, then

$$\sup_{B_r(x_0)} u \ge \sup_{B_{r/2}(x_1)} u \ge u(x_1) + \frac{(r/2)^2}{2n} \ge u(x_0) + \frac{r^2}{8n}.$$

If it happens that $u \le 0$ in $B_{r/2}(x_0)$, from subharmonicity of u and the strong maximum principle we will have that either $u = 0$ identically in $B_{r/2}(x_0)$, or $u < 0$ in $B_{r/2}(x_0)$. The former case is impossible, as $x_0 \in \Omega(u)$, and the latter case implies that $B_{r/2}(x_0) \subset \Omega(u)$ and therefore $\Delta u = 1$ in $B_{r/2}(x_0)$. Then Lemma 8.3 finishes the proof in this case and we obtain

$$\sup_{B_r(x_0)} u \ge \sup_{B_{r/2}(x_0)} u \ge u(x_0) + \frac{r^2}{8n}.$$

3) Finally, for $x_0 \in \overline{\Omega(u)}$, we take a sequence $x_n \in \Omega(u)$ such that $x_n \to x_0$ and pass to the limit in the corresponding nondegeneracy inequality at x_n. □

Even though the proof above does not work for Problem B in general, we still have the nondegeneracy.

Lemma 8.4 (Nondegeneracy: Problem B)**.** *Let u be a solution of Problem B in D. Then we have the inequality*

$$\sup_{\partial B_r(x_0)} u \ge u(x_0) + \frac{r^2}{2n}, \quad \text{for any } x_0 \in \overline{\Omega(u)} \tag{8.3}$$

provided $B_r(x_0) \subset\subset D$.

Proof. By continuity, it suffices to obtain the estimate (8.3) only for points $x_0 \in \Omega(u)$. Note that at those points we have $|\nabla u(x_0)| \neq 0$. Consider now the same auxiliary function w as in (8.2). Then we claim

$$\sup_{\overline{B_r(x_0)}} w = \sup_{\partial B_r(x_0)} w. \tag{8.4}$$

Indeed, if this fails, then the supremum of w is attained at some interior point $y \in B_r(x_0)$, and we would have that $|\nabla w(y)| = 0$, which implies that

$$|\nabla u(y)| = \frac{|y - x_0|}{n}.$$

First off, this implies that $y \neq x_0$, otherwise we would have $|\nabla u(x_0)| = 0$. Therefore $|\nabla u(y)| > 0$ and consequently $y \in \Omega(u)$. Since w is harmonic in $\Omega(u)$, by the strong maximum principle w is constant in a neighborhood of y. Thus, the set of maxima of w is both relatively open and closed in $B_r(x_0)$, which implies that w is constant there and (8.4) is trivially satisfied. □

Finally, in Problem C we have nondegeneracy in both phases, provided $\lambda_\pm > 0$.

Lemma 8.5 (Nondegeneracy: Problem C)**.** *If u is a solution of Problem C in D, then we have*

$$\sup_{\partial B_r(x_0)} u \geq u(x_0) + \lambda_+ \frac{r^2}{2n}, \quad \text{for any } x_0 \in \overline{\Omega_+(u)} \tag{8.5}$$

$$\inf_{\partial B_r(x_0)} u \leq u(x_0) - \lambda_- \frac{r^2}{2n}, \quad \text{for any } x_0 \in \overline{\Omega_-(u)} \tag{8.6}$$

provided $B_r(x_0) \subset\subset D$.

Proof. To prove these inequlities, we consider the auxiliary functions

$$w(x) = u(x) - u(x_0) - \lambda_\pm \frac{|x - x_0|^2}{2n}$$

and argue similarly to part 1) of the proof of Lemma 8.1. We leave the details to the reader.

□

Corollary 8.6 (Nondegeneracy of the gradient)**.** *Under the conditions of Lemmas 8.1, 8.4, 8.5 the following inequality holds*

$$\sup_{B_r(x_0)} |\nabla u| \geq c_0 r,$$

for a positive c_0, depending only on n in Problems A, B, and also on $\lambda_\pm$ for Problem C.

The proof is an application of the mean value theorem and is left as an exercise to the reader.

8.2. **Porosity of the Free Boundary and its Lebesgue measure.**

Definition 8.7. We say that the measurable set $E \subset \mathbb{R}^n$ is *porous* with porosity constant $0 < \delta < 1$ if every ball $B = B_r(x)$ contains a smaller ball $B' = B_{\delta r}(y)$ such that

$$B_{\delta r}(y) \subset B_r(x) \setminus E.$$

We say that E is *locally porous* in D if $E \cap K$ is porous (with possibly different porosity constants) for any $K \subset\subset D$.

It is clear that the Lebesgue upper density of a porous set E

$$d(x) := \limsup_{r\to 0} \frac{|E \cap B_r(x_0)|}{|B_r(x_0)|} \le 1 - \delta^n < 1,$$

which implies that E must have Lebesgue measure zero.

Proposition 8.8. *If $E \subset \mathbb{R}^n$ is porous then $|E| = 0$. If E is locally porous in D, then $|E \cap D| = 0$.* □

An immediate corollary of the nondegeneracy and the $C^{1,1}$ regularity is the following result.

Lemma 8.9 (Porosity of the free boundary)**.** *Let u be a solution of Problem A, B in an open set $D \subset \mathbb{R}^n$. Then $\Gamma(u)$ is locally porous for any $K \subset\subset D$.*

If u is solution of Problem C, then $\Gamma'(u) = \Gamma(u) \cap \{|\nabla u| = 0\}$ is locally porous.

Proof. For Problems A, B, Let $x_0 \in \Gamma(u)$ and $B_r(x_0) \subset\subset D$. Using the nondegeneracy of the gradient (Corollary 8.6), one can find $x_1 \in \overline{B_{r/2}(x_0)}$ such that

$$|\nabla u(x_1)| \ge \frac{c_0}{2}\, r.$$

Now, using that $M = \|D^2 u\|_{L^\infty(D)} < \infty$, we will have

$$\inf_{B_{\delta r}(x_1)} |\nabla u| \ge \left(\frac{c_0}{2} - M\delta\right) r \ge \frac{c_0}{4}\, r, \quad \text{if } \delta = \frac{c_0}{4M}.$$

This implies that

$$B_{\tilde\delta r}(x_1) \subset B_r(x_0) \cap \Omega(u) \subset B_r(x_0) \setminus \Gamma,$$

where $\tilde\delta = \min\{\delta, 1\}$. This implies the porosity condition is satisfied for any ball centered at $\Gamma(u)$. It is a now simple exercise to show that porosity condition is satisfied for any ball $B \subset\subset D$ and therefore $\Gamma(u)$ is locally porous.

For Problem C, the same argument as above shows that

$$B_{\tilde\delta r}(x_1) \subset B_r(x_0) \cap [\Omega(u) \cup \Gamma''(u)] \subset B_r(x_0) \setminus \Gamma'(u),$$

which implies the local porosity of $\Gamma'(u)$. □

Corollary 8.10 (Lebesgue measure of Γ.)**.** *Let u be a solution of Problem A, B, and C in D. Then $\Gamma(u)$ has Lebesgue measure zero.*

Proof. In case of Problems A, B the statement follows immediately from the local porosity of $\Gamma(u)$ and Proposition 8.8.

In the case of Problem C, we obtain $|\Gamma'(u)| = 0$. On the other hand $\Gamma''(u)$ is locally a $C^{1,\alpha}$ curve and therefore also has a Lebesgue measure zero. Hence, $|\Gamma(u)| = 0$ also in this case. □

We finish this subsection with the following observation.

Lemma 8.11 (Density of Ω)**.** *Let u be a solution of Problem A, B, and C in D and $x_0 \in \Gamma(u)$. Then*

$$\frac{|B_r(x_0) \cap \Omega(u)|}{|B_r(x_0)|} \ge \beta, \tag{8.7}$$

provided $B_r(x_0) \subset D$, where β depends only on $\|D^2u\|_{L^\infty(D)}$ and n for Problems A, B and additionally on $\lambda_\pm$ for Problem C.

Proof. The proof of Lemma 8.9 shows that
$$\frac{|B_r(x_0) \cap \Omega(u)|}{|B_r(x_0)|} \geq \tilde{\delta}^n$$
in case of Problems A and B and
$$\frac{|B_r(x_0) \cap \Omega(u)|}{|B_r(x_0)|} = \frac{|B_r(x_0) \cap [\Omega(u) \cup \Gamma''(u)]|}{|B_r(x_0)|} \geq \tilde{\delta}^n$$
in case of Problem C. This completes the proof. □

8.3. **Hausdorff Measure of the Free Boundary.** The porosity if the free boundary not only implies that the its Lebesgue measure is zero but also that it actually has a Hausdorff dimension less than n. A stronger result is as follows.

Lemma 8.12 (Hausdorff measure of Γ)**.** *Let u be a $C^{1,1}$ solution of Problem A, B, or C in an open set $D \subset \mathbb{R}^n$. Then $\Gamma(u)$ is a set of finite $(n-1)$-dimensional Hausdorff measure locally in D.*

Proof. Let
$$v_i = \partial_{x_i} u, \qquad E_\varepsilon = \{0 < |\nabla u| < \varepsilon\}.$$
Observe that
$$c_0 \leq |\Delta u|^2 \leq c_n \sum_{i=1}^n |\nabla v_i|^2 \quad \text{in } \Omega,$$
where $c_0 = 1$ in case of Problems A, B and $c_0 = \min\{\lambda_+^2, \lambda_-^2\}$ for Problem C. Thus, for an arbitrary ball $B \subset\subset D$ we have
$$c_0\,|B \cap E_\epsilon| \leq c_n \int_{B\cap E_\epsilon} \sum_i |\nabla v_i|^2 dx \leq c_n \sum_i \int_{B\cap\{0<|v_i|<\epsilon\}} |\nabla v_i|^2 dx.$$
To estimate the right hand side here we apply Lemma 6.2 (from Lecture 6) noticing that $M_1 = M_2 = 0$ for Problems A, B, and C. It gives
$$\int_D \nabla v_i^\pm \nabla\eta\, dx \leq 0, \quad i = 1, ..., n$$
for any non-negative $\eta \in C_0^\infty(D)$. These inequalities continue to hold for non-negative $\eta \in W_0^{1,2}(D)$. Take $\eta = \psi_\epsilon(v_i^\pm)\phi$, with
$$\psi_\varepsilon(t) = \begin{cases} 0, & t \leq 0 \\ \epsilon^{-1} t, & 0 \leq t \leq \epsilon \\ 1, & t \geq \epsilon \end{cases}$$
and $\phi \in C_0^\infty(D)$, $\phi \geq 0$. We obtain
$$\epsilon^{-1} \int_{B\cap\{0<|v_i|<\epsilon\}} |\nabla v_i|^2 \phi\, dx \leq \int_D |\nabla v_i||\nabla\phi|\, dx \leq c_n M \int_D |\nabla\phi|\, dx$$
In particular, taking $\phi = 1$ on B, after summation by i, we arrive at the estimate
$$c_0\,|B \cap E_\epsilon| \leq C\,\epsilon\, M, \tag{8.8}$$
where $C = C(n, B, D)$.

Consider now a covering of $\Gamma \cap B$ by balls B_i of radius ϵ with centers on $\Gamma \cap B$ and with property that at most N balls may overlap. For Problems A and B, we use (8.7), (8.8) and observe $|\nabla u| \leq M\epsilon$ for in each B_i, which gives

$$\begin{aligned}\sum_i |B_i| \leq \frac{1}{\beta}\sum_i |B_i \cap \Omega| \leq \frac{1}{\beta}\sum_i |B_i \cap E_{M\epsilon}| \\ \leq \frac{N}{\beta}|B \cap E_{M\epsilon}| \leq \frac{cNM\epsilon}{c_0\beta}.\end{aligned}$$

This gives the estimate

$$H^{n-1}(\Gamma(u) \cap B) \leq C(n, M, B, D).$$

For Problem C, the same proof works for Γ', and the rest part of the free boundary Γ'' is smooth. $\square$

Remark 8.13. The estimate (8.8) essentially means

$$|\Omega \cap \{|\nabla u| < \epsilon\}| \leq C\epsilon,$$

which we will use later. In particular, it gives $|\Omega \cap \{|\nabla u| = 0\}| = 0$.