Sven Hagströmer's & Mats Qviberg's Foundation Symposium on Geometry and Regularity of Free Boundaries

February 28 – March 2, 2000
Royal Institute of Technology, Stockholm, Sweden

Abstracts

David R. Adams (Lexington)
Stability of the biharmonic obstacle problem with varying obstacles.

Abstract: We consider the classical obstacle problem for the biharmonic operator on a bounded smooth domain in Euclidean n-space (with zero Dirichlet boundary conditions) and for a smooth obstacle function given on the domain which is negative at the boundary of the domain. The problem addressed is that of determining conditions that insure that the solution of the biharmonic obstacle problem for a sequence of obstacles converges to the solution of the biharmonic obstacle problem for the limit obstacle. It is well known that if the convergence of the sequence of obstacles is too weak, then the corresponding solutions will not converge to the desired limit problem. Such questions have been treated by Gamma convergence methods. However, it is not known when that theory yields results of the type outlined here. The methods discussed in this talk will produce the desired limit. They make use of certain maximal functions from Harmonic Analysis. Several interesting examples result from this approach.

Donatella Danielli (Baltimore)
Singular perturbations of a parabolic free boundary problem arising in combustion theory

Abstract: In this talk I will discuss the uniform properties, and the limit as $\varepsilon \to 0$, of solutions $u^\varepsilon(x,t)$ of the following parabolic singular perturbation problem:

$$Lu^\varepsilon = \operatorname{div} A(x,t)Du^\varepsilon - \partial_t u^\varepsilon + \mathbf{b}(x,t) \cdot Du^\varepsilon + c(x,t)u^\varepsilon = \beta_\varepsilon(u^\varepsilon) \tag{1}$$

in a domain $\mathcal{D} \subset \mathbb{R}^{m+1}$. Here $A(x,t)$ is an uniformly elliptic symmetric matrix with coefficients in $C^1(\mathbb{R}^{m+1})$, $\mathbf{b}(x,t), c(x,t)$ are bounded and measurable functions and, for $\varepsilon > 0$, $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ is smooth, nonnegative and bounded, with $\beta_\varepsilon(s) = 0$ for $s \leq 0$ and support in a small neighborhood of $s = 0$. In particular, my main objective is to show that, under suitable assumptions, (1) is an approximation (as $\varepsilon \to 0$) for a free boundary problem that naturally arises in combustion theory in the analysis of the propagation of curved flames in non-homogeneous media, specifically in the description of laminar flames as an asymptotic limit for high activation energy.

Qing Han (Notre Dame)
Singular sets of solutions to elliptic equations

Abstract: We will discuss the geometric structure and geometric measure of singular sets of solutions to general elliptic equations. We will show that such sets are countably rectifiable under very weak assumptions on coefficients. We will also show the measure estimates on these sets if we have the smoothness of the coefficients. The measure estimates are obtained in terms

of frequency of solutions. A key ingredient is the monotonicity formula by Garofalo-Lin. The method is from algebraic geometry and geometric measure theory. Last we will indicate how to improve such estimates for analytic solutions, especially harmonic functions

Håkan Hedenmalm (Lund)
The Hele-Shaw exponential mapping

Abstract: The exponential mapping of differential geometry is a local homeomorphism from the tangent plane to the manifold, defined so that geodesics (i. e., straight lines) through the origin in the tangent plane are mapped to geodesics through the given point, so that distances are preserved. A classical theorem of Hadamard (1898) states that in two dimensions, the exponential mapping is a global homeomorphism onto the surface provided the surface is simply connected and has negative Gaussian curvature everywhere. We extends Hadamard's theorem to the context of Hele-Shaw flow (the classical theorem concerns metric flow), which corresponds to a Newtonian fluid instead of a photonic gas. New problems arise, such as that of ergodicity of Wrapped Hele-Shaw flow on compact negatively curved surfaces.

David Jerison (Cambridge)
Existence and regularity for two-phase free boundary problems

Abstract: We will discuss joint work with L. A. Caffarelli and C. E. Kenig on existence and regularity of elliptic free boundary problems. These problems are motivated by the example of the Prandtl-Batchelor problem in two variables. The novelty is that the inhomogeneous term can be discontinuous across the free boundary.

Lavi Karp (Haifa)
A free boundary problem and Newtonian potential

Abstract: Suppose Ω is a bounded domain in $\mathbb{R}^n$ with real analytic boundary $\partial\Omega$, and suppose f is a real analytic function in a neighborhood of Ω. Then, as a consequence of Cauchy Kovalewskii's theorem, the Newtonian potential of the density f over Ω (which is harmonic out of Ω), has a harmonic continuation into Ω.
We are considering the inverse problem, that is, suppose the Newtonian potential of f has a harmonic continuation into Ω, does it implies that $\partial\Omega$ is real analytic. We shall see that the answer to that question rely upon the investigation of *global solutions*:

$$\begin{cases} \Delta u = 1 & \text{in } \Omega, \\ u = |\nabla u| = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \\ u(x) = O(|x|^2) & \text{as } x \to \infty. \end{cases}$$

We will also discuss the classification of global solutions and its relation to Newton's theorem which asserts that homeoidal ellipsoid induces no gravity force in the cavity. This talk based on a joint work with L. Caffarelli and H. Shahgholian.

Bernd Kawohl (Köln)
Isoperimetric inequalities for plate eigenvalues

Abstract: The well-known Faber-Krahn inequality states that among all clamped membranes of given area the circular one has minimal fundamental frequency. In my lecture I shall recall

different methods of proof for it and report on related results for plate eigenvalues. These lead to new questions on the regularity of sets which minimize eigenvalues.

Claudia Lederman (Buenos Aires)
Uniqueness and agreement of solution in a two phase free boundary problem from combustion

Abstract: We consider the following two phase free boundary problem: find a function $u(x,t)$, defined in $\mathcal{D} \subset \mathbb{R}^N \times (0,T)$, satisfying that

$$\begin{aligned} &\Delta u + \sum a_i(x,t)\, u_{x_i} - u_t = 0 \quad \text{in} \quad \mathcal{D} \cap \{u \neq 0\}, \\ &u = 0 \;\;,\;\; |\nabla_x u^+|^2 - |\nabla_x u^-|^2 = 2M \quad \text{on } \mathcal{D} \cap \partial\{u > 0\}, \end{aligned}$$

($M > 0$ constant). In addition, Dirichlet or Neumann data are specified on the parabolic boundary of $\mathcal{D}$.
This problem arises in combustion theory as a limit situation in the propagation of premixed flames (high activation energy limit).
The problem admits classical solutions only for good data and for small times. Different generalized concepts of solution have been proposed, among them the concepts of limit solution and viscosity solution. In the work we present here we find conditions under which the concepts of classical, limit and viscosity solution agree and produce a unique solution. This is a joint work with J.L. Vazquez and N. Wolanski.

Peter A. Markowich (Vienna)
Singular limits of mean field equations

Abstract: Mean field equations describe thermal equilibrium states of interacting species of particles in plasmas. Typically, these equations take the form of a semilinear elliptic equation with a nonlocal L1-nonlinearity. We show that mean field equations arise as large time limits of (singular) diffusion equations with Poisson coupling. The main tool in this proof are generalized Sobolev inequalities and the main difficulty in the proof lies in the limited regularity due to the occurance of free boundaries in the porous medium case. Also, the singular perturbation limit is analysed - leading, in certain parameter regimes, to a double obstacle problem. Here, the free boundaries are of particular physical importance since they separate vacuum from non-vacuum regions. The singular perturbation analysis is based on the dual variational formulation with a functional which is not necessarily coercive (in the fast diffusion case).Particular emphasis is given to the convergence (and geometry) of the free boundary in the singular perturbation limit (porous medium case). The talk is based on a joint papers with L. Caffarelli, J. Dolbeault, C. Schmeiser and A. Unterreiter.

Jorge Salazar (Lisbon)
Regularity of viscosity solutions of fully nonlinear elliptic equations with free boundaries

Abstract: We study some properties of viscosity solutions of fully nonlinear elliptic equations of the form

$$F(D^2u, x) = g(x,u) \;\;\text{in}\;\; \{\nabla u \neq 0\}$$

where the solution u is continuous. $\{\nabla u \neq 0\}$ is interpreted as a condition on the test functions, which is a very weak condition.

We establish existence of solutions under some restrictions on F. As a first step in the study of regularity, we eliminate the condition on the gradient by showing that the solutions verify two elliptic inequalities. This result allows us to apply the powerful machinery of the nonlinear elliptic theory and obtain the Alexandroff-Bakelman-Pucci estimate, Harnack's inequality and C^α regularity. We also discuss $W^{2,p}$ regularity, using the notion of L^p-viscosity solutions. In the case of equation $\Delta u = cu$ on $\{\nabla u \neq 0\}$, where c is a positive constant, we prove that u is $C^{1,1}$. When the right hand side g of the equation is strictly positive, we prove that the $(n-1)$-dimensional Hausdorff measure of the free boundary is finite. As an application, we study the equation $\Delta u = u$ on $\{\nabla u \neq 0\}$, on a bounded, convex, plane domain Ω, such that $u \equiv 1$ on $\partial\Omega$. We prove that there is at most one connected component of $\{|\nabla u| = 0\}$ with non empty interior. Besides, such a component is convex.

Sandro Salsa (Milan)
A class of free boundary problems with application to the theory of conductivity

Abstract: We describe some recent results obtained in a joint work with Athanasopoulos and Caffarelli concerning a class of free boundary problems and their application to the uniqueness in a classical inverse problem of conductivity. The regularity of the free boundary in analyzed in details under minimal conditions.

Nina Ural'tseva (St. Petersburg)
How the free boundary can meet the prescribed one

Abstract: The behaviour of free boundaries near the prescribed ones is studied for some classes of free boundary problems. In the case of zero obstacle problem it is proved that the free boundary touches tangentially the given boundary provided the latter is smooth and the boundary values of the solution vanish in a neighbourhood of a contact point. For more general free boundary problem, without sign restrictions on a solution, the same behaviour of the free boundary near contact points is established under certain conditions.

Juan Luis Vazquez (Madrid)
Regularity and dynamics of interfaces in nonlinear parabolic equations

Abstract: Many nonlinear parabolic equations representing processes of diffusion or thermal propagation enjoy the property of finite speed of propagation. Therefore, and speaking of nonnegative solutions, sharp fronts arise separating the regions $\{u > 0\}$ and $\{u = 0\}$. They are called interfaces, free boundaries or moving boundaries. The questions of regularity of the solutions and the fronts, the geometry of the latter have been investigated for years but a rigorous understanding is still in progress. The study of travelling waves has played a key role. The underlying dynamics (Stefan type, Darcy type, combustion type).
In the talk I will address equations of the form

$$u_t = \phi(u)_{xx} + f(u)$$

and show how the combination of the study of travelling waves, the usual methods of a priori estimates plus comparison of lap number (intersection comparison) allow to completely classify the behaviour of solutions and interfaces in the model case $\phi(u) = u^m$ and $f(u) = -u^p$. Novelties appear in the range $m + p \leq 2$ where Darcy's law ceases to explain the movement of the fronts. The technique is explained in the references below

I will show how the technique applies to other one-dimensional problems, among them first-order equations or diffusion equations with first order terms.
A number of related open questions are posed: how to deal with problems in several space dimensions? how to deal with higher order equations. Traveling wave studies show in these cases the expected dynamics and regularity.

References

[1] V.A. Galaktionov, S.I. Shmarev, J.L. Vazquez, Regularity of interfaces in diffusion processes under the influence of strong absorption, *Arch. Rat. Mech. Anal.*, to appear.

[2] V.A. Galaktionov, S.I. Shmarev, J.L. Vazquez, Second-order interface equations for nonlinear diffusion with very strong absorption, *Comm. Contemp. Math.*, **1**, No. 1 (1999), 51–64.

Georg S. Weiss (Tokyo)

The free boundary of a thermal wave in a strongly absorbing medium

Abstract: In dimension $n \geq 2$ we obtain regularity of the free boundary $\partial\{u > 0\}$ of non-negative solutions of the heat equation with strong absorbtion

$$\partial_t u - \Delta u = -\frac{1+\gamma}{2}\, u^\gamma,\ \gamma \in (0,1).$$

Our approach is motivated by methods in Liapunov's stability theory and by results concerning the Plateau problem. We use an *epiperimetric inequality* in order to obtain a *decay estimate* for the energy associated with the equation. By a monotonicity formula this leads to homogeneity improvement and to *unique tangent cones*. The regularity of the free boundary follows then from topological arguments.