

ALMOST MINIMIZERS FOR THE THIN OBSTACLE PROBLEM

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ABSTRACT. We consider Anzellotti-type almost minimizers for the thin obstacle (or Signorini) problem with zero thin obstacle and establish their $C^{1,\beta}$ regularity on the either side of the thin manifold, the optimal growth away from the free boundary, the $C^{1,\gamma}$ regularity of the regular part of the free boundary, as well as a structural theorem for the singular set. The analysis of the free boundary is based on a successful adaptation of energy methods such as a one-parameter family of Weiss-type monotonicity formulas, Almgren-type frequency formula, and the epiperimetric and logarithmic epiperimetric inequalities for the solutions of the thin obstacle problem.

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1. INTRODUCTION AND MAIN RESULTS

1.1. The thin obstacle (or Signorini) problem. Let $D \subset \mathbb{R}^n$ be an open set and $\mathcal{M} \subset \mathbb{R}^n$ a smooth $(n-1)$ -dimensional manifold (the *thin space*) and consider the problem of minimizing the Dirichlet energy

$$(1.1) \quad J_D(u) := \int_D |\nabla u(x)|^2 dx$$

among all functions $u \in W^{1,2}(D)$ satisfying

$$u = g \text{ on } \partial D, \quad u \geq \psi \text{ on } \mathcal{M} \cap D,$$

where $\psi : \mathcal{M} \rightarrow \mathbb{R}$ is the so-called *thin obstacle* and $g : \partial D \rightarrow \mathbb{R}$ is the prescribed boundary data with $g \geq \psi$ on $\mathcal{M} \cap \partial D$. This problem is known as *the thin obstacle problem*. In other words, it is a constrained minimization problem for the energy functional J_D on a closed convex set

$$\mathfrak{K}_{\psi,g}(D, \mathcal{M}) := \{u \in W^{1,2}(D) : u = g \text{ on } \partial D, u \geq \psi \text{ on } \mathcal{M} \cap D\}.$$

This problem can be viewed as a scalar version of the Signorini problem with unilateral constraint from elastostatics [Sig59] and is often referred to as *the Signorini problem*. It goes back to the origins of variational inequalities and is considered as one of the prototypical examples of such problems, see [DL76]. An equivalent formulation is given in the form

$$\begin{aligned} \Delta u &= 0 && \text{on } D \setminus \mathcal{M}, \\ u &= g && \text{on } \partial D, \\ u \geq \psi, \quad \partial_{\nu^+} u + \partial_{\nu^-} u &\geq 0, \quad (\partial_{\nu^+} u + \partial_{\nu^-} u)(u - \psi) &= 0 && \text{on } \mathcal{M} \cap D, \end{aligned}$$

where the conditions on $\mathcal{M} \cap D$ are known as the *Signorini complementarity* (or *ambiguous*) *conditions*. Here, ∂_{ν^\pm} are the exterior normal derivatives from the either side of \mathcal{M} . In particular, at points on $\mathcal{M} \cap D$ we must have one of the two boundary conditions satisfied: either $u = \psi$ or $\partial_{\nu^+} u + \partial_{\nu^-} u = 0$. The set

$$(1.2) \quad \Gamma(u) := \partial_{\mathcal{M}} \{x \in \mathcal{M} \cap D : u(x) = \psi(x)\},$$

which separates the regions where different boundary conditions are satisfied, is known as the *free boundary* and plays a central role in the analysis of the problem.

Because of the presence of the thin obstacle, it is not hard to realize that the solutions u of the Signorini problem are at most Lipschitz across \mathcal{M} , even if both \mathcal{M} and ψ are smooth, as we may have $\partial_{\nu^+} u + \partial_{\nu^-} u > 0$ at some points on \mathcal{M} ¹. However, it has been known since the works [Caf79, Kin81, Ura85] that the solutions of the thin obstacle problem are $C^{1,\beta}$ on \mathcal{M} and consequently on the either side of \mathcal{M} , up to \mathcal{M} . In recent years, there has been a renewed interest in this problem, following the breakthrough result of Athanassopoulos and Caffarelli [AC04] on the optimal $C^{1,1/2}$ regularity of the minimizers (on the either side of \mathcal{M}) as well as its relation to the obstacle-type problems for the fractional Laplacian through the Caffarelli-Silvestre extension [CS07]. There has also been a significant effort in understanding the structure and the regularity of the free boundary. The results have been obtained in many settings, such as for the equations with variable coefficients, time-dependent versions, problems for fractional Laplacian and other nonlocal equations, both regarding the regularity of minimizers, as well as the properties of the free boundary;

¹This can be seen on the explicit example $u(x) = \operatorname{Re}(x_1 + i|x_n|)^{3/2}$, which is a solution of the obstacle problem with $\psi = 0$ on $\mathcal{M} = \{x_n = 0\}$.

see e.g., [Sil07, ACS08, CSS08, GP09, GSVG14, KPS15, PP15, PZ15, DS16, GPS16, KRS16, BSZ17, CRS17, DGPT17, KRS17a, KRS17b, CSV17, ACM18, DPP18, FS18a] and many others.

1.2. Almost minimizers. In [Anz83], Anzellotti introduced the notion of almost minimizers for energy functionals. Given $r_0 > 0$, we say that $\omega : (0, r_0) \rightarrow [0, \infty)$ is a modulus of continuity or a *gauge function*, if $\omega(r)$ is monotone nondecreasing in r and $\omega(0+) = 0$.

Definition 1.1 (Almost minimizers). Given $r_0 > 0$ and a gauge function $\omega(r)$ on $(0, r_0)$, we say that $u \in W_{\text{loc}}^{1,2}(D)$ is an *almost minimizer* (or ω -*minimizer*) for the functional J_D , if, for any ball $B_r(x_0) \Subset D$ with $0 < r < r_0$, we have

$$(1.3) \quad J_{B_r(x_0)}(u) \leq (1 + \omega(r))J_{B_r(x_0)}(v) \quad \text{for any } v \in u + W_0^{1,2}(B_r(x_0)).$$

The idea is that the Dirichlet energy of u on the ball $B_r(x_0)$ is not necessarily minimal among all competitors $v \in u + W_0^{1,2}(B_r(x_0))$ but *almost minimal* in the sense that it cannot decrease more than by a factor of $1 + \omega(r)$. In the specific case of the energy functional J_D in (1.1), i.e., the Dirichlet energy, we refer to the almost minimizers of J_D as *almost harmonic functions* in D .

Results on almost minimizers for more general energy functionals can be found in [DGG00, ELM04, EM99, Min06]. Similar notions were considered earlier in the context of the geometric measure theory [Alm76, Bom82], see also [Amb97]. Almost minimizers are also related to quasiminimizers, introduced in [GG82, GG84], see also [Giu03]. For energy functionals exhibiting free boundaries, almost minimizers have been considered only recently in [DT15, DET17, dQT18, DS18, DS19].

Almost minimizers can be viewed as perturbations of minimizers of various nature, but their study is motivated also by the observation that the minimizers with certain constraints, such as the ones with fixed volume or solutions of the obstacle problem, are realized as almost minimizers of unconstrained problems, see e.g. [Anz83]. Yet another motivation is that the study of almost minimizers reveals a unique perspective on the problem and leads to the development of methods relying on less technical assumptions, thus allowing further generalization.

In this paper we extend the notion of almost minimizers to the thin obstacle problem. Essentially, in (1.3), we restrict the function u and its competitors v to stay above the thin obstacle ψ on \mathcal{M} .

Definition 1.2 (Almost minimizer for the thin obstacle (or Signorini) problem). Given $r_0 > 0$ and a gauge function $\omega(r)$ on $(0, r_0)$, we say that $u \in W_{\text{loc}}^{1,2}(D)$ is an *almost minimizer for the thin obstacle* (or *Signorini*) *problem*, if $u \geq \psi$ on $\mathcal{M} \cap D$ and, for any ball $B_r(x_0) \Subset D$ with $0 < r < r_0$, we have

$$(1.4) \quad J_{B_r(x_0)}(u) \leq (1 + \omega(r))J_{B_r(x_0)}(v), \quad \text{for any } v \in \mathfrak{K}_{\psi,u}(B_r(x_0), \mathcal{M}).$$

Note that in the case when $\mathcal{M} \cap B_r(x_0) = \emptyset$, the condition (1.4) is the same as (1.3) and thus almost minimizers of the Signorini problem are almost harmonic in $D \setminus \mathcal{M}$. As in the case of the solutions of the Signorini problem, we are interested in the regularity properties of almost minimizers as well as the structure and the regularity of the free boundary $\Gamma(u) \subset \mathcal{M}$ as defined in (1.2).

Some examples of almost minimizers are given in Appendix A. We would also like to mention here that a related notion of almost minimizers for the fractional obstacle problem has been considered by the authors in [JP19].

1.3. Main results. Because of the technical nature of the problem, in this paper we restrict ourselves only to the case when $\omega(r) = r^\alpha$ for some $\alpha > 0$, \mathcal{M} is flat, specifically $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, and the thin obstacle $\psi = 0$. As we are mainly interested in local properties of almost minimizers and their free boundaries, we assume that D is the unit ball B_1 , $u \in W^{1,2}(B_1)$, and the constant $r_0 = 1$ in Definition 1.2. We also assume that u is even in x_n -variable:

$$u(x', x_n) = u(x', -x_n), \quad \text{for any } x = (x', x_n) \in B_1.$$

Our first main result is then as follows.

Theorem A ($C^{1,\beta}$ -regularity of almost minimizers). *Let u be an almost minimizer for the Signorini problem in B_1 , under the assumptions above. Then, $u \in C_{\text{loc}}^{1,\beta}(B_1^\pm \cup B_1')$ for $\beta = \beta(\alpha, n)$ and*

$$\|u\|_{C^{1,\beta}(K)} \leq C \|u\|_{W^{1,2}(B_1)},$$

for any $K \Subset B_1^\pm \cup B_1'$ and $C = C(n, \alpha, K)$.

The proof is obtained by using Morrey and Campanato space estimates, following the original idea of Anzellotti [Anz83]. However, in our case the proof is much more elaborate and, in a sense, based on the idea that the solutions of the Signorini problem are 2-valued harmonic functions, as we have to work with both even and odd extensions of u and ∇u from B_1^+ to B_1 .

While the optimal regularity for the minimizer (or solutions) of the Signorini problem is $C^{1,1/2}$, we do not expect such regularity for almost minimizers. However, we are able to establish the optimal growth for almost minimizers, which then allows to study the local properties of the free boundary

$$\Gamma(u) = \partial\{u(\cdot, 0) = 0\} \cap B_1'.$$

Theorem B (Optimal growth near free boundary). *Let u be as in Theorem A. Then,*

$$\int_{\partial B_r(x_0)} u^2 \leq C(n, \alpha) \|u\|_{W^{1,2}(B_1)}^2 r^{n+2},$$

for $x_0 \in B_{1/2}' \cap \Gamma(u)$, $0 < r < r_0(n, \alpha)$.

One of the ingredients in the proof is an Almgren-type monotonicity formula, which we describe below. For an almost minimizer u , *Almgren's frequency* [Alm00] is defined by

$$N(r, u, x_0) := \frac{r \int_{B_r(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)} u^2}, \quad x_0 \in \Gamma(u).$$

It is one of the most important monotone quantities in the analysis of the free boundary for the Signorini problem, see e.g. [PSU12, Chapter 9]. We show that for almost minimizers a small modification of N is monotone.

Theorem C (Monotonicity of the truncated frequency). *Let u be as in Theorem A. Then for any $\kappa_0 \geq 2$, there is $b = b(n, \alpha, \kappa_0)$ such that*

$$r \mapsto \widehat{N}_{\kappa_0}(r, u, x_0) := \min \left\{ \frac{1}{1 - br^\alpha} N(r, u, x_0), \kappa_0 \right\}$$

is monotone for $x_0 \in B_{1/2}' \cap \Gamma(u)$, and $0 < r < r_0(n, \alpha, \kappa_0)$. Moreover, we have that either

$$\widehat{N}_{\kappa_0}(0+, u, x_0) = 3/2 \quad \text{or} \quad \widehat{N}_{\kappa_0}(0+, u, x_0) \geq 2.$$

We give an indirect proof of this fact, based on an one-parametric family of Weiss-type energy functionals $\{W_\kappa\}_{0 < \kappa < \kappa_0}$, see Theorem 5.1, that go back to the work [GP09] for the solutions of the Signorini problem and Weiss [Wei99b] for the classical obstacle problem. The fact that $\widehat{N} \geq 3/2$ at free boundary points is crucial for the proof of the optimal growth (Theorem B), however, the proof of Theorem B requires also an application of so-called *epiperimetric inequality* for Weiss's energy functional $W_{3/2}$ (see [GPS16]), to remove a remaining logarithmic term.

Our next result concerns the subset of the free boundary

$$\mathcal{R}(u) := \{x_0 \in \Gamma(u) : \widehat{N}(0+, u, x_0) = 3/2\},$$

where Almgren's frequency is minimal, known as the *regular set* of u .

Theorem D (Regularity of the regular set). *Let u be as in Theorem A. Then $\mathcal{R}(u)$ is a relatively open subset of the free boundary $\Gamma(u)$ and is a $(n-2)$ -dimensional manifold of class $C^{1,\gamma}$.*

Our proof of this theorem is based on the use of the epiperimetric inequality and is similar to the one for the solutions of the Signorini problem in [GPS16].

Finally, we state our main result for the so-called *singular set*. A free boundary point $x_0 \in \Gamma(u)$ is called *singular* if the *coincidence set* $\Lambda(u) := \{u(\cdot, 0) = 0\}$ has H^{n-1} -density zero at x_0 , i.e.,

$$\lim_{r \rightarrow 0^+} \frac{H^{n-1}(\Lambda(u) \cap B'_r(x_0))}{H^{n-1}(B'_r)} = 0.$$

If $\widehat{N}_{\kappa_0}(0+, u, x_0) = \kappa < \kappa_0$, then x_0 is singular if and only if $\kappa = 2m$, $m \in \mathbb{N}$ (see Proposition 10.2). For such κ , we then define

$$\Sigma_\kappa(u) := \{x_0 \in \Gamma(u) : \widehat{N}_{\kappa_0}(0+, u, x_0) = \kappa\}.$$

Theorem E (Structure of the singular set). *Let u be as in Theorem A. Then, for any $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, $\Sigma_\kappa(u)$ is contained in a countable union of $(n-2)$ -dimensional manifolds of class $C^{1,\log}$.*

A more refined version of this theorem is given in Theorem 10.13. The proof is based on the *logarithmic epiperimetric inequality* of Colombo-Spolaor-Velichkov [CSV17] for Weiss's energy functional W_κ , with $\kappa = 2m$, $m \in \mathbb{N}$. We also point out that this inequality is instrumental in the proof of the optimal growth at singular points, which is rather immediate for the solutions of the Signorini problem, but far more complicated for the almost minimizers (see Lemmas 10.5–10.8).

1.3.1. *Proofs of Theorems A–E.* While we don't give formal proofs of Theorems A–E, in the main body of the paper, they follow from the combination of results there. More specifically,

- Theorem A follows by combining Theorems 3.1 and 4.6.
- The statement of Theorem B is contained in that of Lemma 7.4.
- Theorem C follows by combining Theorem 5.4 and Corollary 9.4.
- The statement of Theorem D is contained in that of Theorem 9.7.
- The statement of Theorem E is contained in that of Theorem 10.13.

1.4. Notation. Throughout the paper we use the following notation. \mathbb{R}^n stands for the n -dimensional Euclidean space. We denote the points of \mathbb{R}^n by $x = (x', x_n)$, where $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$. We routinely identify $x' \in \mathbb{R}^{n-1}$ with $(x', 0) \in \mathbb{R}^{n-1} \times \{0\}$. \mathbb{R}_\pm^n stand for open halfspaces $\{x \in \mathbb{R}^n : \pm x_n > 0\}$.

For $x \in \mathbb{R}^n$, $r > 0$, we use the following notations for balls of radius r , centered at x .

$$\begin{aligned} B_r(x) &= \{y \in \mathbb{R}^n : |x - y| < r\}, & \text{ball in } \mathbb{R}^n, \\ B_r^\pm(x') &= B_r(x', 0) \cap \{\pm x_n > 0\}, & \text{half-ball in } \mathbb{R}^n, \\ B'_r(x') &= B_r(x', 0) \cap \{x_n = 0\}, & \text{ball in } \mathbb{R}^{n-1}, \text{ or thin ball.} \end{aligned}$$

We typically drop the center from the notation if it is the origin. Thus, $B_r = B_r(0)$, $B'_r = B'_r(0)$, etc.

Next, for a direction $e \in \mathbb{R}^n$, we denote

$$\partial_e u = \nabla u \cdot e,$$

the directional derivative of u in the direction e . For the standard coordinate directions e_i , $i = 1, \dots, n$, we simply write

$$u_{x_i} = \partial_{x_i} u = \partial_{e_i} u.$$

Moreover, by $\partial_{x_n}^\pm u(x', 0)$ we mean the limit of $\partial_{x_n} u$ from within B_r^\pm , specifically,

$$\begin{aligned} \partial_{x_n}^+ u(x', 0) &= \lim_{\substack{y \rightarrow (x', 0) \\ y \in B_r^+}} \partial_{x_n} u(y) = -\partial_{\nu^+} u(x', 0), \\ \partial_{x_n}^- u(x', 0) &= \lim_{\substack{y \rightarrow (x', 0) \\ y \in B_r^-}} \partial_{x_n} u(y) = \partial_{\nu^-} u(x', 0), \end{aligned}$$

where $\nu^\pm = \mp e_n$ are unit outward normal vectors for B_r^\pm on B'_r .

In integrals, we often drop the variable and the measure of integration if it is with respect to the Lebesgue measure or the surface measure. Thus,

$$\int_{B_r} u = \int_{B_r} u(x) dx, \quad \int_{\partial B_r} u = \int_{\partial B_r} u(x) dS_x,$$

where S_x stands for the surface measure.

We indicate by $\langle u \rangle_{x,r}$ the integral mean value of a function u over $B_r(x)$. That is,

$$\langle u \rangle_{x,r} := \fint_{B_r(x)} u = \frac{1}{\omega_n r^n} \int_{B_r(x)} u,$$

where $\omega_n = |B_1|$ is the volume of unit ball in \mathbb{R}^n . Similarly to the other notations, we drop the origin if it is 0 and write $\langle u \rangle_r$ for $\langle u \rangle_{0,r}$.

2. ALMOST HARMONIC FUNCTIONS

In this section we recall some results of Anzellotti [Anz83] on almost harmonic functions, i.e., almost minimizers of the Dirichlet integral $J_D(v) = \int_D |\nabla v|^2$. We also state here some of the relevant auxiliary results that we will need also in the treatment of almost minimizers for the Signorini problem.

Theorem 2.1. *Let u be an almost harmonic function in an open set D with a gauge function ω . Then*

- (i) *u is locally almost Lipschitz, i.e., $u \in C_{\text{loc}}^{0,\sigma}(D)$ for all $\sigma \in (0, 1)$.*

(ii) If $\omega(r) \leq Cr^\alpha$ for some $\alpha \in (0, 2)$, then $u \in C_{\text{loc}}^{1, \alpha/2}(D)$.

While we refer to [Anz83] for the full proof of this theorem, we would like to outline the key steps in Anzellotti's argument. The idea to prove $C^{0, \sigma}$ and $C^{1, \alpha/2}$ regularity of u is through the Morrey and Campanato space estimates, namely, by establishing that

$$(2.1) \quad \int_{B_\rho(x_0)} |\nabla u|^2 \leq C\rho^{n-2+2\sigma}$$

$$(2.2) \quad \int_{B_\rho(x_0)} |\nabla u - \langle \nabla u \rangle_{x_0, \rho}|^2 \leq C\rho^{n+\alpha}$$

for $x_0 \in K \Subset D$, and $0 < \rho < \rho_0$, with C and ρ_0 depending on $n, r_0, d = \text{dist}(K, \partial D)$, the gauge function ω , and $\|u\|_{W^{1,2}(D)}$.

To obtain the estimates above, one starts by choosing a special competitor v in (1.3). Namely, we take $v = h$ which solves the Dirichlet problem

$$\Delta h = 0 \quad \text{in } B_r(x_0), \quad h = u \quad \text{on } \partial B_r(x_0).$$

Equivalently, h is the minimizer of the Dirichlet energy $\int_{B_r(x_0)} |\nabla v|^2$ among all functions in $u + W_0^{1,2}(B_r(x_0))$. We call this h the *harmonic replacement* of u in $B_r(x_0)$. We then have the following concentric ball estimates for h .

Proposition 2.2. *Let h be harmonic in $B_r(x_0)$ and $0 < \rho < r$. Then*

$$(2.3) \quad \int_{B_\rho(x_0)} |\nabla h|^2 \leq \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla h|^2$$

$$(2.4) \quad \int_{B_\rho(x_0)} |\nabla h - \langle \nabla h \rangle_{x_0, \rho}|^2 \leq \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |\nabla h - \langle \nabla h \rangle_{x_0, r}|^2.$$

Proof. The estimates above follow from the monotonicity in ρ of the quantities

$$\frac{1}{\rho^n} \int_{B_\rho(x_0)} |\nabla h|^2, \quad \frac{1}{\rho^{n+2}} \int_{B_\rho(x_0)} |\nabla h - \langle \nabla h \rangle_{x_0, \rho}|^2.$$

Noticing that $\langle \nabla h \rangle_{x_0, \rho} = \nabla h(x_0)$, an easy proof is obtained by decomposing h into the sum of the series of homogeneous harmonic polynomials. \square

We next use the almost minimizing property of u to deduce perturbed versions of the estimates above.

Proposition 2.3. *Let u be an almost harmonic function in D . Then for any ball $B_r(x_0) \Subset D$ with $r < r_0$ and $0 < \rho < r$ we have*

$$(2.5) \quad \int_{B_\rho(x_0)} |\nabla u|^2 \leq 2 \left[\left(\frac{\rho}{r}\right)^n + \omega(r) \right] \int_{B_r(x_0)} |\nabla u|^2$$

$$(2.6) \quad \int_{B_\rho(x_0)} |\nabla u - \langle \nabla u \rangle_{x_0, \rho}|^2 \leq 9 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |\nabla u - \langle \nabla u \rangle_{x_0, r}|^2 \\ + 24\omega(r) \int_{B_r(x_0)} |\nabla u|^2.$$

Proof. If h is a harmonic replacement of u in $B_r(x_0)$, we first note that

$$\int_{B_r(x_0)} |\nabla(u - h)|^2 = \int_{B_r(x_0)} |\nabla u|^2 - |\nabla h|^2 - 2 \int_{B_r(x_0)} \nabla h \nabla(u - h)$$

$$= \int_{B_r(x_0)} |\nabla u|^2 - |\nabla h|^2 \leq \omega(r) \int_{B_r(x_0)} |\nabla h|^2 \leq \omega(r) \int_{B_r(x_0)} |\nabla u|^2.$$

Then, combined with (2.3), we estimate

$$\begin{aligned} \int_{B_\rho(x_0)} |\nabla u|^2 &\leq 2 \int_{B_\rho(x_0)} |\nabla h|^2 + 2 \int_{B_\rho(x_0)} |\nabla(u-h)|^2 \\ &\leq 2 \left[\left(\frac{\rho}{r}\right)^n + \omega(r) \right] \int_{B_r(x_0)} |\nabla u|^2, \end{aligned}$$

which gives (2.5). To obtain (2.6), we argue very similarly by using additionally that by Jensen's inequality

$$\int_{B_\rho(x_0)} |\langle \nabla u \rangle_{x_0, \rho} - \langle \nabla h \rangle_{x_0, \rho}|^2 \leq \int_{B_\rho(x_0)} |\nabla u - \nabla h|^2.$$

For more details we refer to the proof of Theorem 4.6, Case 1.1. \square

From here, one deduces the estimates (2.1)–(2.2) with the help of the following useful lemma. The proof can be found e.g. in [HL97].

Lemma 2.4. *Let $r_0 > 0$ be a positive number and let $\varphi : (0, r_0) \rightarrow (0, \infty)$ be a nondecreasing function. Let a, β , and γ be such that $a > 0$, $\gamma > \beta > 0$. There exist two positive numbers $\varepsilon = \varepsilon(a, \gamma, \beta)$, $c = c(a, \gamma, \beta)$ such that, if*

$$\varphi(\rho) \leq a \left[\left(\frac{\rho}{r}\right)^\gamma + \varepsilon \right] \varphi(r) + b r^\beta$$

for all ρ, r with $0 < \rho \leq r < r_0$, where $b \geq 0$, then one also has, still for $0 < \rho < r < r_0$,

$$\varphi(\rho) \leq c \left[\left(\frac{\rho}{r}\right)^\beta \varphi(r) + b \rho^\beta \right].$$

We can now give a formal proof of Theorem 2.1.

Proof of Theorem 2.1. (i) Taking r_0 small enough so that $\omega(r_0) < \varepsilon$, a direct application of Lemma 2.4 to (2.5) produces the estimate (2.1), which in turn implies that $u \in C_{\text{loc}}^{0, \sigma}(D)$, by the Morrey space embedding theorem.

(ii) Using that $\omega(r) \leq Cr^\alpha$, combined with the estimate (2.1), we first obtain

$$\int_{B_\rho(x_0)} |\nabla u - \langle \nabla u \rangle_{x_0, \rho}|^2 \leq 9 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |\nabla u - \langle \nabla u \rangle_{x_0, r}|^2 + Cr^{n-2+2\sigma+\alpha}.$$

If σ is so that $\alpha' = -2 + 2\sigma + \alpha > 0$, Lemma 2.4 implies that

$$\int_{B_\rho(x_0)} |\nabla u - \langle \nabla u \rangle_{x_0, \rho}|^2 \leq C\rho^{n+\alpha'}.$$

By the Campanato space embedding, we therefore obtain that $\nabla u \in C_{\text{loc}}^{0, \alpha'}(D)$. However, it is easy to bootstrap the regularity up to $C_{\text{loc}}^{0, \alpha/2}$ by noticing that we now know that ∇u is locally bounded in D and thus $\int_{B_r(x_0)} |\nabla u|^2 \leq Cr^n$. Plugging that in the last term of (2.6), we obtain that

$$\int_{B_\rho(x_0)} |\nabla u - \langle \nabla u \rangle_{x_0, \rho}|^2 \leq 9 \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |\nabla u - \langle \nabla u \rangle_{x_0, r}|^2 + Cr^{n+\alpha}$$

and repeating the arguments above conclude that $u \in C_{\text{loc}}^{1, \alpha/2}$. \square

3. ALMOST LIPSCHITZ REGULARITY OF ALMOST MINIMIZERS

In this section we prove the first regularity results for the almost minimizers for the Signorini problem, see Definition 1.2. Recall that we assume $D = B_1$, $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, $\psi = 0$, $r_0 = 1$, and $\omega(r) = r^\alpha$ for some $\alpha > 0$. Furthermore we assume that u is even symmetric in x_n -variable.

Theorem 3.1. *Let u be an almost minimizer for the Signorini problem in B_1 . Then $u \in C^{0,\sigma}(B_1)$ for all $0 < \sigma < 1$. Moreover, for any $K \Subset B_1$,*

$$(3.1) \quad \|u\|_{C^{0,\sigma}(K)} \leq C \|u\|_{W^{1,2}(B_1)}$$

with $C = C(n, \alpha, \sigma, K)$.

The idea of the proof is to follow that of Anzellotti [Anz83] that we outlined in Section 2 and to prove an estimate similar to (2.5). The proof of the latter estimate followed by a perturbation argument from a similar estimate for the harmonic replacement of u . However, in the case of the Signorini problem, the harmonic replacements are not necessarily admissible competitors. Instead, for $B_r(x_0) \Subset B_1$, we consider the *Signorini replacements* h of u in $B_r(x_0)$, which solve the Signorini problem in $B_r(x_0)$ with the thin obstacle 0 on \mathcal{M} and boundary values $h = u$ on $\partial B_r(x_0)$. Equivalently, Signorini replacements are the minimizers of $J_{B_r(x_0)}$ on the constraint set $\mathfrak{K}_{0,u}(B_r(x_0), \mathcal{M})$ and they also satisfy the variational inequality²

$$(3.2) \quad \int_{B_r(x_0)} \nabla h \cdot \nabla(v - h) \geq 0 \quad \text{for any } v \in \mathfrak{K}_{0,u}(B_r(x_0), \mathcal{M}).$$

We then have the following concentric ball estimates for Signorini replacements similar to the one for harmonic replacements, at least when the center of the balls is on $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$.

Proposition 3.2. *Let $x_0 \in \mathcal{M}$ and let h be a solution of the Signorini problem in $B_r(x_0)$ with zero obstacle on \mathcal{M} , even in x_n -variable. Then,*

$$(3.3) \quad \int_{B_\rho(x_0)} |\nabla h|^2 \leq \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |\nabla h|^2, \quad 0 < \rho < r.$$

Proof. We claim that $|\nabla h|^2$ is subharmonic in $B_r(x_0)$. This follows from the fact that $h_{x_i}^\pm$, $i = 1, \dots, n-1$, are subharmonic in $B_r(x_0)$, see [PSU12], and similarly that the even extensions $\tilde{h}_{x_n}^\pm$ of $h_{x_n}^\pm$ in x_n -variable from $B_R^+(x_0)$ to all of $B_R(x_0)$ are also subharmonic. These are all consequences of the fact that a continuous nonnegative function, subharmonic in its positivity set is subharmonic, see [PSU12, Ex. 2.6].

The subharmonicity of $|\nabla h|^2$ in $B_r(x_0)$ then implies, by the sub-mean value property, that the function

$$\rho \mapsto \frac{1}{\rho^n} \int_{B_\rho(x_0)} |\nabla h|^2$$

is monotone nondecreasing. This readily implies (3.3). \square

We next have the perturbed version of Proposition 3.2.

²which follows from the inequality $\int_{B_r(x_0)} |\nabla h|^2 \leq \int_{B_r(x_0)} |\nabla((1-\varepsilon)h + \varepsilon v)|^2$, $\varepsilon \in (0, 1)$ by a first variation argument.

Proposition 3.3. *Let u be an almost minimizer for the Signorini problem in B_1 , and $B_r(x_0) \subset B_1$. Then, there is $C_1 = C_1(n) > 1$ such that*

$$(3.4) \quad \int_{B_\rho(x_0)} |\nabla u|^2 \leq C_1 \left[\left(\frac{\rho}{r} \right)^n + r^\alpha \right] \int_{B_r(x_0)} |\nabla u|^2, \quad 0 < \rho < r.$$

Proof. By using the continuity argument, we may assume that $B_r(x_0) \Subset B_1$. We first prove the estimate when x_0 is in the thin space, i.e., $x_0 \in B'_1$ and then extend it to arbitrary $x_0 \in B_1$.

Case 1. Suppose $x_0 \in B'_1$ and let h be the Signorini replacement of u in $B_r(x_0)$. Recall that h satisfies (3.2). Then, plugging $v = u$, we obtain

$$(3.5) \quad \int_{B_r(x_0)} \nabla h \cdot \nabla u - |\nabla h|^2 \geq 0.$$

Using this, we can estimate

$$(3.6) \quad \begin{aligned} \int_{B_r(x_0)} |\nabla(u-h)|^2 &= \int_{B_r(x_0)} \left(|\nabla u|^2 + |\nabla h|^2 - 2\nabla u \cdot \nabla h \right) \\ &\leq \int_{B_r(x_0)} |\nabla u|^2 - \int_{B_r(x_0)} |\nabla h|^2 \\ &\leq (1+r^\alpha) \int_{B_r(x_0)} |\nabla h|^2 - \int_{B_r(x_0)} |\nabla h|^2 \\ &= r^\alpha \int_{B_r(x_0)} |\nabla h|^2 \leq r^\alpha \int_{B_r(x_0)} |\nabla u|^2, \end{aligned}$$

where in the very last step we have used that h minimizes the Dirichlet integral among all functions in $\mathfrak{K}_{0,u}(B_r(x_0), \mathcal{M})$.

Next, we use the same perturbation argument as in the proof of (2.5). By using (3.3) and (3.6), we estimate

$$\begin{aligned} \int_{B_\rho(x_0)} |\nabla u|^2 &\leq 2 \int_{B_\rho(x_0)} |\nabla h|^2 + 2 \int_{B_\rho(x_0)} |\nabla(u-h)|^2 \\ &\leq 2 \left(\frac{\rho}{r} \right)^n \int_{B_r(x_0)} |\nabla h|^2 + 2r^\alpha \int_{B_r(x_0)} |\nabla u|^2 \\ &\leq 2 \left[\left(\frac{\rho}{r} \right)^n + r^\alpha \right] \int_{B_r(x_0)} |\nabla u|^2. \end{aligned}$$

Thus, (3.4) follows in this case.

Case 2. Consider now the case $x_0 \in B_1^+$. If $\rho \geq r/4$, then we simply have

$$\int_{B_\rho(x_0)} |\nabla u|^2 \leq 4^n \left(\frac{\rho}{r} \right)^n \int_{B_r(x_0)} |\nabla u|^2.$$

Thus, we may assume $\rho < r/4$. Then, let $d := \text{dist}(x_0, B'_1) > 0$ and choose $x_1 \in \partial B_d(x_0) \cap B'_1$.

Case 2.1. If $\rho \geq d$, then we use $B_\rho(x_0) \subset B_{2\rho}(x_1) \subset B_{r/2}(x_1) \subset B_r(x_0)$ and the result of Case 1 to write

$$\int_{B_\rho(x_0)} |\nabla u|^2 \leq \int_{B_{2\rho}(x_1)} |\nabla u|^2 \leq C \left[\left(\frac{2\rho}{r/2} \right)^n + (r/2)^\alpha \right] \int_{B_{r/2}(x_1)} |\nabla u|^2$$

$$\leq C \left[\left(\frac{\rho}{r} \right)^n + r^\alpha \right] \int_{B_r(x_0)} |\nabla u|^2.$$

Case 2.2. Suppose now $d > \rho$. If $d > r$, then $B_r(x_0) \Subset B_1^+$. Since u is almost harmonic in B_1^+ , we can apply Proposition 2.3 to obtain

$$\int_{B_\rho(x_0)} |\nabla u|^2 \leq 2 \left[\left(\frac{\rho}{r} \right)^n + r^\alpha \right] \int_{B_r(x_0)} |\nabla u|^2.$$

Thus, we may assume $d \leq r$. Then we note that $B_d(x_0) \subset B_1^+$ and by a limiting argument from the previous estimate, we obtain

$$\int_{B_\rho(x_0)} |\nabla u|^2 \leq 2 \left[\left(\frac{\rho}{d} \right)^n + r^\alpha \right] \int_{B_d(x_0)} |\nabla u|^2.$$

Case 2.2.1. If $r/4 \leq d$, then

$$\int_{B_d(x_0)} |\nabla u|^2 \leq 4^n \left(\frac{d}{r} \right)^n \int_{B_r(x_0)} |\nabla u|^2,$$

which immediately implies (3.4).

Case 2.2.2. It remains to consider the case $\rho < d < r/4$. Using Case 1 again, we have

$$\begin{aligned} \int_{B_d(x_0)} |\nabla u|^2 &\leq \int_{B_{2d}(x_1)} |\nabla u|^2 \leq C \left[\left(\frac{2d}{r/2} \right)^n + (r/2)^\alpha \right] \int_{B_{r/2}(x_1)} |\nabla u|^2 \\ &\leq C \left[\left(\frac{d}{r} \right)^n + r^\alpha \right] \int_{B_r(x_0)} |\nabla u|^2, \end{aligned}$$

which also implies (3.4). This concludes the proof of the proposition. \square

We can now give the proof of the almost Lipschitz regularity of almost minimizers.

Proof of Theorem 3.1. Let $K \Subset B_1$ and $x_0 \in K$. Take $\delta = \delta(n, \alpha, \sigma, K) > 0$ such that $\delta < \text{dist}(K, \partial B_1)$ and $\delta^\alpha \leq \varepsilon(C_1, n, n + 2\sigma - 2)$, where $\varepsilon = \varepsilon(C_1, n, n + 2\sigma - 2)$ is as in Lemma 2.4. Then for all $0 < \rho < r < \delta$, by (3.4),

$$\int_{B_\rho(x_0)} |\nabla u|^2 \leq C_1 \left[\left(\frac{\rho}{r} \right)^n + \varepsilon \right] \int_{B_r(x_0)} |\nabla u|^2.$$

By applying Lemma 2.4, we obtain

$$\int_{B_\rho(x_0)} |\nabla u|^2 \leq C(n, \sigma) \left(\frac{\rho}{r} \right)^{n+2\sigma-2} \int_{B_r(x_0)} |\nabla u|^2.$$

Taking $r \nearrow \delta$, we can therefore conclude

$$(3.7) \quad \int_{B_\rho(x_0)} |\nabla u|^2 \leq C(n, \alpha, \sigma, K) \|\nabla u\|_{L^2(B_1)}^2 \rho^{n+2\sigma-2}.$$

From here, we use the Morrey space embedding to obtain $u \in C^{0,\sigma}(K)$ with the norm estimate

$$\|u\|_{C^{0,\sigma}(K)} \leq C(n, \alpha, \sigma, K) \|u\|_{W^{1,2}(B_1)},$$

as required. \square

4. $C^{1,\beta}$ REGULARITY OF ALMOST MINIMIZERS

In this section we establish the $C^{1,\beta}$ regularity of almost minimizers for some $\beta > 0$. The idea is again to use Signorini replacements and an appropriate version of the concentric ball estimate (2.4) for solutions of the Signorini problem.

As we saw in the proof of the almost Lipschitz regularity of almost minimizers, it is enough to obtain such estimates when balls are centered at x_0 on the thin space $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$. It turns out that to prove a proper version of (2.4), we have to work with both even and odd extensions in x_n -variable of Signorini replacements h from $B_r^+(x_0)$ to $B_r(x_0)$. The reason is that even extensions are harmonic across the positivity set $\{h(\cdot, 0) > 0\}$, while the odd extensions are harmonic across the interior of the coincidence set $\{h(\cdot, 0) = 0\}$.

We start with the borderline case when the center x_0 of concentric balls is on the free boundary $\Gamma(h) = B_r^+(x_0) \cap \partial_{\mathbb{R}^{n-1}}\{h(\cdot, 0) = 0\}$.

Proposition 4.1. *Let h be a solution of the Signorini problem in $B_r(x_0)$ with $x_0 \in \mathcal{M}$, even in x_n , and define*

$$\tilde{h}(x', x_n) := \begin{cases} h(x', x_n), & x_n \geq 0 \\ -h(x', -x_n), & x_n < 0, \end{cases}$$

the odd extension in x_n -variable of h from $B_r^+(x_0)$ to $B_r(x_0)$.

Suppose that $x_0 \in \Gamma(h)$. Then, for any $0 < \alpha < 1$, there exists $C = C(n, \alpha)$ such that for any $0 < \rho < s < (3/4)r$ we have

$$(4.1) \quad \int_{B_\rho(x_0)} |\nabla h - \langle \nabla h \rangle_{x_0, \rho}|^2 \leq \left(\frac{\rho}{s}\right)^{n+\alpha} \int_{B_s(x_0)} |\nabla h - \langle \nabla h \rangle_{x_0, s}|^2 + C \left(\int_{B_r(x_0)} h^2 \right) \frac{s^{n+1}}{r^{n+3}},$$

$$(4.2) \quad \int_{B_\rho(x_0)} |\nabla \tilde{h} - \langle \nabla \tilde{h} \rangle_{x_0, \rho}|^2 \leq \left(\frac{\rho}{s}\right)^{n+\alpha} \int_{B_s(x_0)} |\nabla \tilde{h} - \langle \nabla \tilde{h} \rangle_{x_0, s}|^2 + C \left(\int_{B_r(x_0)} h^2 \right) \frac{s^{n+1}}{r^{n+3}}.$$

Remark 4.2. We note here that $\langle \nabla h \rangle_{x_0, \rho}$ has its n -th component zero because of odd symmetry of h_{x_n} , while $\langle \nabla \tilde{h} \rangle_{x_0, \rho}$ has its first $(n-1)$ components zero because of odd symmetry of \tilde{h}_{x_i} , $i = 1, \dots, n-1$.

Proof. Without loss of generality we may assume $x_0 = 0$. For $0 < t < (3/4)r$, define

$$\varphi(t) := \frac{1}{t^{n+\alpha}} \int_{B_t} |\nabla h - \langle \nabla h \rangle_t|^2.$$

Then

$$\begin{aligned} \varphi(t) &= \frac{1}{t^{n+\alpha}} \int_{B_t} |\nabla h - \langle \nabla h \rangle_t|^2 \\ &= \frac{1}{t^{n+\alpha}} \left[\int_{B_t} |\nabla h|^2 - 2 \langle \nabla h \rangle_t \int_{B_t} \nabla h + \int_{B_t} \langle \nabla h \rangle_t^2 \right] \\ &= \frac{1}{t^{n+\alpha}} \left[\int_{B_t} |\nabla h|^2 - \frac{1}{\omega_n t^n} \left(\int_{B_t} \nabla h \right)^2 \right]. \end{aligned}$$

Thus,

$$(4.3) \quad \varphi'(t) = \frac{1}{t^{n+\alpha}} \left[-\frac{n+\alpha}{t} \int_{B_t} |\nabla h|^2 + \int_{\partial B_t} |\nabla h|^2 + \frac{2n+\alpha}{\omega_n t^{n+1}} \left(\int_{B_t} \nabla h \right)^2 - \frac{2}{\omega_n t^n} \left(\int_{B_t} \nabla h \right) \left(\int_{\partial B_t} \nabla h \right) \right].$$

We next recall that h is $C^{1,1/2}$ regular in $B_r^\pm \cup B'_r$ and we have the estimate

$$(4.4) \quad \|\nabla h\|_{C^{0,1/2}(B_{(3/4)r}^\pm \cup B'_{(3/4)r})} \leq C(n) r^{-\frac{n+3}{2}} \|h\|_{L^2(B_r^+)},$$

see e.g. Theorem 9.13 in [PSU12]. Then, using $\nabla h(0) = 0$, we obtain

$$\frac{n+\alpha}{t^{n+\alpha+1}} \int_{B_t} |\nabla h|^2 \leq \frac{C(n, \alpha)}{t^\alpha r^{n+3}} \left(\int_{B_r} h^2 \right).$$

We can similarly estimate the other term with a negative sign in (4.3) to obtain

$$\varphi'(t) \geq -\frac{C(n, \alpha)}{t^\alpha r^{n+3}} \left(\int_{B_r} h^2 \right).$$

Thus,

$$\begin{aligned} \varphi(s) - \varphi(\rho) &\geq -C \left(\int_{B_r} h^2 \right) \frac{1}{r^{n+3}} \int_\rho^s t^{-\alpha} dt \\ &\geq -C \left(\int_{B_r} h^2 \right) \frac{s^{1-\alpha}}{r^{n+3}}. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 &= \rho^{n+\alpha} \varphi(\rho) \leq \rho^{n+\alpha} \left(\varphi(s) + C \left(\int_{B_r} h^2 \right) \frac{s^{1-\alpha}}{r^{n+3}} \right) \\ &\leq \left(\frac{\rho}{s} \right)^{n+\alpha} \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2 + C \left(\int_{B_r} h^2 \right) \frac{s^{n+1}}{r^{n+3}}. \end{aligned}$$

This proves the first estimate in the statement of the lemma. For the second estimate, involving $\nabla \tilde{h}$, we essentially repeat the above argument with

$$\tilde{\varphi}(t) := \frac{1}{t^{n+\alpha}} \int_{B_t} |\nabla \tilde{h} - \langle \nabla \tilde{h} \rangle_t|^2. \quad \square$$

We next consider the case when the center $x_0 \notin \Gamma(h)$. We have to distinguish the cases x_0 is in $\{h(\cdot, 0) > 0\}$ or the interior of $\{h(\cdot, 0) = 0\}$.

Proposition 4.3. *Let h be a solution of the Signorini problem in $B_r(x_0)$ with $x_0 \in \mathcal{M}$, even in x_n -variable. Suppose $x_0 \in B'_r(x_0) \setminus \Gamma(h)$. Let $d := \text{dist}(x_0, \Gamma(h)) > 0$ and fix $\alpha \in (0, 1)$. Then there are $C_1 = C_1(n, \alpha)$, $C_2 = C_2(n, \alpha)$ such that for $0 < \rho < s < r$ the following inequalities hold.*

(i) *If $B'_d(x_0) \subset \{h(\cdot, 0) > 0\}$, then*

$$\begin{aligned} \int_{B_\rho(x_0)} |\nabla h - \langle \nabla h \rangle_{x_0, \rho}|^2 &\leq C_1 \left(\frac{\rho}{s} \right)^{n+\alpha} \int_{B_s(x_0)} |\nabla h - \langle \nabla h \rangle_{x_0, s}|^2 \\ &\quad + C_2 \left(\int_{B_r(x_0)} h^2 \right) \frac{s^{n+1}}{r^{n+3}}. \end{aligned}$$

(ii) If $B'_d(x_0) \subset \{h(\cdot, 0) = 0\}$, then

$$\begin{aligned} \int_{B_\rho(x_0)} |\nabla \tilde{h} - \langle \nabla \tilde{h} \rangle_{x_0, \rho}|^2 &\leq C_1 \left(\frac{\rho}{s}\right)^{n+\alpha} \int_{B_s(x_0)} |\nabla \tilde{h} - \langle \nabla \tilde{h} \rangle_{x_0, s}|^2 \\ &\quad + C_2 \left(\int_{B_r(x_0)} h^2 \right) \frac{s^{n+1}}{r^{n+3}}. \end{aligned}$$

Proof. Without loss of generality we may assume $x_0 = 0$.

(i) First consider the case $B'_d \subset \{h(\cdot, 0) > 0\}$. If $d \geq s$, then h is harmonic in B_s and hence

$$\int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 \leq \left(\frac{\rho}{s}\right)^{n+2} \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2 \leq \left(\frac{\rho}{s}\right)^{n+\alpha} \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2.$$

We can therefore assume $0 < d < s$

Case 1. $s/4 \leq d < s$.

Case 1.1. Suppose $0 < \rho < d$. We first observe that

$$\int_{B_d} |\nabla h - \langle \nabla h \rangle_d|^2 = \min_{C \in \mathbb{R}^n} \int_{B_d} |\nabla h - C|^2 \leq \int_{B_d} |\nabla h - \langle \nabla h \rangle_s|^2 \leq \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2.$$

Now using that h is harmonic in B_d , we obtain

$$\begin{aligned} \int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 &\leq \left(\frac{\rho}{d}\right)^{n+2} \int_{B_d} |\nabla h - \langle \nabla h \rangle_d|^2 \\ &\leq \left(\frac{4\rho}{s}\right)^{n+\alpha} \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2 \\ &= 4^{n+\alpha} \left(\frac{\rho}{s}\right)^{n+\alpha} \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2. \end{aligned}$$

Case 1.2. If $\rho \geq d$, then we use $\rho/s \geq 1/4$ to obtain

$$\begin{aligned} \int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 &\leq \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2 \\ &\leq 4^{n+\alpha} \left(\frac{\rho}{s}\right)^{n+\alpha} \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2. \end{aligned}$$

Case 2. $0 < d < s/4$.

Case 2.1. If $0 < \rho < d$, take $x_1 \in \partial B'_d \cap \Gamma(h)$. We first use the harmonicity of h in B_d and then applying Proposition 4.1 for balls $B_{2d}(x_1) \subset B_{s/2}(x_1) \subset B_{(2/3)r}(x_1)$, to obtain

$$\begin{aligned} \int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 &\leq \left(\frac{\rho}{d}\right)^{n+\alpha} \int_{B_d} |\nabla h - \langle \nabla h \rangle_d|^2 \\ &\leq \left(\frac{\rho}{d}\right)^{n+\alpha} \int_{B_{2d}(x_1)} |\nabla h - \langle \nabla h \rangle_{x_1, 2d}|^2 \\ &\leq \left(\frac{\rho}{d}\right)^{n+\alpha} \left[\left(\frac{4d}{s}\right)^{n+\alpha} \int_{B_{s/2}(x_1)} |\nabla h - \langle \nabla h \rangle_{x_1, s/2}|^2 \right. \\ &\quad \left. + C_2 \left(\int_{B_{(2/3)r}(x_1)} h^2 \right) \frac{s^{n+1}}{r^{n+3}} \right] \end{aligned}$$

$$\leq C\left(\frac{\rho}{s}\right)^{n+\alpha} \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2 + C_2 \left(\int_{B_r} h^2 \right) \frac{s^{n+1}}{r^{n+3}},$$

where in the last step we have used that $B_{s/2}(x_1) \subset B_s$ and $B_{(2/3)r}(x_1) \subset B_r$.

Case 2.2. If $d \leq \rho < s/4$, then we apply Proposition 4.1 with $B_{2\rho}(x_1) \subset B_{s/2}(x_1) \subset B_{(2/3)r}(x_1)$ as in Case 2.1:

$$\begin{aligned} \int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 &\leq \int_{B_{2\rho}(x_1)} |\nabla h - \langle \nabla h \rangle_{x_1, 2\rho}|^2 \\ &\leq \left(\frac{4\rho}{s}\right)^{n+\alpha} \int_{B_{s/2}(x_1)} |\nabla h - \langle \nabla h \rangle_{x_1, s/2}|^2 \\ &\quad + C_2 \left(\int_{B_{(2/3)r}(x_1)} h^2 \right) \frac{s^{n+1}}{r^{n+3}} \\ &\leq C\left(\frac{\rho}{s}\right)^{n+\alpha} \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2 + C_2 \left(\int_{B_r} h^2 \right) \frac{s^{n+1}}{r^{n+3}}. \end{aligned}$$

Case 2.3. If $\rho \geq s/4$, then $\rho/s \geq 1/4$ and can we repeat the argument in Case 1.2.

This completes the proof of part (i).

(ii) Now suppose that $h = 0$ in $B'_d(x_0)$. Notice that in the proof of part (i) only harmonicity of h in B_d and Proposition 4.1 were used. In the present case, it is the odd reflection \tilde{h} that is harmonic in B_d , and we can repeat the same arguments as in part (i) with ∇h replaced by $\nabla \tilde{h}$. \square

Now we have the following estimate combining the two preceding ones.

Proposition 4.4. *Let h be a solution of the Signorini problem in $B_r(x_0)$ with $x_0 \in \mathcal{M}$, even in x_n -variable. Define*

$$\widehat{\nabla} h := \begin{cases} \nabla h(x', x_n), & x_n \geq 0 \\ \nabla h(x', -x_n), & x_n < 0, \end{cases}$$

the even extension of ∇h from $B_r^+(x_0)$ to $B_r(x_0)$. Then for $0 < \alpha < 1$, there are $C_1 = C_1(n, \alpha)$, $C_2 = C_2(n, \alpha)$ such that for all $0 < \rho < s \leq (3/4)r$,

$$\begin{aligned} \int_{B_\rho(x_0)} |\widehat{\nabla} h - \langle \widehat{\nabla} h \rangle_{x_0, \rho}|^2 &\leq C_1 \left(\frac{\rho}{s}\right)^{n+\alpha} \int_{B_s(x_0)} |\widehat{\nabla} h - \langle \widehat{\nabla} h \rangle_{x_0, s}|^2 \\ &\quad + C_2 \left(\int_{B_r(x_0)} h^2 \right) \frac{s^{n+1}}{r^{n+3}}. \end{aligned}$$

Remark 4.5. We explicitly warn the reader that $\nabla \tilde{h}$ should not be confused with $\widehat{\nabla} h$. In $\nabla \tilde{h}$, the first $n-1$ components are odd and the last one is even in x_n -variable, while in $\widehat{\nabla} h$ all components are even in x_n .

Proof. Let $d := \text{dist}(x_0, \Gamma(h))$. Without loss of generality we may assume that $d > 0$, as the case $d = 0$ will follow by continuity. Also, without loss of generality, assume $x_0 = 0$.

(i) First consider the case when $B'_d \subset \{h > 0\}$. By the odd symmetry of h_{x_n} in x_n -variable, we have $\langle h_{x_n} \rangle_\rho = 0$. Thus, for

$$\widehat{h_{x_n}}(x) = \begin{cases} h_{x_n}(x', x_n), & x_n \geq 0 \\ h_{x_n}(x', -x_n), & x_n < 0 \end{cases},$$

we obtain

$$\begin{aligned} \int_{B_\rho} |\widehat{h_{x_n}} - \langle \widehat{h_{x_n}} \rangle_\rho|^2 &= \int_{B_\rho} \widehat{h_{x_n}}^2 - \frac{1}{|B_\rho|} \left(\int_{B_\rho} \widehat{h_{x_n}} \right)^2 \\ &= \int_{B_\rho} |h_{x_n} - \langle h_{x_n} \rangle_\rho|^2 - \frac{1}{|B_\rho|} \left(\int_{B_\rho} \widehat{h_{x_n}} \right)^2. \end{aligned}$$

Further, if $\widehat{h_{x_i}}$ denotes the i -th component of $\widehat{\nabla h}$, we have $\widehat{h_{x_i}} = h_{x_i}$ for $i = 1, \dots, n-1$, and hence arrive at

$$(4.5) \quad \int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 = \int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 - \frac{1}{|B_\rho|} \left(\int_{B_\rho} \widehat{h_{x_n}} \right)^2.$$

Similarly, we have

$$(4.6) \quad \int_{B_s} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_s|^2 = \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2 - \frac{1}{|B_s|} \left(\int_{B_s} \widehat{h_{x_n}} \right)^2.$$

Then, by (4.5), (4.6), and Proposition 4.3, we obtain

$$\begin{aligned} \int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 &\leq \int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 \\ &\leq C_1 \left(\frac{\rho}{s} \right)^{n+\alpha} \int_{B_s} |\nabla h - \langle \nabla h \rangle_s|^2 + C_2 \left(\int_{B_r} h^2 \right) \frac{s^{n+1}}{r^{n+3}} \\ &\leq C_1 \left(\frac{\rho}{s} \right)^{n+\alpha} \int_{B_s} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_s|^2 + \frac{C_1}{|B_s|} \left(\int_{B_s} \widehat{h_{x_n}} \right)^2 + C_2 \left(\int_{B_r} h^2 \right) \frac{s^{n+1}}{r^{n+3}}. \end{aligned}$$

From $h(0) > 0$, we have $h_{x_n}(0) = 0$. Thus, using (4.4), we obtain

$$\frac{1}{|B_s|} \left(\int_{B_s} \widehat{h_{x_n}} \right)^2 \leq C \left(\int_{B_r} h^2 \right) \frac{s^{n+1}}{r^{n+3}}.$$

This completes the proof in this case.

(ii) Suppose now $B'_d \subset \{h = 0\}$. In this case, we use Proposition 4.1 for $\widetilde{\nabla h}$, which differs from $\widehat{\nabla h}$ in the first $(n-1)$ components by their symmetry, and has the same even n -th component. Arguing as above, we obtain error terms

$$\frac{1}{|B_s|} \left(\int_{B_s} h_{x_i} \right)^2, \quad i = 1, \dots, n-1,$$

up to a factor of $C(n, \alpha)$. Then, using that $h_{x_i}(0) = 0$, $i = 1, \dots, n-1$ and (4.4), we conclude that

$$\frac{1}{|B_s|} \left(\int_{B_s} h_{x_i} \right)^2 \leq C \left(\int_{B_r} h^2 \right) \frac{s^{n+1}}{r^{n+3}}.$$

This completes the proof. \square

We now prove the $C^{1,\beta}$ regularity of almost minimizers.

Theorem 4.6. *Let u be an almost minimizer of the Signorini problem in B_1 . Define*

$$\widehat{\nabla}u(x', x_n) := \begin{cases} \nabla u(x', x_n), & x_n \geq 0 \\ \nabla u(x', -x_n), & x_n < 0. \end{cases}$$

Then

$$\widehat{\nabla}u \in C^{0,\beta}(B_1) \quad \text{with } \beta = \frac{\alpha}{4(2n + \alpha)}.$$

Moreover, for any $K \Subset B_1$ there holds

$$(4.7) \quad \|\widehat{\nabla}u\|_{C^{0,\beta}(K)} \leq C(n, \alpha, K)\|u\|_{W^{1,2}(B_1)}.$$

Proof. Without loss of generality, we may assume that K is a ball centered at 0. Fix a small $r_0 = r_0(n, \alpha, K) > 0$ to be chosen later. Particularly, we will ask $R_0 := r_0^{\frac{2n}{2n+\alpha}} \leq (1/2) \text{dist}(K, \partial B_1)$, which will imply that

$$\tilde{K} := \{x \in B_1 : \text{dist}(x, K) \leq R_0\} \Subset B_1.$$

Our goal now is to show that for $x_0 \in K$, $0 < \rho < r < r_0$,

$$(4.8) \quad \int_{B_\rho(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, \rho}|^2 \leq C(n, \alpha) \left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, r}|^2 \\ + C(n, \alpha, K)\|u\|_{W^{1,2}(B_1)}^2 r^{n+2\beta},$$

which readily gives the estimate (4.7) by applying Lemma 2.4 and using the Campanato space embedding.

We first prove (4.8) for $x_0 \in K \cap B'_1$, by taking the advantage of the symmetry of $\widehat{\nabla}u$, and then we argue as in the proof of Proposition 3.3 to extend it to all $x_0 \in K$.

Case 1. Suppose $x_0 \in K \cap B'_1$. For notational simplicity, we assume $x_0 = 0$ (by shifting the center of the domain $D = B_1$ to $-x_0$) and let $0 < r < r_0$ be given. Let us also denote

$$\alpha' := 1 - \frac{\alpha}{8n} \in (0, 1), \quad R := r^{\frac{2n}{2n+\alpha}}.$$

We then split our proof into two cases:

$$\sup_{\partial B_R} |u| \leq C_3 R^{\alpha'} \quad \text{and} \quad \sup_{\partial B_R} |u| > C_3 R^{\alpha'},$$

with $C_3 = 2[u]_{0, \alpha', \tilde{K}} = 2 \sup_{\substack{x, y \in \tilde{K} \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha'}}$.

Case 1.1. Assume first that $\sup_{\partial B_R} |u| \leq C_3 R^{\alpha'}$. Let h be the Signorini replacement of u on B_R . Then, for any $0 < \rho < r$, we have

$$\int_{B_\rho} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_\rho|^2 \leq 3 \int_{B_\rho} |\widehat{\nabla}h - \langle \widehat{\nabla}h \rangle_\rho|^2 \\ + 3 \int_{B_\rho} |\widehat{\nabla}u - \widehat{\nabla}h|^2 + 3 \int_{B_\rho} |\langle \widehat{\nabla}u \rangle_\rho - \langle \widehat{\nabla}h \rangle_\rho|^2.$$

Besides, by Jensen's inequality, we have

$$\int_{B_\rho} |\langle \widehat{\nabla}u \rangle_\rho - \langle \widehat{\nabla}h \rangle_\rho|^2 \leq \int_{B_\rho} |\widehat{\nabla}u - \widehat{\nabla}h|^2.$$

Hence, combining the estimates above, we obtain

$$(4.9) \quad \int_{B_\rho} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_\rho|^2 \leq 3 \int_{B_\rho} |\widehat{\nabla}h - \langle \widehat{\nabla}h \rangle_\rho|^2 + 6 \int_{B_\rho} |\widehat{\nabla}u - \widehat{\nabla}h|^2.$$

Similarly

$$(4.10) \quad \int_{B_r} |\widehat{\nabla}h - \langle \widehat{\nabla}h \rangle_r|^2 \leq 3 \int_{B_r} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_r|^2 + 6 \int_{B_r} |\widehat{\nabla}u - \widehat{\nabla}h|^2.$$

Next, note that if $r_0 \leq (3/4)\frac{2n+\alpha}{\alpha}$, then $r \leq (3/4)R$, and thus by Proposition 4.4,

$$(4.11) \quad \int_{B_\rho} |\widehat{\nabla}h - \langle \widehat{\nabla}h \rangle_\rho|^2 \leq C(n, \alpha) \left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r} |\widehat{\nabla}h - \langle \widehat{\nabla}h \rangle_r|^2 \\ + C(n, \alpha) \left(\int_{B_R} h^2\right) \frac{r^{n+1}}{R^{n+3}}.$$

Then, using (4.9), (4.10), and (4.11), we obtain

$$(4.12) \quad \int_{B_\rho} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_\rho|^2 \leq 3 \int_{B_\rho} |\widehat{\nabla}h - \langle \widehat{\nabla}h \rangle_\rho|^2 + 6 \int_{B_\rho} |\widehat{\nabla}u - \widehat{\nabla}h|^2 \\ \leq C(n, \alpha) \left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r} |\widehat{\nabla}h - \langle \widehat{\nabla}h \rangle_r|^2 \\ + C(n, \alpha) \left(\int_{B_R} h^2\right) \frac{r^{n+1}}{R^{n+3}} + 6 \int_{B_\rho} |\widehat{\nabla}u - \widehat{\nabla}h|^2 \\ \leq C(n, \alpha) \left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_r|^2 \\ + C(n, \alpha) \left(\int_{B_R} h^2\right) \frac{r^{n+1}}{R^{n+3}} + C(n, \alpha) \int_{B_r} |\widehat{\nabla}u - \widehat{\nabla}h|^2.$$

Now take $\delta = \delta(n, \alpha, K) > 0$ such that $\delta < \text{dist}(K, \partial B_1)$ and $\delta^\alpha \leq \varepsilon = \varepsilon(C_1, n, n + 2\alpha' - 2)$, where C_1 is as in Theorem 3.1 and ε is as in Lemma 2.4. If $r_0 \leq \delta^{\frac{2n+\alpha}{2n}}$, then $R < \delta$, and therefore by (3.7),

$$\int_{B_R} |\widehat{\nabla}u|^2 \leq C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 R^{n+2\alpha'-2}.$$

Thus, using the above inequality, combined with (3.5), we obtain

$$(4.13) \quad \int_{B_r} |\widehat{\nabla}u - \widehat{\nabla}h|^2 \leq \int_{B_R} |\widehat{\nabla}u - \widehat{\nabla}h|^2 \leq \int_{B_R} |\widehat{\nabla}u|^2 - \int_{B_R} |\widehat{\nabla}h|^2 \\ \leq R^\alpha \int_{B_R} |\widehat{\nabla}h|^2 \leq R^\alpha \int_{B_R} |\widehat{\nabla}u|^2 \\ \leq C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 R^{n+\alpha+2\alpha'-2} \\ = C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 r^{n+\frac{\alpha}{2n+\alpha}(n-\frac{1}{2})}.$$

We next use that h^2 is subharmonic in B_R . (This can be seen for instance by a direct computation $\Delta(h^2) = 2(|\nabla h|^2 + h\Delta h) = 2|\nabla h|^2 \geq 0$, or by using the fact that h^\pm are subharmonic.) Then,

$$(4.14) \quad \langle h^2 \rangle_R \leq \sup_{B_R} h^2 = \sup_{\partial B_R} h^2 = \sup_{\partial B_R} u^2 \leq C_3^2 R^{2\alpha'}.$$

Also note that by (3.1), $C_3 \leq C(n, \alpha, K)\|u\|_{W^{1,2}(B_1)}$. Hence,

$$(4.15) \quad \begin{aligned} \left(\int_{B_R} h^2\right) \frac{r^{n+1}}{R^{n+3}} &= C(n) \langle h^2 \rangle_R \frac{r^{n+1}}{R^3} \\ &\leq C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}^2 r^{n+\frac{\alpha}{2(2n+\alpha)}}. \end{aligned}$$

Now (4.12), (4.13), (4.15) give

$$(4.16) \quad \begin{aligned} \int_{B_\rho} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_\rho|^2 &\leq C(n, \alpha) \left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_r|^2 \\ &\quad + C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}^2 r^{n+\frac{\alpha}{2(2n+\alpha)}} \\ &\quad + C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 r^{n+\frac{\alpha}{2n+\alpha}(n-\frac{1}{2})} \\ &\leq C(n, \alpha) \left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_r|^2 \\ &\quad + C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}^2 r^{n+\frac{\alpha}{2(2n+\alpha)}}. \end{aligned}$$

Case 1.2. Now suppose $\sup_{\partial B_R} |u| > C_3 R^{\alpha'}$. By the choice of $C_3 = 2[u]_{0, \alpha', \tilde{K}}$, we have either $u \geq (C_3/2)R^{\alpha'}$ in all of B_R or $u \leq -(C_3/2)R^{\alpha'}$ in all of B_R . However, from the inequality $u(0) \geq 0$, the only possibility is

$$u \geq \frac{C_3}{2} R^{\alpha'} \quad \text{in } B_R.$$

Let h again be the Signorini replacement of u in B_R . Then from positivity of $h = u > 0$ on ∂B_R and superharmonicity of h in B_R , it follows that $h > 0$ in B_R and is therefore harmonic there. Thus,

$$\int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 \leq \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\nabla h - \langle \nabla h \rangle_r|^2, \quad 0 < \rho < r.$$

Using (4.5) and (4.6) with r in lieu of s , we have for all $0 < \rho < r$

$$(4.17) \quad \begin{aligned} \int_{B_\rho} |\widehat{\nabla}h - \langle \widehat{\nabla}h \rangle_\rho|^2 &\leq \int_{B_\rho} |\nabla h - \langle \nabla h \rangle_\rho|^2 \leq \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\nabla h - \langle \nabla h \rangle_r|^2 \\ &\leq \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |\widehat{\nabla}h - \langle \widehat{\nabla}h \rangle_r|^2 + \frac{1}{|B_r|} \left(\int_{B_r} \widehat{h}_{x_n}\right)^2. \end{aligned}$$

Next, note that if $r_0 \leq (1/2)^{\frac{2n+\alpha}{\alpha}}$, then $r \leq R/2$. Then, for $\gamma := 1 - \frac{3\alpha}{8n}$,

$$\begin{aligned} \sup_{B_{R/2}} |D^2 h| &\leq \frac{C(n)}{R} \sup_{B_{(3/4)R}} |\nabla h| \leq \frac{C(n)}{R^{1+\frac{n}{2}}} \left(\int_{B_R} |\nabla h|^2\right)^{1/2} \\ &\leq \frac{C(n)}{R^{1+\frac{n}{2}}} \left(\int_{B_R} |\nabla u|^2\right)^{1/2} \leq C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)} R^{\gamma-2}, \end{aligned}$$

where the last inequality follows from (3.7). Thus, for $x = (x', x_n) \in B_r$, we have

$$\begin{aligned} |h_{x_n}| &\leq |x_n| \sup_{B_{R/2}} |D^2 h| \\ &\leq C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)} r R^{\gamma-2} \\ &\leq C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)} r^{1+\frac{2n}{2n+\alpha}(\gamma-2)}, \end{aligned}$$

and hence

$$(4.18) \quad \frac{1}{|B_r|} \left(\int_{B_r} \widehat{h_{x_n}} \right)^2 \leq C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 r^{n+2+\frac{4n}{2n+\alpha}(\gamma-2)}$$

$$= C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 r^{n+\frac{\alpha}{2(2n+\alpha)}}.$$

Combining (4.17) and (4.18), we obtain

$$(4.19) \quad \int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 \leq \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_r|^2$$

$$+ C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 r^{n+\frac{\alpha}{2(2n+\alpha)}}.$$

Finally, (4.9), (4.10), (4.13), and (4.19) give

$$(4.20) \quad \int_{B_\rho} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_\rho|^2$$

$$\leq 3 \int_{B_\rho} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_\rho|^2 + 6 \int_{B_\rho} |\widehat{\nabla u} - \widehat{\nabla h}|^2$$

$$\leq 3 \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r} |\widehat{\nabla h} - \langle \widehat{\nabla h} \rangle_r|^2$$

$$+ C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 r^{n+\frac{\alpha}{2(2n+\alpha)}} + 6 \int_{B_\rho} |\widehat{\nabla u} - \widehat{\nabla h}|^2$$

$$\leq 9 \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_r|^2$$

$$+ C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 r^{n+\frac{\alpha}{2(2n+\alpha)}} + 24 \int_{B_r} |\widehat{\nabla u} - \widehat{\nabla h}|^2$$

$$\leq 9 \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_r|^2$$

$$+ C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 r^{n+\frac{\alpha}{2(2n+\alpha)}}$$

$$+ C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 r^{n+\frac{\alpha}{2n+\alpha}(n-\frac{1}{2})}$$

$$\leq 9 \left(\frac{\rho}{r} \right)^{n+2} \int_{B_r} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_r|^2$$

$$+ C(n, \alpha, K) \|\nabla u\|_{L^2(B_1)}^2 r^{n+\frac{\alpha}{2(2n+\alpha)}}$$

From (4.16) and (4.20) we obtain (4.8) for $x_0 \in K \cap B'_1$.

Case 2. To extend (4.8) to any $x_0 \in K$, we now assume $x_0 \in K \cap B_1^+$. We use an argument similar to the one in Case 2 in the proof of Proposition 3.3.

Now, if $\rho \geq r/4$, then

$$\int_{B_\rho(x_0)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_0, \rho}|^2 \leq \int_{B_\rho(x_0)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_0, r}|^2$$

$$\leq 4^{n+\alpha} \left(\frac{\rho}{r} \right)^{n+\alpha} \int_{B_r(x_0)} |\widehat{\nabla u} - \langle \widehat{\nabla u} \rangle_{x_0, r}|^2,$$

and thus we may assume $\rho < r/4$. Let $d := \text{dist}(x_0, B'_1) > 0$ and choose $x_1 \in \partial B_d(x_0) \cap B'_1$. Note that from the assumption that K is a ball centered at 0, we have $x_1 \in K \cap B'_1$.

Case 2.1. If $\rho \geq d$, then from $B_\rho(x_0) \subset B_{2\rho}(x_1) \subset B_{r/2}(x_1) \subset B_r(x_0)$, we have

$$\begin{aligned} \int_{B_\rho(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, \rho}|^2 &\leq \int_{B_{2\rho}(x_1)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_1, 2\rho}|^2 \\ &\leq C(n, \alpha) \left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_{r/2}(x_1)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_1, r/2}|^2 \\ &\quad + C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}^2 r^{n+2\beta} \\ &\leq C(n, \alpha) \left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, r}|^2 \\ &\quad + C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}^2 r^{n+2\beta}, \end{aligned}$$

which gives (4.8) in this case.

Case 2.2. Now we suppose $d > \rho$. If also $d > r$, then $B_r(x_0) \subset B_1^+$ and since u is almost harmonic in B_1^+ , we can apply Proposition 2.3, together with the growth estimate (3.7) in the proof of Theorem 3.1, to conclude

$$\begin{aligned} \int_{B_\rho(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, \rho}|^2 &\leq C(n, \alpha) \left(\frac{\rho}{r}\right)^{n+\alpha} \int_{B_r(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, r}|^2 \\ &\quad + C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}^2 r^{n+2\beta}. \end{aligned}$$

Thus, we may assume $d \leq r$. Then, $B_d(x_0) \subset B_1^+$, and hence, again by the combination of Proposition 2.3 and the growth estimate (3.7), we have

$$\begin{aligned} \int_{B_\rho(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, \rho}|^2 &\leq C(n, \alpha) \left(\frac{\rho}{d}\right)^{n+\alpha} \int_{B_d(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, d}|^2 \\ &\quad + C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}^2 d^{n+2\beta}. \end{aligned}$$

We need to consider further subcases.

Case 2.2.1. If $r/4 \leq d$, then (since also $d \leq r$)

$$\int_{B_d(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, d}|^2 \leq 4^{n+\alpha} \left(\frac{d}{r}\right)^{n+\alpha} \int_{B_r(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, r}|^2$$

and combined with the previous inequality, we obtain (4.8) in this subcase.

Case 2.2.2. If $d < r/4$, then we also have

$$\begin{aligned} \int_{B_d(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, d}|^2 &\leq \int_{B_{2d}(x_1)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_1, 2d}|^2 \\ &\leq C(n, \alpha) \left(\frac{d}{r}\right)^{n+\alpha} \int_{B_{r/2}(x_1)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_1, r/2}|^2 \\ &\quad + C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}^2 r^{n+2\beta} \\ &\leq C(n, \alpha) \left(\frac{d}{r}\right)^{n+\alpha} \int_{B_r(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, r}|^2 \\ &\quad + C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}^2 r^{n+2\beta}. \end{aligned}$$

Hence, the estimate (4.8) has been established in all possible cases.

To complete the proof of the theorem, we now apply Lemma 2.4 to the estimate (4.8) to obtain

$$\int_{B_\rho(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, \rho}|^2 \leq C(n, \alpha) \left[\left(\frac{\rho}{r}\right)^{n+2\beta} \int_{B_r(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, r}|^2 + C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}^2 \rho^{n+2\beta} \right].$$

Taking $r \nearrow r_0 = r_0(n, \alpha, K)$, we have

$$\int_{B_\rho(x_0)} |\widehat{\nabla}u - \langle \widehat{\nabla}u \rangle_{x_0, \rho}|^2 \leq C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}^2 \rho^{n+2\beta}.$$

Then by the Campanato space embedding we conclude that

$$\widehat{\nabla}u \in C^{0,\beta}(K)$$

with

$$\|\widehat{\nabla}u\|_{C^{0,\beta}(K)} \leq C(n, \alpha, K) \|u\|_{W^{1,2}(B_1)}. \quad \square$$

Having the $C^{1,\beta}$ regularity of almost minimizers, we can now talk about pointwise values of

$$\partial_{x_n}^+ u(x', 0) = \lim_{\substack{y \rightarrow (x', 0) \\ y \in B_r^+}} \partial_{x_n} u(y)$$

for $x' \in B'_1$. The following complementarity condition is of crucial importance in the study of the free boundary.

Lemma 4.7 (Complementarity condition). *Let u be an almost minimizer for the Signorini problem in B_1 . Then u satisfies the following complementarity condition*

$$u \partial_{x_n}^+ u = 0 \quad \text{on } B'_1.$$

Moreover, if $x_0 \in \Gamma(u)$ then

$$u(x_0) = 0 \quad \text{and} \quad |\widehat{\nabla}u(x_0)| = 0.$$

Proof. Since $u \geq 0$ on B'_1 , the complementarity condition will follow once we show that $\partial_{x_n}^+ u$ vanishes where $u > 0$ on B'_1 . To this end, let $u(x', 0) > 0$ for some $x' \in B'_1$. By the continuity of u in B_1 , (see Theorem 3.1), we have $u > 0$ in some open neighborhood $U \subset B_1$ of $(x', 0)$. If $B_r(y) \Subset U$ (not necessarily centered on B'_1) and v is a harmonic replacement of u in $B_r(y)$, then by the minimum principle $v > 0$ in $\overline{B_r(y)}$, and particularly $v > 0$ on set $B_r(y) \cap B'_1$. Then $v \in \mathfrak{K}_{0,u}(B_r(y), \mathcal{M})$ and therefore we must have

$$\int_{B_r(y)} |\nabla u|^2 \leq (1 + r^\alpha) \int_{B_r(y)} |\nabla v|^2.$$

This means that u is an almost harmonic function in U . Hence $u \in C^{1,\alpha/2}(U)$ by Theorem 2.1. From the even symmetry of u in x_n , it is then immediate that $\partial_{x_n}^+ u(x', 0) = \partial_{x_n} u(x', 0) = 0$.

The second part of the lemma now follows by the $C^{1,\beta}$ regularity and the complementarity condition. \square

5. WEISS- AND ALMGREN-TYPE MONOTONICITY FORMULAS

In the rest of the paper we study the free boundary of almost minimizers. In this section we introduce important technical tools, so-called *Weiss-* and *Almgren-type monotonicity formulas*, which play a significant role in our analysis.

We start with Weiss-type monotonicity formulas. They go back to the works of Weiss [Wei99b, Wei99a] in the case of the classical obstacle problem and Alt-Caffarelli minimum problem, respectively, and to [GP09] for the solutions of the thin obstacle problems. In the context of almost minimizers, this type of monotonicity formulas has been used in a recent paper [DET17].

Theorem 5.1 (Weiss-type monotonicity formula). *Let u be an almost minimizer for the Signorini problem in B_1 . Suppose $x_0 \in B'_{1/2}$ and $u(x_0) = 0$. For $0 < \kappa < \kappa_0$ with a fixed $\kappa_0 \geq 2$ set*

$$W_\kappa(t, u, x_0) := \frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \left[\int_{B_t(x_0)} |\nabla u|^2 - \kappa \frac{1-bt^\alpha}{t} \int_{\partial B_t(x_0)} u^2 \right],$$

with

$$a = a_\kappa = \frac{n+2\kappa-2}{\alpha}, \quad b = \frac{n+2\kappa_0}{\alpha}.$$

Then, for $0 < t < t_0 = t_0(n, \alpha, \kappa_0)$,

$$\frac{d}{dt} W_\kappa(t, u, x_0) \geq \frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \int_{\partial B_t(x_0)} \left(u_\nu - \frac{\kappa(1-bt^\alpha)}{t} u \right)^2.$$

In particular, $W_\kappa(t, u, x_0)$ is nondecreasing in t for $0 < t < t_0$.

Remark 5.2. It is important to observe that while $a = a_\kappa$ depends on κ , the constant b depends only on α , n and κ_0 . We also note that in our version of Weiss's monotonicity formula, perturbations (from the case of the thin obstacle problem) appear in the form of multiplicative factors, rather than additive errors as in [DET17]. Because of the multiplicative nature of the perturbations, we can then use the one-parametric family of monotonicity formulas $\{W_\kappa\}_{0 < \kappa < \kappa_0}$ to derive an Almgren-type monotonicity formula, see Theorem 5.4.

Remark 5.3. To avoid bulky notations, we will write $W_\kappa(t, u)$ for $W_\kappa(t, u, x_0)$ when $x_0 = 0$ or even simply $W_\kappa(t)$, when both u and x_0 are clear from the context.

Proof. The proof uses an argument similar to the one in Theorem 1.2 in [Wei99a]. Essentially, it follows from a comparison (1.4) with special competitors, described below. Without loss of generality, assume $x_0 = 0$. Then for $t \in (0, 1/2)$, define w by

$$w(x) := \left(\frac{|x|}{t} \right)^\kappa u \left(t \frac{x}{|x|} \right), \quad \text{for } x \in B_t.$$

Note that w is κ -homogeneous in B_t , i.e., $w(\lambda x) = \lambda^\kappa w(x)$ for $\lambda > 0$, $x, \lambda x \in B_t$, and coincides with u on ∂B_t . Also note that $w \geq 0$ on B'_t and is therefore a valid competitor for u in (1.4). We refer to this w as the κ -homogeneous replacement of u in B_t .

Now, in B_t , we have

$$\nabla w(x) = \left(\frac{|x|}{t} \right)^{\kappa-1} \left[\frac{\kappa}{t} u \left(t \frac{x}{|x|} \right) \frac{x}{|x|} + \nabla u \left(t \frac{x}{|x|} \right) - \nabla u \left(t \frac{x}{|x|} \right) \cdot \frac{x}{|x|} \frac{x}{|x|} \right],$$

which gives

$$\begin{aligned}
\int_{B_t} |\nabla w|^2 dx &= \int_0^t \int_{\partial B_r} |\nabla w(x)|^2 dS_x dr \\
&= \int_0^t \int_{\partial B_r} \left(\frac{r}{t}\right)^{2\kappa-2} \left| \frac{\kappa}{t} u \left(\frac{x}{r}\right) \nu - \left(\nabla u \left(\frac{x}{r}\right) \cdot \nu\right) \nu + \nabla u \left(\frac{x}{r}\right) \right|^2 dS_x dr \\
&= \int_0^t \int_{\partial B_t} \left(\frac{r}{t}\right)^{n+2\kappa-3} \left| \frac{\kappa}{t} u \nu - (\nabla u \cdot \nu) \nu + \nabla u \right|^2 dS_x dr \\
&= \frac{t}{n+2\kappa-2} \int_{\partial B_t} \left| \nabla u - (\nabla u \cdot \nu) \nu + \frac{\kappa}{t} u \nu \right|^2 dS_x \\
&= \frac{t}{n+2\kappa-2} \int_{\partial B_t} \left(|\nabla u|^2 - (\nabla u \cdot \nu)^2 + \left(\frac{\kappa}{t}\right)^2 u^2 \right) dS_x.
\end{aligned}$$

The latter equality can be rewritten as

$$(5.1) \quad \int_{\partial B_t} u^2 dS_x = \left(\frac{t}{\kappa}\right)^2 \left[\frac{n+2\kappa-2}{t} \int_{B_t} |\nabla w|^2 dx + \int_{\partial B_t} (u_\nu^2 - |\nabla u|^2) dS_x \right].$$

Since w is a competitor for u , we have

$$(5.2) \quad \int_{B_t} |\nabla w|^2 dx \geq \frac{1}{1+t^\alpha} \int_{B_t} |\nabla u|^2 dx \geq (1-t^\alpha) \int_{B_t} |\nabla u|^2 dx$$

and combining (5.1) and (5.2) yields

$$(5.3) \quad \int_{\partial B_t} u^2 dS_x \geq \left(\frac{t}{\kappa}\right)^2 \left[(n+2\kappa-2) \frac{1-t^\alpha}{t} \int_{B_t} |\nabla u|^2 dx + \int_{\partial B_t} (u_\nu^2 - |\nabla u|^2) dS_x \right].$$

Multiplying this by $\kappa^2 e^{at^\alpha} t^{-n-2\kappa}$ and rearranging terms, we obtain

$$(5.4) \quad \begin{aligned} & \frac{d}{dt} \left(e^{at^\alpha} t^{-n-2\kappa+2} \right) \int_{B_t} |\nabla u|^2 dx \\ &= -(n+2\kappa-2) e^{at^\alpha} t^{-n-2\kappa} (t-t^{\alpha+1}) \int_{B_t} |\nabla u|^2 dx \\ &\geq e^{at^\alpha} t^{-n-2\kappa+2} \int_{\partial B_t} (u_\nu^2 - |\nabla u|^2) dS_x - \kappa^2 e^{at^\alpha} t^{-n-2\kappa} \int_{\partial B_t} u^2 dS_x. \end{aligned}$$

Define now an auxiliary function

$$\psi(t) = \frac{\kappa e^{at^\alpha} (1-bt^\alpha)}{t^{n+2\kappa-1}}.$$

Then we write

$$W_\kappa(t, u, 0) = e^{at^\alpha} t^{-n-2\kappa+2} \int_{B_t} |\nabla u|^2 dx - \psi(t) \int_{\partial B_t} u^2 dS_x$$

and, using (5.4), obtain

$$\begin{aligned}
\frac{d}{dt} W_\kappa(t, u, 0) &= \frac{d}{dt} \left(e^{at^\alpha} t^{-n-2\kappa+2} \right) \int_{B_t} |\nabla u|^2 dx + e^{at^\alpha} t^{-n-2\kappa+2} \int_{\partial B_t} |\nabla u|^2 dS_x \\
&\quad - \psi'(t) \int_{\partial B_t} u^2 dS_x - 2\psi(t) \int_{\partial B_t} uu_\nu dS_x - (n-1) \frac{\psi(t)}{t} \int_{\partial B_t} u^2 dS_x
\end{aligned}$$

$$\begin{aligned}
&\geq e^{at^\alpha} t^{-n-2\kappa+2} \int_{\partial B_t} (u_\nu^2 - |\nabla u|^2) dS_x - \kappa^2 e^{at^\alpha} t^{-n-2\kappa} \int_{\partial B_t} u^2 dS_x \\
&\quad + e^{at^\alpha} t^{-n-2\kappa+2} \int_{\partial B_t} |\nabla u|^2 dS_x - \psi'(t) \int_{\partial B_t} u^2 dS_x \\
&\quad - 2\psi(t) \int_{\partial B_t} uu_\nu dS_x - (n-1) \frac{\psi(t)}{t} \int_{\partial B_t} u^2 dS_x \\
&= e^{at^\alpha} t^{-n-2\kappa+2} \int_{\partial B_t} u_\nu^2 dS_x - 2\psi(t) \int_{\partial B_t} uu_\nu dS_x \\
&\quad - \left(\kappa^2 e^{at^\alpha} t^{-n-2\kappa} + \psi'(t) + (n-1) \frac{\psi(t)}{t} \right) \int_{\partial B_t} u^2 dS_x.
\end{aligned}$$

Now observe that $\psi(t)$ satisfies the inequality

$$-\frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \left(\kappa^2 e^{at^\alpha} t^{-n-2\kappa} + \psi'(t) + (n-1) \frac{\psi(t)}{t} \right) - \psi^2(t) \geq 0$$

for $0 < t < t_0(n, \alpha, \kappa_0)$ and $0 < \kappa < \kappa_0$. Indeed, a direct computation shows that the above inequality is equivalent to

$$2\alpha^2(1 + \kappa_0 - \kappa) - (n + 2\kappa_0)[(n + 2\kappa_0)\kappa - \alpha(n + 2\kappa - 2)]t^\alpha \geq 0,$$

which holds for $0 < \kappa < \kappa_0$ and small $t > 0$ such that

$$2\alpha^2 - 4(n + 2\kappa_0)^2 \kappa_0 t^\alpha \geq 0.$$

Hence, recalling also the formula for $\psi(t)$, we can conclude that

$$\begin{aligned}
\frac{d}{dt} W_\kappa(t, u, 0) &\geq \frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \left[\int_{\partial B_t} u_\nu^2 dS_x - 2 \frac{\kappa(1 - bt^\alpha)}{t} \int_{\partial B_t} uu_\nu dS_x \right. \\
&\quad \left. + \left(\frac{\kappa(1 - bt^\alpha)}{t} \right)^2 \int_{\partial B_t} u^2 dS_x \right] \\
&= \frac{e^{at^\alpha}}{t^{n+2\kappa-2}} \int_{\partial B_t} \left(u_\nu - \frac{\kappa(1 - bt^\alpha)}{t} u \right)^2,
\end{aligned}$$

for $0 < t < t_0(n, \alpha, \kappa_0)$. \square

Next, for an almost minimizer u in B_1 and $x_0 \in B'_{1/2}$, consider the quantity

$$N(t, u, x_0) := \frac{t \int_{B_t(x_0)} |\nabla u|^2}{\int_{\partial B_t(x_0)} u^2}, \quad 0 < t < 1/2$$

which is known as *Almgren's frequency* and goes back to Almgren's *Big Regularity Paper* [Alm00]. This kind of quantities have also been used in unique continuation for a class of elliptic operators [GL86, GL87] and have been instrumental in thin obstacle-type problems, starting with the works [ACS08, CSS08, GP09].

Before proceeding, we observe that Almgren's frequency is well defined when x_0 is a free boundary point, since $\int_{\partial B_t(x_0)} u^2 > 0$. Indeed, otherwise $u = 0$ on $\partial B_t(x_0)$ and we can use $h \equiv 0$ in $B_t(x_0)$ as a competitor, to obtain that $\int_{B_t(x_0)} |\nabla u|^2 \leq (1+t^\alpha)0 = 0$, implying $u \equiv 0$ in $B_t(x_0)$, contradicting the assumption that x_0 is a free boundary point. Next, we also consider a modification of N :

$$\tilde{N}(t, u, x_0) := \frac{1}{1 - bt^\alpha} N(t, u, x_0),$$

where b is as in Theorem 5.1, as well as

$$\widehat{N}_{\kappa_0}(t, u, x_0) := \min\{\widetilde{N}(t), \kappa_0\}, \quad 0 < t < t_0,$$

which we call the *truncated frequency*.

For the frequencies N , \widetilde{N} , and \widehat{N}_{κ_0} , we will follow the same notational conventions as outlined in Remark 5.3 for Weiss's functionals W_κ .

With the Weiss type monotonicity formula at hand, we easily obtain the following monotonicity of \widehat{N}_{κ_0} .

Theorem 5.4 (Almgren-type monotonicity formula). *Let u , κ_0 , and t_0 be as in Theorem 5.1, and x_0 a free boundary point. Then $\widehat{N}_{\kappa_0}(t, u, x_0)$ is nondecreasing in $0 < t < t_0$.*

Proof. We assume $x_0 = 0$. It is quite important to observe that t_0 depends only on n , α , and κ_0 . Then, if $\widetilde{N}(t) < \kappa$ for some $t \in (0, t_0)$ and $\kappa \in (0, \kappa_0)$, then

$$\begin{aligned} W_\kappa(t) &= \frac{e^{at^\alpha}}{t^{n+2\kappa-1}} \left(\int_{\partial B_t} u^2 \right) (N(t) - \kappa(1 - bt^\alpha)) \\ &= \frac{e^{at^\alpha}}{t^{n+2\kappa-1}} \left(\int_{\partial B_t} u^2 \right) (1 - bt^\alpha) (\widetilde{N}(t) - \kappa) < 0. \end{aligned}$$

By Theorem 5.1 we also have $W_\kappa(s) \leq W_\kappa(t) < 0$ for all $s \in (0, t)$, and thus $\widetilde{N}(s) < \kappa$. This completes the proof. \square

Remark 5.5. The proof above is rather indirect and establishes the monotonicity of \widehat{N}_{κ_0} from that of Weiss-type formulas in one-parametric family $\{W_\kappa\}_{0 < \kappa < \kappa_0}$. This kind of relation has been first observed in [GP09].

6. ALMGREN RESCALINGS AND BLOWUPS

In this section we prove a lower bound on Almgren's frequency for almost minimizers at free boundary points. The idea is to consider appropriate rescalings and blowups of almost minimizers to obtain solutions of the Signorini problem, for which a bound $N(0+) \geq 3/2$ is known.

Now, let u be an almost minimizer for the Signorini problem in B_1 , and $x_0 \in B'_{1/2}$ a free boundary point. For $0 < r < 1/2$ consider the *Almgren rescaling*³ of u at x_0

$$u_{x_0, r}^A(x) := \frac{u(rx + x_0)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r(x_0)} u^2 \right)^{\frac{1}{2}}}, \quad x \in B_{1/(2r)}.$$

When $x_0 = 0$, we also write u_r^A instead of $u_{0, r}^A$. The Almgren rescalings have the following normalization and scaling properties

$$\begin{aligned} \|u_{x_0, r}^A\|_{L^2(\partial B_1)} &= 1 \\ N(\rho, u_{x_0, r}^A) &= N(\rho r, u, x_0), \quad \rho < 1/(2r). \end{aligned}$$

We will call the limits of $u_{x_0, r}^A$ over any sequence $r = r_j \rightarrow 0+$ *Almgren blowups* of u at x_0 and denote by $u_{x_0, 0}^A$.

³We use the superscript A to distinguish this rescaling from the other rescalings, namely, homogeneous and almost homogeneous rescalings that we consider later.

Proposition 6.1 (Existence of Almgren blowups). *Let $x_0 \in B'_{1/2} \cap \Gamma(u)$ be such that $\widehat{N}_{\kappa_0}(0+, u, x_0) = \kappa < \kappa_0$. Then every sequence of Almgren rescalings u_{x_0, r_j}^A , with $r_j \rightarrow 0+$ contains a subsequence, still denoted r_j , such that for a function $u_{x_0, 0}^A \in W^{1,2}(B_1) \cap C^1_{\text{loc}}(B_1^\pm \cup B'_1)$*

$$\begin{aligned} u_{x_0, r_j}^A &\rightarrow u_{x_0, 0}^A && \text{in } W^{1,2}(B_1), \\ u_{x_0, r_j}^A &\rightarrow u_{x_0, 0}^A && \text{in } L^2(\partial B_1), \\ u_{x_0, r_j}^A &\rightarrow u_{x_0, 0}^A && \text{in } C^1_{\text{loc}}(B_1^\pm \cup B'_1). \end{aligned}$$

Moreover, $u_{x_0, 0}^A$ is a nonzero solution of the Signorini problem in B_1 , even in x_n , and homogeneous of degree κ in B_1 , i.e.,

$$u_{x_0, 0}^A(\lambda x) = \lambda^\kappa u_{x_0, 0}^A(x),$$

for $\lambda > 0$, provided $x, \lambda x \in B_1$.

Proof. Without loss of generality, we assume $x_0 = 0$. From the fact that $\widehat{N}(0+, u) = \kappa < \kappa_0$, it follows also that $N(0+, u) = \widehat{N}(0+, u) = \kappa$. In particular, $N(r_j, u) < \kappa_0$ for large j . Then, for such j

$$\int_{B_1} |\nabla u_{r_j}^A|^2 = N(1, u_{r_j}^A) = N(r_j, u) \leq \kappa_0$$

and combined with the normalization $\int_{\partial B_1} (u_{r_j}^A)^2 = 1$, we see that the sequence $u_{r_j}^A$ is bounded in $W^{1,2}(B_1)$. Hence, there is a function $u_0^A \in W^{1,2}(B_1)$ such that, over a subsequence,

$$\begin{aligned} u_{r_j}^A &\rightarrow u_0^A && \text{weakly in } W^{1,2}(B_1), \\ u_{r_j}^A &\rightarrow u_0^A && \text{strongly in } L^2(\partial B_1). \end{aligned}$$

In particular, $\int_{\partial B_1} (u_0^A)^2 = 1$, implying that $u_0^A \not\equiv 0$ in B_1 .

Next, we observe that since u is an almost minimizer in B_1 with gauge function $\omega(t) = t^\alpha$, u_r^A is also an almost minimizer in $B_{1/(2r)}$ with gauge function $\omega_r(t) = (rt)^\alpha$. This is rather easy to see, since $u_r^A(x)$ up to a positive constant factor is $u(rx)$ and the multiplication (or the division) by a positive number preserves the almost minimizing property. Since $\omega_r(t) \leq \omega(t)$, Theorem 4.6 is applicable to rescalings $u_{r_j}^A$, from where we can deduce that over yet another subsequence,

$$(6.1) \quad u_{r_j}^A \rightarrow u_0^A \quad \text{in } C^1_{\text{loc}}(B_1^\pm \cup B'_1).$$

Now, we claim that since the gauge functions $\omega_r(t) = (rt)^\alpha \rightarrow 0$ as $r \rightarrow 0$, the blowup u_0^A is a solution of the Signorini problem in B_1 . Indeed, for a fixed r_j , let h_{r_j} be the Signorini replacement of $u_{r_j}^A$ in B_1 . Then, by repeating the argument as in the proof of Proposition 3.3

$$\int_{B_1} |\nabla(u_{r_j}^A - h_{r_j})|^2 \leq r_j^\alpha \int_{B_1} |\nabla u_{r_j}^A|^2.$$

This implies that $h_{r_j} \rightarrow u_0^A$ weakly in $W^{1,2}(B_1)$. On the other hand, by the boundedness of the sequence h_{r_j} in $W^{1,2}(B_1)$, we have also boundedness in $C^{1,1/2}_{\text{loc}}(B_1^\pm \cup B'_1)$

and hence, over a subsequence, $h_{r_j} \rightarrow u_0^A$ in $C_{\text{loc}}^1(B_1^\pm \cup B'_1)$. By this convergence we then conclude that u_0^A satisfies

$$\begin{aligned} \Delta u_0^A &= 0 \quad \text{in } B_1 \setminus B'_1 \\ u_0^A &\geq 0, \quad -\partial_{x_n}^+ u_0^A \geq 0, \quad u_0^A \partial_{x_n}^+ u_0^A = 0 \quad \text{on } B'_1, \end{aligned}$$

and hence u_0^A itself solves the Signorini problem in B_1 .

Using the C_{loc}^1 convergence again, we have that for any $0 < \rho < 1$

$$N(\rho, u_0^A) = \lim_{r_j \rightarrow 0} N(\rho, u_{r_j}^A) = \lim_{r_j \rightarrow 0} N(\rho r_j, u) = N(0+, u) = \kappa.$$

Thus, the Almgren frequency of u_0^A is constant κ , which is possible only if u_0^A is a κ -homogeneous solution of the Signorini problem in B_1 , see [PSU12, Theorem 9.4]. \square

In what follows, it will be sufficient for us to fix $\kappa_0 \geq 2$ (say $\kappa_0 = 2$), in the definition of \widehat{N}_{κ_0} and we will simply write

$$\widehat{N} = \widehat{N}_{\kappa_0}.$$

Lemma 6.2 (Minimal frequency). *Let u be an almost minimizer for the Signorini problem in B_1 . If $x_0 \in B'_{1/2} \cap \Gamma(u)$, then*

$$\widehat{N}(0+, u, x_0) = \lim_{r \rightarrow 0+} \widehat{N}(r, u, x_0) \geq \frac{3}{2}.$$

Consequently, we also have

$$\widehat{N}(t, u, x_0) \geq 3/2 \quad \text{for } 0 < t < t_0.$$

Proof. As before, let $x_0 = 0$. Assume to the contrary that $\widehat{N}(0+, u) = \kappa < 3/2$. Since $\kappa < \kappa_0$ we can apply Proposition 6.1 to obtain that over a sequence $r_j \rightarrow 0+$, $u_{r_j}^A \rightarrow u_0^A$ in $C_{\text{loc}}^1(B_1^\pm \cup B'_1)$, where u_0^A is a nonzero κ -homogeneous solution of the Signorini problem in B_1 , even in x_n . Moreover, since $0 \in \Gamma(u)$, by Lemma 4.7 we have that $u(0) = |\widehat{\nabla}u(0)| = 0$, implying that $u_{r_j}^A(0) = |\widehat{\nabla}u_{r_j}^A(0)| = 0$ and, by passing to the limit, $u_0^A(0) = |\widehat{\nabla}u_0^A(0)| = 0$. Now, to arrive at a contradiction, we argue as in the proof of [PSU12, Proposition 9.9] to reduce the problem to dimension $n = 2$, where we can classify all possible homogeneous solutions of the Signorini problem, even in x_n . The only nonzero homogeneous solutions with $\kappa < 3/2$ in dimension $n = 2$ are possible for $\kappa = 1$ and have the form $u_0^A(x) = -cx_n$ for some $c > 0$, but they fail to satisfy the condition $|\widehat{\nabla}u_0^A(0)| = 0$. Thus, we arrived at contradiction, implying that $\widehat{N}(0+, u) \geq 3/2$. Finally, applying Theorem 5.4, we obtain $\widehat{N}(t, u) \geq \widehat{N}(0+, u) \geq 3/2$, for $0 < t < t_0$. \square

Corollary 6.3. *Let u be an almost minimizer for the Signorini problem in B_1 and x_0 a free boundary point. Then*

$$W_{3/2}(t, u, x_0) \geq 0 \quad \text{for } 0 < t < t_0.$$

Proof. We simply observe that $\widetilde{N}(t) \geq \widehat{N}(t) \geq 3/2$ for $0 < t < t_0$ and hence

$$W_{3/2}(t, u, x_0) = \frac{e^{at^\alpha}}{t^{n+2\kappa-1}} \left(\int_{\partial B_t} u^2 \right) (1 - bt^\alpha) \left(\widetilde{N}(t) - \frac{3}{2} \right) \geq 0. \quad \square$$

7. GROWTH ESTIMATES

An important step in the study of the free boundary in the Signorini problem (and in many other free boundary problems) is the proof of the optimal regularity of solutions, which in this case is $C^{1,1/2}$ on each side of the thin space. This allows to make proper blowup arguments to establish the regularity of the so-called regular part of the free boundary. However, in the case of almost minimizers, we only know $C^{1,\beta}$ regularity for some small $\beta > 0$ and do not expect to have anything better. Yet, in this section, we establish the *optimal growth* of the almost minimizers at free boundary points with the help of the Weiss-type monotonicity formula and the epiperimetric inequality.

Finally, we want to point out that the results in this section are rather immediate in the case of minimizers, as they follow easily from the differentiation formulas for the quantities involved in the Almgren's frequency formula. This is completely unavailable for almost minimizers.

We start by defining a new type of rescalings. Fix $\kappa \geq 3/2$. For a free boundary point x_0 in $B'_{1/2}$ and $r > 0$, we define the κ -homogeneous rescaling by

$$u_{x_0,r}(x) := u_{x_0,r}^{(\kappa)}(x) = \frac{u(rx + x_0)}{r^\kappa}, \quad x \in B_{1/(2r)}.$$

To take advantage of the Weiss-type monotonicity formula, we need a slight modification of this rescaling. With the help of an auxiliary function

$$\phi(r) = \phi_\kappa(r) := e^{-(\kappa b/\alpha)r^\alpha}, \quad r > 0,$$

which is a solution of the differential equation

$$\phi'(r) = \kappa \phi(r) \frac{1 - br^\alpha}{r}, \quad r > 0$$

we define the κ -almost homogeneous rescalings by

$$u_{x_0,r}^\phi(x) := \frac{u(rx + x_0)}{\phi(r)}, \quad x \in B_{1/(2r)}.$$

Lemma 7.1 (Weak growth estimate). *Let u be an almost minimizer of the Signorini problem in B_1 and $x_0 \in B'_{1/2} \cap \Gamma(u)$ be such that $\widehat{N}(0+, u, x_0) \geq \kappa$ for $\kappa \leq \kappa_0$. Then*

$$\begin{aligned} \int_{\partial B_t(x_0)} u^2 &\leq C(n, \alpha, \kappa_0) \|u\|_{W^{1,2}(B_1)}^2 \left(\log \frac{1}{t}\right) t^{n+2\kappa-1}, \\ \int_{B_t(x_0)} |\nabla u|^2 &\leq C(n, \alpha, \kappa_0) \|u\|_{W^{1,2}(B_1)}^2 \left(\log \frac{1}{t}\right) t^{n+2\kappa-2}, \end{aligned}$$

for $0 < t < t_0 = t_0(n, \alpha, \kappa_0)$.

Proof. Without loss of generality, assume $x_0 = 0$. We first note that the condition $\widehat{N}(0+, u) \geq \kappa$ implies that $\widehat{N}(t, u) \geq \kappa$ for $0 < t < t_0 = t_0(n, \alpha, \kappa_0)$. Then also $\widetilde{N}(t, u) \geq \kappa$ for such t and consequently,

$$W_\kappa(t, u) = \frac{e^{at^\alpha}}{t^{n+2\kappa-1}} \left(\int_{\partial B_t} u^2 \right) (1 - bt^\alpha) \left(\widetilde{N}(t, u) - \kappa \right) \geq 0.$$

Next, for $\phi = \phi_\kappa$, we have that

$$\frac{d}{dr} u_r^\phi(x) = \frac{\nabla u(rx) \cdot x}{\phi(r)} - \frac{u(rx)[\phi'(r)/\phi(r)]}{\phi(r)}$$

$$= \frac{1}{\phi(r)} \left(\nabla u(rx) \cdot x - \frac{\kappa(1-br^\alpha)}{r} u(rx) \right).$$

Now let

$$m(r) = \left(\int_{\partial B_1} (u_r^\phi(\xi))^2 dS_\xi \right)^{1/2}, \quad r > 0.$$

Then,

$$m'(r) = \left(\int_{\partial B_1} u_r^\phi(\xi) \frac{d}{dr} u_r^\phi(\xi) dS_\xi \right) \left(\int_{\partial B_1} (u_r^\phi(\xi))^2 dS_\xi \right)^{-1/2}$$

and consequently, by Cauchy-Schwarz,

$$|m'(r)| \leq \left(\int_{\partial B_1} \left[\frac{d}{dr} u_r^\phi(\xi) \right]^2 dS_\xi \right)^{1/2}.$$

Hence,

$$\begin{aligned} |m'(r)| &\leq \frac{1}{\phi(r)} \left(\int_{\partial B_1} \left(\nabla u(r\xi) \cdot \xi - \frac{\kappa(1-br^\alpha)}{r} u(r\xi) \right)^2 dS_\xi \right)^{1/2} \\ &= \frac{1}{\phi(r)} \left(\frac{1}{r^{n-1}} \int_{\partial B_r} \left(\partial_\nu u(x) - \frac{\kappa(1-br^\alpha)}{r} u(x) \right)^2 dS_x \right)^{1/2} \\ &\leq \frac{1}{\phi(r)} \left(\frac{1}{r^{n-1}} \frac{r^{n+1}}{e^{ar^\alpha}} \frac{d}{dr} W_\kappa(r) \right)^{1/2} = \frac{e^{cr^\alpha}}{r^{1/2}} \left(\frac{d}{dr} W_\kappa(r) \right)^{1/2}, \quad c = \kappa \frac{b}{\alpha} - \frac{a}{2}, \end{aligned}$$

for $0 < r < t_0 = t_0(n, \alpha, \kappa_0)$. Thus, we have shown

$$|m'(r)| \leq \frac{e^{cr^\alpha}}{r^{1/2}} \left(\frac{d}{dr} W_\kappa(r) \right)^{1/2}, \quad 0 < r < t_0.$$

Integrating in r over the interval $(s, t) \subset (0, t_0)$, we obtain

$$\begin{aligned} |m(t) - m(s)| &\leq \int_s^t \frac{e^{cr^\alpha}}{r^{1/2}} \left(\frac{d}{dr} W_\kappa(r) \right)^{1/2} dr \\ &\leq \left(\int_s^t \frac{e^{2cr^\alpha}}{r} dr \right)^{1/2} \left(\int_s^t \frac{d}{dr} W_\kappa(r) \right)^{1/2} \\ &\leq C_0 \left(\log \frac{t}{s} \right)^{1/2} [W_\kappa(t) - W_\kappa(s)]^{1/2}. \end{aligned}$$

In particular (recalling that $W_\kappa(s) \geq 0$), we obtain

$$m(t) \leq m(t_0) + C_0 \left(\log \frac{t_0}{t} \right)^{1/2} [W_\kappa(t_0)]^{1/2}.$$

Varying t_0 by an absolute factor, we can guarantee that

$$m(t_0) \leq C(n, \alpha, \kappa_0) \|u\|_{L^2(B_1)}, \quad W_\kappa(t_0) \leq C(n, \alpha, \kappa_0) \|u\|_{W^{1,2}(B_1)}^2.$$

Hence, we can conclude

$$\int_{\partial B_t} u^2 \leq C(n, \alpha, \kappa_0) \|u\|_{W^{1,2}(B_1)}^2 \left(\log \frac{1}{t} \right) t^{n+2\kappa-1},$$

for $0 < t < t_0 = t_0(n, \alpha, \kappa_0)$. This implies the first bound. The second bound follows immediately from the first one by using that $W_\kappa(t, u) \leq W_\kappa(t_0, u)$:

$$\begin{aligned} \frac{1}{t^{n+2\kappa-2}} \int_{B_t} |\nabla u|^2 &\leq \frac{\kappa(1-bt^\alpha)}{t^{n+2\kappa-1}} \int_{\partial B_t} u^2 + e^{-at^\alpha} W_\kappa(t_0, u) \\ &\leq C(n, \alpha, \kappa_0) \|u\|_{W^{1,2}(B_1)}^2 \left(\log \frac{1}{t} \right) + \frac{e^{at_0^\alpha}}{t_0^{n+2\kappa-2}} \int_{B_{t_0}} |\nabla u|^2 \\ &\leq C(n, \alpha, \kappa_0) \|u\|_{W^{1,2}(B_1)}^2 \left(\log \frac{1}{t} \right). \quad \square \end{aligned}$$

The logarithmic term in Lemma 7.1 does not allow to conclude that the sequence of κ -homogeneous or almost homogeneous rescaling is uniformly bounded say in $W^{1,2}(B_1)$. In the rest of this section we show that in the case of the minimal frequency $\kappa = 3/2$ we can do that with the help of the so-called epiperimetric inequality for the Signorini problem for the Weiss energy

$$W_{3/2}^0(w) := \int_{B_1} |\nabla w|^2 - \frac{3}{2} \int_{\partial B_1} w^2.$$

To state this result, we let

$$(7.1) \quad \mathcal{A} := \{w \in W^{1,2}(B_1) : w \geq 0 \text{ on } B'_1, w(x', x_n) = w(x', -x_n)\}$$

Theorem 7.2 (Epiperimetric inequality). *There exists $\eta \in (0, 1)$ such that if $w \in \mathcal{A}$ is homogeneous of degree $3/2$ in B_1 , then there exists $v \in \mathcal{A}$ with $v = w$ on ∂B_1 such that*

$$W_{3/2}^0(v) \leq (1 - \eta) W_{3/2}^0(w).$$

This kind of inequalities go back to the work of Weiss [Wei99b], in the case of the classical obstacle problem. For the Signorini problem, a version of this theorem was proved in [GPS16] and [FS16]. In fact, the theorem above is the version in [RS17]. The inequality in [GPS16] and [FS16] requires w to be close to the blowup profile, but this can be easily removed by a scaling argument (see [RS17]). We also refer to [CSV17], for a more direct proof of this inequality with an explicit constant $\eta = 1/(2n + 3)$.

Now, with the help of the epiperimetric inequality, we can prove a decay estimate for the Weiss-type energy functional $W_{3/2}$. For the rest of the section, we will assume

$$\kappa_0 = 2,$$

which will make some of the constants independent of κ_0 , but the results hold also for any other value of $\kappa_0 \geq 2$, with possible added dependence of constants on κ_0 .

Lemma 7.3. *Let $x_0 \in B'_{1/2}$ be a free boundary point. Then, there exist $\delta = \delta(n, \alpha) > 0$ such that*

$$0 \leq W_{3/2}(t, u, x_0) \leq Ct^\delta, \quad 0 < t < t_0 = t_0(n, \alpha),$$

with $C = C(n, \alpha) \|u\|_{W^{1,2}(B_1)}^2$.

Proof. As before, without loss of generality we assume that $x_0 = 0$.

The proof will follow from a differential inequality that we derive by using our earlier computations and the epiperimetric inequality. Recalling the proof of the Weiss-type monotonicity formula (Theorem 5.1), for small $t > 0$, we have

$$\begin{aligned}
\frac{d}{dt}W_{3/2}(t, u) &= \frac{e^{at^\alpha}}{t^{n+1}} \int_{\partial B_t} |\nabla u|^2 - \frac{(n+1)(1-t^\alpha)e^{at^\alpha}}{t^{n+2}} \int_{B_t} |\nabla u|^2 \\
&\quad - \psi'(t) \int_{\partial B_t} u^2 - (n-1) \frac{\psi(t)}{t} \int_{\partial B_t} u^2 - 2\psi(t) \int_{\partial B_t} u \partial_\nu u \\
&= -\frac{(n+1)(1-t^\alpha)}{t} W_{3/2}(t, u) + \frac{e^{at^\alpha}}{t^{n+1}} \int_{\partial B_t} |\nabla u|^2 \\
&\quad - \left([(n+1)(1-t^\alpha) + (n-1)] \frac{\psi(t)}{t} + \psi'(t) \right) \int_{\partial B_t} u^2 \\
&\quad - 2\psi(t) \int_{\partial B_t} u \partial_\nu u \\
&\geq -\frac{(n+1)(1-t^\alpha)}{t} W_{3/2}(t, u) \\
&\quad + \frac{e^{at^\alpha}(1-bt^\alpha)}{t^{n+1}} \int_{\partial B_t} \left(|\nabla u|^2 - \frac{3}{t} u \partial_\nu u \right. \\
&\quad \quad \left. - \frac{3}{2t} \left[\frac{(n+1)(1-t^\alpha) + (n-1)}{t} + \frac{\psi'(t)}{\psi(t)} \right] u^2 \right).
\end{aligned}$$

To proceed, note that

$$\frac{(n+1)(1-t^\alpha) + (n-1)}{t} + \frac{\psi'(t)}{\psi(t)} = \frac{(n-2) + O(t^\alpha)}{t}.$$

Now, for the homogeneous rescalings

$$u_t(x) = \frac{u(tx)}{t^{3/2}},$$

we can write

$$\begin{aligned}
&\int_{\partial B_t} |\nabla u|^2 - \frac{3}{t} u \partial_\nu u - \frac{3}{2} \frac{(n-2) + O(t^\alpha)}{t^2} u^2 \\
&= t^n \int_{\partial B_1} |\nabla u_t|^2 - 3u_t \partial_\nu u_t - \frac{3}{2} [(n-2) + O(t^\alpha)] u_t^2 \\
&= t^n \int_{\partial B_1} \left(\partial_\nu u_t - \frac{3}{2} u_t \right)^2 + (\partial_\tau u_t)^2 - \frac{3}{2} \left[\left(n - \frac{1}{2} \right) + O(t^\alpha) \right] u_t^2,
\end{aligned}$$

where $\partial_\tau u_t$ is the tangential component of ∇u_t on the unit sphere. We can summarize for now that

$$\begin{aligned}
\frac{d}{dt}W_{3/2}(t, u) &\geq -\frac{(n+1)(1-t^\alpha)}{t} W_{3/2}(t, u) \\
&\quad + \frac{e^{at^\alpha}(1-bt^\alpha)}{t} \int_{\partial B_1} \left[\left(\partial_\nu u_t - \frac{3}{2} u_t \right)^2 + (\partial_\tau u_t)^2 - \frac{3}{2} \left(n - \frac{1}{2} \right) u_t^2 \right] \\
&\quad + O(t^{\alpha-1}) \int_{\partial B_1} u_t^2.
\end{aligned}$$

On the other hand, if w_t is a 3/2-homogeneous replacement of u_t in B_1 , i.e.,

$$w_t(x) = |x|^{3/2}u_t(x/|x|)$$

then

$$\begin{aligned} \int_{\partial B_1} (\partial_\tau u_t)^2 - \frac{3}{2} \left(n - \frac{1}{2} \right) u_t^2 &= \int_{\partial B_1} (\partial_\tau w_t)^2 - \frac{3}{2} \left(n - \frac{1}{2} \right) w_t^2 \\ &= (n+1)W_{3/2}^0(w_t), \end{aligned}$$

where

$$W_{3/2}^0(w_t) = \int_{B_1} |\nabla w_t|^2 - \frac{3}{2} \int_{\partial B_1} w_t^2.$$

The last equality follows by repeating the arguments in the beginning of the proof of Theorem 5.1 with $\kappa = 3/2$. Let v_t be the solution of the Signorini problem in B_1 with $v_t = u_t = w_t$ on ∂B_1 . Then by the epiperimetric inequality

$$W_{3/2}^0(v_t) \leq (1 - \eta)W_{3/2}^0(w_t).$$

On the other hand, since u is an almost minimizer, we have

$$\int_{B_1} |\nabla u_t|^2 \leq (1 + t^\alpha) \int_{B_1} |\nabla v_t|^2$$

and since also $u_t = v_t$ on ∂B_1 , we have

$$\begin{aligned} W_{3/2}(t, u) &= \frac{e^{at^\alpha}}{t^{n+1}} \left[\int_{B_t} |\nabla u|^2 - \frac{(3/2)(1 - bt^\alpha)}{t} \int_{\partial B_t} u^2 \right] \\ &\leq (1 + O(t^\alpha))W_{3/2}^0(v_t) + O(t^\alpha) \int_{\partial B_1} u_t^2 \\ &\leq \left(1 - \frac{\eta}{2} \right) W_{3/2}^0(w_t) + O(t^\alpha) \int_{\partial B_1} u_t^2, \quad \text{for } 0 < t < t_0 = t_0(n, \alpha). \end{aligned}$$

We can therefore write

$$\begin{aligned} \frac{d}{dt} W_{3/2}(t, u) &\geq -\frac{(n+1)(1 - t^\alpha)}{t} W_{3/2}(t, u) \\ &\quad + \frac{(n+1)e^{at^\alpha}(1 - bt^\alpha)}{t} W_{3/2}^0(w_t) + O(t^{\alpha-1}) \int_{\partial B_1} u_t^2 \\ &\geq \frac{n+1}{t} \left(-1 + \frac{1}{1 - \eta/2} + O(t^\alpha) \right) W_{3/2}(t, u) + \frac{O(t^\alpha)}{t^{n+3}} \int_{\partial B_t} u^2 \\ &\geq \frac{\eta}{4t} W_{3/2}(t, u) - Ct^{\alpha/2-1}, \end{aligned}$$

for small t , where we have also used the growth estimate in Lemma 7.1. Taking now δ such that

$$0 < \delta < \min \left\{ \frac{\eta}{4}, \frac{\alpha}{2} \right\},$$

we have

$$\frac{d}{dt} \left[W_{3/2}(t, u)t^{-\delta} + \frac{C}{\alpha/2 - \delta} t^{\alpha/2 - \delta} \right]$$

$$\begin{aligned}
&= t^{-\delta} \left(\frac{d}{dt} W_{3/2}(t, u) - \frac{\delta}{t} W_{3/2}(t, u) \right) + Ct^{\alpha/2-\delta-1} \\
&\geq t^{-\delta-1} \left[\frac{\eta}{4} - \delta \right] W_{3/2}(t, u) - Ct^{\alpha/2-\delta-1} + Ct^{\alpha/2-\delta-1} \\
&\geq 0,
\end{aligned}$$

for small t , where we have used again that $W_{3/2}(t, u) \geq 0$. Thus, we can conclude that

$$0 \leq W_{3/2}(t, u) \leq Ct^\delta, \quad 0 < t < t_0 = t_0(n, \alpha),$$

with $C = C(n, \alpha) \|u\|_{W^{1,2}(B_1)}^2$. \square

Using the estimate on $W_{3/2}(t, u)$ in Lemma 7.3, we can improve on Lemma 7.1 in the case $\kappa = 3/2$.

Lemma 7.4 (Optimal growth estimate). *Let $x_0 \in B'_{1/2}$ be a free boundary point. Then, for $0 < t < t_0 = t_0(n, \alpha)$,*

$$\begin{aligned}
\int_{\partial B_t(x_0)} u^2 &\leq C(n, \alpha) \|u\|_{W^{1,2}(B_1)}^2 t^{n+2}, \\
\int_{B_t(x_0)} |\nabla u|^2 &\leq C(n, \alpha) \|u\|_{W^{1,2}(B_1)}^2 t^{n+1}.
\end{aligned}$$

Proof. We proceed as in the proof of Lemma 7.1 up to the estimate

$$|m(t) - m(s)| \leq C_0 \left(\log \frac{t}{s} \right)^{1/2} [W_{3/2}(t, u) - W_{3/2}(s, u)]^{1/2}.$$

From there, using Lemma 7.3, we now have an improved bound

$$|m(t) - m(s)| \leq C \left(\log \frac{t}{s} \right)^{1/2} t^{\delta/2}, \quad s < t < t_0,$$

with $C = C(n, \alpha) \|u\|_{W^{1,2}(B_1)}$. Then, by a dyadic argument, we can conclude that

$$|m(t) - m(s)| \leq Ct^{\delta/2}.$$

Indeed, let $k = 0, 1, 2, \dots$ be such that $t/2^{k+1} \leq s < t/2^k$. Then,

$$\begin{aligned}
|m(t) - m(s)| &\leq \sum_{j=1}^k |m(t/2^{j-1}) - m(t/2^j)| + |m(t/2^k) - m(s)| \\
&\leq C(\log 2)^{1/2} \sum_{j=1}^{k+1} (t/2^{j-1})^{\delta/2} \leq C(\log 2)^{1/2} \frac{t^{\delta/2}}{1 - 2^{-\delta/2}} = Ct^{\delta/2}.
\end{aligned}$$

In particular, we have

$$m(t) \leq m(t_0) + Ct_0^{\delta/2} \leq C(n, \alpha) \|u\|_{W^{1,2}(B_1)}, \quad t < t_0.$$

This implies the first bound. The second bound follows immediately from the first one by using that $W_{3/2}(t, u) \leq W_{3/2}(t_0, u)$:

$$\frac{1}{t^{n+1}} \int_{B_t} |\nabla u(x)|^2 dx \leq \frac{(3/2)(1 - bt^\alpha)}{t^{n+2}} \int_{\partial B_t} u(x)^2 dS_x + e^{-at^\alpha} W_{3/2}(t_0, u)$$

$$\begin{aligned}
&\leq C(n, \alpha) \|u\|_{W^{1,2}(B_1)}^2 + \frac{e^{at_0^\alpha}}{t_0^{n+1}} \int_{B_{t_0}} |\nabla u(x)|^2 dx \\
&\leq C(n, \alpha) \|u\|_{W^{1,2}(B_1)}^2. \quad \square
\end{aligned}$$

8. 3/2-HOMOGENEOUS BLOWUPS

For a free boundary point $x_0 \in B'_{1/2}$, we consider again the 3/2-almost homogeneous rescalings

$$u_{x_0, t}^\phi(x) = \frac{u(tx + x_0)}{\phi(t)}, \quad x \in B_{1/(2t)},$$

with $\phi = \phi_{3/2}$. We now observe that the optimal growth estimates in Lemma 7.4 implies the boundedness of this family of rescalings in $W^{1,2}(B_R)$ for any $R > 1$. Indeed, the rescalings above will be defined in B_R if $t < 1/(2R)$, and by Lemma 7.4, we will have

$$\begin{aligned}
\int_{B_R} |\nabla u_{x_0, t}^\phi|^2 &= \frac{e^{\frac{3b}{\alpha} t^\alpha}}{t^{n+1}} \int_{B_{Rt}(x_0)} |\nabla u|^2 \leq C(n, \alpha) \|u\|_{W^{1,2}(B_1)}^2 R^{n+1}, \\
\int_{\partial B_R} (u_{x_0, t}^\phi)^2 &= \frac{e^{\frac{3b}{\alpha} t^\alpha}}{t^{n+2}} \int_{\partial B_{Rt}(x_0)} u^2 \leq C(n, \alpha) \|u\|_{W^{1,2}(B_1)}^2 R^{n+2},
\end{aligned}$$

for $0 < t < t_0/R$. Arguing as in the proof of Proposition 6.1, we have for a sequence $t = t_j \rightarrow 0+$

$$u_{x_0, t_j}^\phi \rightarrow u_{x_0, 0}^\phi \quad \text{in } C_{\text{loc}}^1(B_R^\pm \cup B'_R).$$

By letting $R \rightarrow \infty$ and using Cantor's diagonal argument, we therefore have that over a subsequence $t = t_j \rightarrow 0+$

$$u_{x_0, t_j}^\phi \rightarrow u_{x_0, 0}^\phi \quad \text{in } C_{\text{loc}}^1(\mathbb{R}_\pm^n \cup \mathbb{R}^{n-1}).$$

We call such $u_{x_0, 0}^\phi$ a 3/2-homogeneous blowup of u at x_0 . The name is explained by the fact that

$$\lim_{t \rightarrow 0} \frac{\phi(t)}{t^{3/2}} = 1,$$

which implies that if we consider the 3/2-homogeneous rescalings

$$u_{x_0, t}^{(3/2)}(x) = \frac{u(tx + x_0)}{t^{3/2}},$$

then we will have

$$u_{x_0, 0}^\phi = \lim_{t_j \rightarrow 0} u_{x_0, t_j}^\phi = \lim_{t_j \rightarrow 0} u_{x_0, t_j}^{(3/2)} =: u_{x_0, 0}^{(3/2)}$$

and thus $u_{x_0, 0}^\phi = u_{x_0, 0}^{(3/2)}$.

Remark 8.1. Because of the logarithmic term in the weak growth estimates in Lemma 7.1, at the moment we are unable to consider κ -homogeneous blowups as above for frequencies other than $\kappa = 3/2$. However, once the logarithmic term is removed, the same construction as for $\kappa = 3/2$ applies. In particular, we note that in Lemma 10.8 we prove the optimal growth estimates for frequencies $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, enabling us to consider the κ -homogeneous blowups for these values of κ .

We show next that the 3/2-homogeneous blowups are unique at free boundary points. This is achieved by the control on the ‘‘rotation’’ of the rescalings $u_{x_0, r}^\phi(x)$.

Lemma 8.2 (Rotation estimate). *Let u be an almost minimizer for the Signorini problem in B_1 , $x_0 \in B'_{1/2}$ a free boundary point, and δ as in Lemma 7.3. Then for $\kappa = 3/2$ and $\phi = \phi_{3/2}$*

$$\int_{\partial B_1} |u_{x_0,t}^\phi - u_{x_0,s}^\phi| \leq Ct^{\delta/2}, \quad s < t < t_0 = t_0(n, \alpha),$$

for $C = C(n, \alpha) \|u\|_{W^{1,2}(B_1)}$.

Proof. The proof uses computations similar to the proof of Lemma 7.1 combined with the growth estimated for $W_{3/2}(t, u)$ in Lemma 7.3. We assume $x_0 = 0$, and have

$$\begin{aligned} \int_{\partial B_1} |u_t^\phi - u_s^\phi| &\leq \int_{\partial B_1} \int_s^t \left| \frac{d}{dr} u_r^\phi \right| dr = \int_s^t \int_{\partial B_1} \left| \frac{d}{dr} u_r^\phi \right| dr \\ &\leq C_n \int_s^t \left(\int_{\partial B_1} \left| \frac{d}{dr} u_r^\phi \right|^2 \right)^{1/2} \\ &\leq C_n \left(\int_s^t \frac{1}{r} dr \right)^{1/2} \left(\int_s^t r \int_{\partial B_1} \left| \frac{d}{dr} u_r^\phi \right|^2 \right)^{1/2} \\ &\leq C_n e^{ct^\alpha} \left(\log \frac{t}{s} \right)^{1/2} \left(\int_s^t \frac{d}{dr} W_{3/2}(r, u) dr \right)^{1/2}, \quad c = \frac{3b}{2\alpha} - \frac{a}{2}, \end{aligned}$$

where we have re-used the computation made in the proof of Lemma 7.1. Thus, we obtain

$$\begin{aligned} \int_{\partial B_1} |u_t^\phi - u_s^\phi| &\leq C(n, \alpha) \left(\log \frac{t}{s} \right)^{1/2} (W_{3/2}(t, u) - W_{3/2}(s, u))^{1/2} \\ &\leq C \left(\log \frac{t}{s} \right)^{1/2} t^{\delta/2}. \end{aligned}$$

Then, using a dyadic argument as Lemma 7.4, we can conclude that

$$\int_{\partial B_1} |u_t^\phi - u_s^\phi| \leq Ct^{\delta/2}, \quad s < t < t_0,$$

as required. Indeed, let $k = 0, 1, 2, \dots$ be such that $t/2^{k+1} \leq s < t/2^k$. Then

$$\begin{aligned} \int_{\partial B_1} |u_t^\phi - u_s^\phi| &\leq \sum_{j=1}^k \int_{\partial B_1} |u_{t/2^{j-1}}^\phi - u_{t/2^j}^\phi| + \int_{\partial B_1} |u_{t/2^k}^\phi - u_s^\phi| \\ &\leq C(\log 2)^{1/2} \sum_{j=1}^{k+1} (t/2^{j-1})^{\delta/2} \leq C(\log 2)^{1/2} \frac{t^{\delta/2}}{1 - 2^{-\delta/2}}. \end{aligned}$$

This completes the proof. \square

The uniqueness of 3/2-homogeneous blowup now follows.

Lemma 8.3. *Let $u_{x_0,0}^\phi$ be a blowup at a free boundary point $x_0 \in B'_{1/2}$. Then for $\kappa = 3/2$*

$$\int_{\partial B_1} |u_{x_0,t}^\phi - u_{x_0,0}^\phi| \leq Ct^{\delta/2}, \quad 0 < t < t_0,$$

where $C = C(n, \alpha, \|u\|_{W^{1,2}(B_1)})$ and $\delta = \delta(n, \alpha) > 0$ are as in Lemma 8.2. In particular, the blowup $u_{x_0,0}^\phi$ is unique.

Proof. If $u_{x_0,0}$ is the limit of u_{x_0,t_j}^ϕ for $t_j \rightarrow 0$, then first part of the lemma follows immediately from Lemma 8.2, by taking $s = t_j \rightarrow 0$ and passing to the limit.

To see the uniqueness of blowup, we observe that $u_{x_0,0}^\phi$ is a solution of the Signorini problem in B_1 , by arguing as in the proof of Lemma 6.2 for Almgren blowups. Now, if $\tilde{u}_{x_0,0}^\phi$ is another blowup, then from the first part of the lemma we will have

$$\int_{\partial B_1} |\tilde{u}_{x_0,0}^\phi - u_{x_0,0}^\phi|^2 = 0,$$

implying that both $\tilde{u}_{x_0,0}^\phi$ and $u_{x_0,0}^\phi$ are solutions of the Signorini problem in B_1 with the same boundary values on ∂B_1 . By the uniqueness of such solutions, we have $\tilde{u}_{x_0,0}^\phi = u_{x_0,0}^\phi$ in B_1 . The equality propagates to all of \mathbb{R}^n by the unique continuation of harmonic functions in \mathbb{R}_+^n . \square

We next show that not only the blowups are unique, but also depend continuously on a free boundary point.

Lemma 8.4 (Continuous dependence of blowups). *There exists $\rho = \rho(n, \alpha) > 0$ such that if $x_0, y_0 \in B'_\rho$ are free boundary points, then*

$$\int_{\partial B_1} |u_{x_0,0}^\phi - u_{y_0,0}^\phi| \leq C|x_0 - y_0|^\gamma,$$

with $C = C(n, \alpha, \|u\|_{W^{1,2}(B_1)})$ and $\gamma = \gamma(n, \alpha) > 0$.

Proof. Let $d = |x_0 - y_0|$ and $d^\mu \leq r \leq 2d^\mu$ with $\mu \in (0, 1]$ to be determined. By Lemma 8.3 we have

$$\begin{aligned} \int_{\partial B_1} |u_{x_0,0}^\phi - u_{y_0,0}^\phi| &\leq 2Cr^{\delta/2} + \int_{\partial B_1} |u_{x_0,r}^\phi - u_{y_0,r}^\phi| \\ &\leq Cd^{\mu\delta/2} + \frac{C}{d^{\mu(n+1/2)}} \int_{\partial B_r} |u(x_0 + z) - u(y_0 + z)| dS_z \end{aligned}$$

and taking the average over $d^\mu \leq r \leq 2d^\mu$, we have

$$\int_{\partial B_1} |u_{x_0,0}^\phi - u_{y_0,0}^\phi| \leq Cd^{\mu\delta/2} + \frac{C}{d^{\mu(n+3/2)}} \int_{B_{2d^\mu} \setminus B_{d^\mu}} |u(x_0 + z) - u(y_0 + z)| dz.$$

On the other hand, by using Lemma 7.4,

$$\int_{B_{2d^\mu} \setminus B_{d^\mu}} |u(x_0 + z) - u(y_0 + z)| dz$$

$$\begin{aligned}
&\leq \int_{B_{2d^\mu} \setminus B_{d^\mu}} \left| \int_0^1 \frac{d}{ds} u(z + x_0(1-s) + y_0s) ds \right| dz \\
&\leq |x_0 - y_0| \int_0^1 \int_{B_{2d^\mu}} |\nabla u(z + x_0(1-s) + y_0s)| dz ds \\
&\leq d \int_0^1 \left(\int_{B_{2d^\mu}(x_0(1-s)+y_0s)} |\nabla u| \right) ds \\
&\leq d \int_{B_{2d^\mu+d}(x_0)} |\nabla u| \leq d \int_{B_{3d^\mu}(x_0)} |\nabla u| \\
&\leq Cd^{1+\mu n/2} \left(\int_{B_{3d^\mu}(x_0)} |\nabla u|^2 \right)^{1/2} \leq Cd^{1+\mu n/2} d^{\mu(n+1)/2} \\
&\leq Cd^{1+\mu(n+1/2)},
\end{aligned}$$

provided $3d^\mu < t_0$, which will hold if $d < \rho(n, \alpha)$.

Combining the estimates, we infer that

$$\int_{\partial B_1} |u_{x_0,0}^\phi - u_{y_0,0}^\phi| \leq Cd^{\mu\delta/2} + Cd^{1-\mu}.$$

Now choosing μ so that $\mu\delta/2 = 1 - \mu$, that is $\mu = 1/(1 + \delta/2)$, we obtain

$$\int_{\partial B_1} |u_{x_0,0}^\phi - u_{y_0,0}^\phi| \leq C|x_0 - y_0|^\gamma, \quad x_0, y_0 \in B'_\rho$$

with

$$\gamma = \frac{\delta}{\delta + 2}. \quad \square$$

9. REGULARITY OF THE REGULAR SET

In this section we establish one of the main result of this paper, the $C^{1,\gamma}$ regularity of the regular set. In fact, the most technical part of the proof has already been done in the previous section, where we proved the uniqueness of the 3/2-homogeneous blowups, as well as their Hölder continuous dependence on the free boundary points.

We start by defining the regular set.

Definition 9.1 (Regular points). For an almost minimizer u for the Signorini problem in B_1 , we say that a free boundary point x_0 is *regular* if

$$\widehat{N}(0+, u, x_0) = 3/2.$$

Note that since $3/2 < 2 \leq \kappa_0$, we will have that $\widehat{N}(r) < \kappa_0$ for small $r > 0$, implying that $\widetilde{N}(r) = \widehat{N}(r)$ for such r and consequently that

$$N(0+) = \widetilde{N}(0+) = \widehat{N}(0+) = 3/2.$$

In particular, the condition above does not depend on the choice of $\kappa_0 \geq 2$.

We denote the set of all regular points of u by $\mathcal{R}(u)$ and call it the *regular set*.

An important ingredient in the analysis of the regular set is the following nondegeneracy lemma.

Lemma 9.2 (Nondegeneracy at regular points). *Let $x_0 \in B'_{1/2} \cap \mathcal{R}(u)$ for an almost minimizer u for the Signorini problem in B_1 . Then, for $\kappa = 3/2$,*

$$\liminf_{t \rightarrow 0} \int_{\partial B_t} (u_{x_0, t}^\phi)^2 = \liminf_{t \rightarrow 0} \frac{1}{t^{n+2}} \int_{\partial B_t(x_0)} u^2 > 0.$$

Proof. As before, assume $x_0 = 0$. In terms of the quantities defined in the proofs of Lemmas 7.1 and 7.4, we want to prove that

$$\liminf_{t \rightarrow 0} m(t) > 0.$$

Assume, towards a contradiction, that $m(t_j) \rightarrow 0$ for some sequence $t_j \rightarrow 0$. Recall that by the proof of Lemma 7.4, we have

$$|m(t) - m(s)| \leq Ct^{\delta/2}, \quad 0 < s < t < t_0.$$

Now, setting $s = t_j \rightarrow 0$, we conclude that

$$|m(t)| \leq Ct^{\delta/2}, \quad 0 < t < t_0.$$

Equivalently, we can rewrite this as

$$\int_{\partial B_t} u^2 \leq Ct^{n+2+\delta}.$$

Next, take $\tilde{\kappa} = 3/2 + \delta/4$ and consider Weiss's monotonicity formula

$$W_{\tilde{\kappa}}(t, u) = \frac{e^{a_{\tilde{\kappa}} t^\alpha}}{t^{n+2\tilde{\kappa}-2}} \left[\int_{B_t} |\nabla u|^2 - \tilde{\kappa} \frac{1 - bt^\alpha}{t} \int_{\partial B_t} u^2 \right].$$

Now observe that

$$\frac{1}{t^{n+2\tilde{\kappa}-1}} \int_{\partial B_t} u^2 \leq Ct^{\delta/2} \rightarrow 0,$$

which readily implies that

$$W_{\tilde{\kappa}}(0+, u) \geq 0.$$

In particular, by monotonicity, $W_{\tilde{\kappa}}(t, u) \geq 0$, for small $t > 0$, which also implies that $\tilde{N}(t, u) \geq \tilde{\kappa}$. But then $N(0+, u) = \tilde{N}(0+, u) \geq \tilde{\kappa} = 3/2 + \delta/4$ contrary to the assumption in the lemma. This completes the proof. \square

The next result provides two important facts: a gap in possible values of Almgren's frequency $N(0+)$ as well as the classification of 3/2-homogeneous blowups.

Proposition 9.3. *If $\widehat{N}(0+, u, x_0) = \kappa < 2$, then $\kappa = 3/2$ and*

$$u_{x_0, 0}^\phi(x) = a_{x_0} \operatorname{Re}(x' \cdot \nu_{x_0} + i|x_n|)^{3/2}$$

for some $a_{x_0} > 0$, $\nu_{x_0} \in \partial B'_1$.

Proof. Without loss of generality, we may assume $x_0 = 0$. Let $r_j \rightarrow 0+$ be a sequence such that $u_{r_j}^\phi \rightarrow u_0^\phi$ in $C_{\text{loc}}^1(\mathbb{R}_\pm^n \cup \mathbb{R}^{n-1})$. Comparing 3/2-almost homogeneous and Almgren rescalings, we have

$$u_r^\phi(x) = u_r^A(x) \mu(r), \quad \mu(r) := \frac{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2 \right)^{1/2}}{\phi(r)}.$$

By the optimal growth estimate (Lemma 7.4) and the nondegeneracy at regular points (Lemma 9.2) we have

$$0 < \liminf_{r \rightarrow 0+} \mu(r) \leq \limsup_{r \rightarrow 0+} \mu(r) < \infty.$$

Thus, we may assume that, over a subsequence of r_j , $\mu(r_j) \rightarrow \mu_0 \in (0, \infty)$, and therefore

$$u_{r_j}^\phi \rightarrow \mu_0 u_0^A \quad \text{in } C_{\text{loc}}^1(B_1^\pm \cup B_1'),$$

where u_0^A is an Almgren blowup of u at $x_0 = 0$. Now, since $\kappa < \kappa_0$, we can apply Proposition 6.1 to obtain that u_0^A is a nonzero κ -homogeneous solution of the Signorini problem in B_1 , even in x_n -variable. Next, applying Lemma 6.2, we have $3/2 \leq \kappa < 2$ and thus by [PSU12, Proposition 9.9], we must have $\kappa = 3/2$ and

$$u_0^A(x) = C_n \operatorname{Re}(x' \cdot \nu_0 + i|x_n|)^{3/2}$$

for some $C_n > 0$, $\nu_0 \in \partial B_1'$. (The constant C_n comes from the normalization $\int_{\partial B_1} (u_0^A)^2 = 1$.) Thus,

$$u_0^\phi(x) = a_0 \operatorname{Re}(x' \cdot \nu_0 + i|x_n|)^{3/2} \quad \text{in } B_1$$

with $a_0 = C_n \mu_0$. By the unique continuation of harmonic functions in \mathbb{R}_\pm^n , we obtain that the above formula for u_0^ϕ propagates to all of \mathbb{R}^n . \square

Proposition 9.3 has an immediate corollary.

Corollary 9.4 (Almgren frequency gap). *Let u be an almost minimizer for the Signorini problem in B_1 and x_0 a free boundary point. Then either*

$$\widehat{N}(0+, u) = 3/2 \quad \text{or} \quad \widehat{N}(0+, u) \geq 2.$$

Yet another important fact is as follows.

Corollary 9.5. *The regular set $\mathcal{R}(u)$ is a relatively open subset of the free boundary.*

Proof. For a fixed $0 < t < t_0$, the mapping $x \mapsto \widehat{N}(t, u, x)$ is continuous on $\Gamma(u)$. Then, by the monotonicity of \widehat{N} , the mapping $x \mapsto \widehat{N}(0+, u, x_0)$ is upper semicontinuous on $\Gamma(u)$. Moreover, by Proposition 9.3,

$$\mathcal{R}(u) = \{x \in \Gamma(u) : \widehat{N}(0+, u, x) < 2\},$$

which implies that $\mathcal{R}(u)$ is relatively open in $\Gamma(u)$. \square

The combination of Proposition 9.3 and Lemma 8.4 implies the following lemma.

Lemma 9.6. *Let u be an almost minimizer for the Signorini problem in B_1 , and $x_0 \in \mathcal{R}(u)$. Then there exists $\rho > 0$, depending on x_0 such that $B'_\rho(x_0) \cap \Gamma(u) \subset \mathcal{R}(u)$ and if $u_{\bar{x}, 0}^\phi(x) = a_{\bar{x}} \operatorname{Re}(x' \cdot \nu_{\bar{x}} + i|x_n|)^{3/2}$ is the unique $3/2$ -homogeneous blowup at $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$, then*

$$\begin{aligned} |a_{\bar{x}} - a_{\bar{y}}| &\leq C_0 |\bar{x} - \bar{y}|^\gamma, \\ |\nu_{\bar{x}} - \nu_{\bar{y}}| &\leq C_0 |\bar{x} - \bar{y}|^\gamma, \end{aligned}$$

for any $\bar{x}, \bar{y} \in B'_\rho(x_0) \cap \Gamma(u)$ with a constant C_0 depending on x_0 .

Proof. The proof follows by repeating the argument in Lemma 7.5 in [GPS16]. \square

Now we are ready to prove the main result on the regularity of the regular set.

Theorem 9.7 ($C^{1,\gamma}$ regularity of the regular set). *Let u be an almost minimizer for the Signorini problem in B_1 . Then, if $x_0 \in B'_{1/2} \cap \mathcal{R}(u)$, there exists $\rho > 0$, depending on x_0 such that, after a possible rotation of coordinate axes in \mathbb{R}^{n-1} , one has $B'_\rho(x_0) \cap \Gamma(u) \subset \mathcal{R}(u)$, and*

$$B'_\rho(x_0) \cap \Gamma(u) = B'_\rho(x_0) \cap \{x_{n-1} = g(x_1, \dots, x_{n-2})\},$$

for $g \in C^{1,\gamma}(\mathbb{R}^{n-2})$ with an exponent $\gamma = \gamma(n, \alpha) \in (0, 1)$.

Proof. The proof of the theorem is similar to that of Theorem 1.2 in [GPS16]. However, we provide full details since there are technical differences.

Step 1. By relative openness of $\mathcal{R}(u)$ in $\Gamma(u)$, for small $\rho > 0$ we have $B'_{2\rho}(x_0) \cap \Gamma(u) \subset \mathcal{R}(u)$. We then claim that for any $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that for $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$, $r < r_\varepsilon$, we have that for $\phi = \phi_{3/2}$

$$\|u_{\bar{x},r}^\phi - u_{\bar{x},0}^\phi\|_{C^1(\overline{B_1^\pm})} < \varepsilon.$$

Assuming the contrary, there is a sequence of points $\bar{x}_j \in B'_\rho(x_0) \cap \Gamma(u)$ and radii $r_j \rightarrow 0$ such that

$$\|u_{\bar{x}_j,r_j}^\phi - u_{\bar{x}_j,0}^\phi\|_{C^1(\overline{B_1^\pm})} \geq \varepsilon_0$$

for some $\varepsilon_0 > 0$. Taking a subsequence, if necessary, we may assume $\bar{x}_j \rightarrow \bar{x}_0 \in \overline{B'_\rho(x_0)} \cap \Gamma(u)$. Using estimates (3.1), (4.7) and Lemma 7.4, we can see that $u_{\bar{x}_j,r_j}^\phi$ are uniformly bounded in $C^{1,\beta}(B_2^\pm \cup B_2)$. Thus, we may assume that for some w

$$u_{\bar{x}_j,r_j}^\phi \rightarrow w \quad \text{in } C^1(\overline{B_1^\pm}).$$

By arguing as in the proof of Proposition 6.1, we see that the limit w is a solution of the Signorini problem in B_1 . Further, by Lemma 8.3, we have

$$\|u_{\bar{x}_j,r_j}^\phi - u_{\bar{x}_j,0}^\phi\|_{L^1(\partial B_1)} \rightarrow 0.$$

On the other hand, by Lemma 9.6, we have

$$u_{\bar{x}_j,0}^\phi \rightarrow u_{\bar{x}_0,0}^\phi \quad \text{in } C^1(\overline{B_1^\pm}),$$

and thus

$$w = u_{\bar{x}_0,0}^\phi \quad \text{on } \partial B_1.$$

Since both w and $u_{\bar{x}_0,0}^\phi$ are solutions of the Signorini problem, they must coincide also in B_1 . Therefore

$$u_{\bar{x}_j,r_j}^\phi \rightarrow u_{\bar{x}_0,0}^\phi \quad \text{in } C^1(\overline{B_1^\pm}),$$

implying also that

$$\|u_{\bar{x}_j,r_j}^\phi - u_{\bar{x}_j,0}^\phi\|_{C^1(\overline{B_1^\pm})} \rightarrow 0,$$

which contradicts our assumption.

Step 2. As [GPS16], for a given $\varepsilon > 0$ and a unit vector $\nu \in \mathbb{R}^{n-1}$ define the cone

$$\mathcal{C}_\varepsilon(\nu) = \{x' \in \mathbb{R}^{n-1} : x' \cdot \nu > \varepsilon|x'|\}.$$

By Lemma 9.6, we may assume $a_{\bar{x}} \geq \frac{a_{x_0}}{2}$ for $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$ by taking ρ small. For such ρ we then claim that for any $\varepsilon > 0$ there is $r_\varepsilon > 0$ such that for any $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$ we have

$$\bar{x} + (\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}) \subset \{u(\cdot, 0) > 0\}.$$

Indeed, denoting $\mathcal{K}_\varepsilon(\nu) = \mathcal{C}_\varepsilon \cap \partial B'_{1/2}$, we have for some universal $C_\varepsilon > 0$

$$\mathcal{K}_\varepsilon(\nu_{\bar{x}}) \Subset \{u_{\bar{x},0}^\phi(\cdot, 0) > 0\} \cap B'_1 \quad \text{and} \quad u_{\bar{x},0}^\phi(\cdot, 0) \geq a_{\bar{x}} C_\varepsilon \geq \frac{a_{x_0}}{2} C_\varepsilon \quad \text{on } \mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

Since $\frac{a_{x_0}}{2} C_\varepsilon$ is independent of \bar{x} , by Step 1 we can find $r_\varepsilon > 0$ such that for $r < 2r_\varepsilon$,

$$u_{\bar{x},r}^\phi(\cdot, 0) > 0 \quad \text{on } \mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

This implies that for $r < 2r_\varepsilon$,

$$u(\cdot, 0) > 0 \quad \text{on} \quad \bar{x} + r\mathcal{K}_\varepsilon(\nu_{\bar{x}}) = \bar{x} + \left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap \partial B'_{r/2} \right).$$

Taking the union over all $r < 2r_\varepsilon$, we obtain

$$u(\cdot, 0) > 0 \quad \text{on} \quad \bar{x} + \left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon} \right).$$

Step 3. We claim that for given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for any $\bar{x} \in B'_\rho(x_0) \cap \Gamma(u)$ we have $\bar{x} - (\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon}) \subset \{u(\cdot, 0) = 0\}$.

Indeed, we first note that

$$-\partial_{x_n}^+ u_{\bar{x},0}^\phi \geq a_{\bar{x}} C_\varepsilon > \left(\frac{a_{x_0}}{2} \right) C_\varepsilon \quad \text{on} \quad -\mathcal{K}_\varepsilon(\nu_{\bar{x}})$$

for a universal constant $C_\varepsilon > 0$. From Step 1, there exists $r_\varepsilon > 0$ such that for $r < 2r_\varepsilon$,

$$-\partial_{x_n}^+ u_{\bar{x},r}^\phi(\cdot, 0) > 0 \quad \text{on} \quad -\mathcal{K}_\varepsilon(\nu_{\bar{x}}).$$

By arguing as in Step 2, we obtain

$$-\partial_{x_n}^+ u(\cdot, 0) > 0 \quad \text{on} \quad \bar{x} - \left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon} \right).$$

By the complementarity condition in Lemma 4.7, we therefore conclude that

$$\bar{x} - \left(\mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon} \right) \subset \{-\partial_{x_n}^+ u(\cdot, 0) > 0\} \subset \{u(\cdot, 0) = 0\}.$$

Step 4. By rotation in \mathbb{R}^{n-1} we may assume $\nu_{x_0} = e_{n-1}$. For any $\varepsilon > 0$, by Lemma 9.6 again, we can take $\rho_\varepsilon = \rho(x_0, \varepsilon)$, possibly smaller than ρ in the previous steps, such that

$$\mathcal{C}_{2\varepsilon}(e_{n-1}) \cap B'_{r_\varepsilon} \subset \mathcal{C}_\varepsilon(\nu_{\bar{x}}) \cap B'_{r_\varepsilon} \quad \text{for } \bar{x} \in B'_{\rho_\varepsilon}(x_0) \cap \Gamma(u).$$

By Step 2 and Step 3, for $\bar{x} \in B'_{\rho_\varepsilon}(x_0) \cap \Gamma(u)$,

$$\begin{aligned} \bar{x} + \left(\mathcal{C}_{2\varepsilon}(e_{n-1}) \cap B'_{r_\varepsilon} \right) &\subset \{u(\cdot, 0) > 0\}, \\ \bar{x} - \left(\mathcal{C}_{2\varepsilon}(e_{n-1}) \cap B'_{r_\varepsilon} \right) &\subset \{u(\cdot, 0) = 0\}. \end{aligned}$$

Now, fixing $\varepsilon = \varepsilon_0$, by the standard arguments, we conclude that there exists a Lipschitz function $g : \mathbb{R}^{n-2} \rightarrow \mathbb{R}$ with $|\nabla g| \leq C_n/\varepsilon_0$ such that

$$\begin{aligned} B'_{\rho_{\varepsilon_0}}(x_0) \cap \{u(\cdot, 0) = 0\} &= B'_{\rho_{\varepsilon_0}}(x_0) \cap \{x_{n-1} \leq g(x'')\}, \\ B'_{\rho_{\varepsilon_0}}(x_0) \cap \{u(\cdot, 0) > 0\} &= B'_{\rho_{\varepsilon_0}}(x_0) \cap \{x_{n-1} > g(x'')\}. \end{aligned}$$

Step 5. Taking $\varepsilon \rightarrow 0$ in Step 4, $\Gamma(u)$ is differentiable at x_0 with normal ν_{x_0} . Recentering at any $\bar{x} \in B'_{\rho_{\varepsilon_0}}(x_0) \cap \Gamma(u)$, we see that $\Gamma(u)$ has a normal $\nu_{\bar{x}}$ at \bar{x} . By Lemma 9.6, we conclude that g in Step 4 is $C^{1,\gamma}$. This completes the proof of the theorem. \square

10. SINGULAR POINTS

In this section we study the set of so-called singular free boundary points. An important technical tool to accomplish this is the logarithmic epiperimetric inequality of [CSV17]. We use it for two purposes: to establish the optimal growth at singular points as well as the rate of convergence of rescalings to blowups, ultimately implying a structural theorem for the singular set.

Definition 10.1 (Singular points). Let u be an almost minimizer for the Signorini problem in B_1 . We say that a free boundary point x_0 is *singular* if the coincidence set $\Lambda(u) = \{u(\cdot, 0) = 0\} \subset B'_1$ has zero H^{n-1} -density at x_0 , i.e.,

$$\lim_{r \rightarrow 0^+} \frac{H^{n-1}(\Lambda(u) \cap B'_r(x_0))}{H^{n-1}(B'_r(x_0))} = 0.$$

By using Almgren's rescalings $u_{x_0, r}^A$, we can rewrite this condition as

$$\lim_{r \rightarrow 0^+} H^{n-1}(\Lambda(u_{x_0, r}^A) \cap B'_1) = 0.$$

We denote the set of all singular points by $\Sigma(u)$ and call it the *singular set*.

Throughout the section we will assume that

$$\kappa_0 > 2.$$

We can take κ_0 as large as we like, however, we have to remember that the constants in $\widehat{N} = \widehat{N}_{\kappa_0}$ and W_κ do depend on κ_0 .

We then have the following characterization of singular points, similar to Proposition 9.22 in [PSU12] for the solutions of the Signorini problem.

Proposition 10.2 (Characterization of singular points). *Let u be an almost minimizer for the Signorini problem in B_1 , and $x_0 \in B'_{1/2} \cap \Gamma(u)$ be such that $\widehat{N}(0^+, u, x_0) = \kappa < \kappa_0$. Then the following statements are equivalent.*

- (i) $x_0 \in \Sigma(u)$.
- (ii) any Almgren blowup of u at x_0 is a nonzero polynomial from the class

$\mathcal{Q}_\kappa = \{q : q \text{ is homogeneous polynomial of degree } \kappa \text{ such that}$

$$\Delta q = 0, \quad q(x', 0) \geq 0, \quad q(x', x_n) = q(x', -x_n)\}.$$

- (iii) $\kappa = 2m$ for some $m \in \mathbb{N}$.

Note that for $\kappa < \kappa_0$, the condition $\widehat{N}(0^+) = \kappa$ is equivalent to $N(0^+) = \kappa$.

Proof. Without loss of generality we may assume $x_0 = 0$. By Proposition 6.1, any Almgren blowup u_0^A of u at 0 is a nonzero global solution of the Signorini problem, homogeneous of degree κ . Moreover u_0^A is a C^1_{loc} limit of Almgren rescalings $u_{r_j}^A$ in $B_1^\pm \cup B'_1$. Because of that, most parts of the proof of this proposition are just the repetitions of Proposition 9.22 in [PSU12]. Thus, by following Proposition 9.22 in [PSU12], we can easily see the implications (ii) \Rightarrow (iii), (iii) \Rightarrow (ii), (ii) \Rightarrow (i). Moreover, in the proof of the remaining implication (i) \Rightarrow (ii), the only nontrivial part is that any blowup u_0^A is harmonic in B_1 . But this comes from the complementarity condition in Lemma 4.7. Indeed, assuming (i), we claim that

$$\partial_{x_n}^+ u_0^A = 0 \quad \text{in } B'_1.$$

Otherwise,

$$H^{n-1}(\{-\partial_{x_n}^+ u_0^A(\cdot, 0) > 0\} \cap B'_1) \geq \delta$$

for some $\delta > 0$. Then using the continuity from the below we also have that for some $\rho > 0$,

$$H^{n-1}(\{-\partial_{x_n}^+ u_0^A(\cdot, 0) > \rho\} \cap B'_{1-\rho}) \geq \delta/2.$$

Using C_{loc}^1 convergence $u_{r_j}^A \rightarrow u_0^A$ in $B_1^\pm \cup B'_1$ and applying the complementarity condition in Lemma 4.7 to rescalings $u_{r_j}^A$, we obtain that for small r_j ,

$$H^{n-1}(\Lambda(u_{r_j}^A) \cap B'_1) \geq H^{n-1}(\{-\partial_{x_n}^+ u_{r_j}^A(\cdot, 0) > 0\} \cap B'_1) \geq \delta/4,$$

which contradicts (i). Now recalling that u_0^A is a solution of the Signorini problem, even in x_n -variable, it satisfies

$$\Delta u_0^A = 2(\partial_{x_n}^+ u_0^A) H^{n-1}|_{\Lambda(u_0^A)} = 0 \quad \text{in } B_1.$$

By homogeneity, we obtain that u_0^A is harmonic in all of \mathbb{R}^n , and we complete the proof as in [PSU12]. \square

In order to study the singular set, in view of Proposition 10.2, we need to refine the growth estimate in Lemma 7.1 by removing the logarithmic term in the case when $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$. In the case $\kappa = 3/2$ we were able to do so by proving a decay estimate for $W_{3/2}$ with the help of the epiperimetric inequality. In the case $\kappa = 2m$ we will use the so-called *logarithmic epiperimetric inequality* for the Weiss energy

$$W_\kappa^0(w) = \int_{B_1} |\nabla w|^2 - \kappa \int_{\partial B_1} w^2, \quad \kappa = 2m, \quad m \in \mathbb{N}$$

that first appeared in [CSV17]. To state this result, we recall the notation

$$\mathcal{A} = \{w \in W^{1,2}(B_1) : w \geq 0 \text{ on } B'_1, w(x', x_n) = w(x', -x_n)\}.$$

Theorem 10.3 (Logarithmic epiperimetric inequality). *Let $\kappa = 2m$, $m \in \mathbb{N}$ and $w \in \mathcal{A}$ be homogeneous of degree κ in B_1 such that $w \in W^{1,2}(\partial B_1)$ and*

$$\int_{\partial B_1} w^2 \leq 1, \quad |W_\kappa^0(w)| \leq 1.$$

There is constant $\varepsilon = \varepsilon(n, \kappa) > 0$ and a function $v \in \mathcal{A}$ with $v = w$ on ∂B_1 such that

$$W_\kappa^0(v) \leq W_\kappa^0(w)(1 - \varepsilon |W_\kappa^0(w)|^\gamma), \quad \text{where } \gamma = \frac{n-2}{n}.$$

To simplify the notations, in the results below all constants will depend on n , α , κ , κ_0 , as well as $\|u\|_{W^{1,2}(B_1)}$, unless stated otherwise, in addition to other quantities. Thus, when we write $C = C(\sigma)$, we mean $C = C(n, \alpha, \kappa, \kappa_0, \|u\|_{W^{1,2}(B_1)}, \sigma)$.

The next lemma allows to apply the logarithmic epiperimetric inequality, without the constraints.

Lemma 10.4. *Let u be an almost minimizer for the Signorini problem in B_1 such that $0 \in \Gamma(u)$ and $\tilde{N}(0+, u) = \kappa < \kappa_0$, $\kappa = 2m$, $m \in \mathbb{N}$. For $0 < r < 1$, let*

$$u_r(x) = u_r^{(\kappa)}(x) = \frac{u(rx)}{r^\kappa}, \quad w_r(x) = |x|^\kappa u_r\left(\frac{x}{|x|}\right).$$

Suppose that for a given $0 \leq \sigma \leq 1$, there is $C = C(\sigma)$ such that

$$\int_{\partial B_r} u^2 \leq C \left(\log \frac{1}{r} \right)^\sigma r^{n+2\kappa-1}.$$

Then there is a constant $\varepsilon = \varepsilon(\sigma) > 0$ and $h \in \mathcal{A}$ with $h = w_r$ on ∂B_1 such that

(i) If $|W_\kappa^0(w_r)| \geq \int_{\partial B_1} w_r^2$, then

$$W_\kappa^0(h) \leq (1 - \varepsilon) W_\kappa^0(w_r)$$

(ii) If $|W_\kappa^0(w_r)| \leq 2 \int_{\partial B_1} w_r^2$, then

$$W_\kappa^0(h) \leq W_\kappa^0(w_r) \left(1 - \varepsilon \left(\log \frac{1}{r} \right)^{-\sigma\gamma} |W_\kappa^0(w_r)|^\gamma \right), \quad \text{where } \gamma = \frac{n-2}{n}.$$

Proof. Let $A = \int_{\partial B_1} w_r^2 + |W_\kappa^0(w_r)|$. Then by Theorem 10.3 applied to $w_r/A^{1/2}$, there is $h \in \mathcal{A}$ such that $h = w_r$ on ∂B_1 and

$$W_\kappa^0(h) \leq W_\kappa(w_r)^0 (1 - \varepsilon A^{-\gamma} |W_\kappa^0(w_r)|^\gamma).$$

If $|W_\kappa^0(w_r)| \geq \int_{\partial B_1} w_r^2$, then $A \leq 2|W_\kappa^0(w_r)|$, implying

$$W_\kappa^0(h) \leq W_\kappa^0(w_r) (1 - \varepsilon 2^{-\gamma}).$$

If $|W_\kappa^0(w_r)| \leq 2 \int_{\partial B_1} w_r^2$, then

$$A \leq 3 \int_{\partial B_1} w_r^2 = \frac{3}{r^{n+2\kappa-1}} \int_{\partial B_r} u^2 \leq C(\sigma) \left(\log \frac{1}{r} \right)^\sigma.$$

This completes the proof. \square

Now we show that the logarithmic epiperimetric inequality, combined with a growth estimate for u , implies a growth estimate on $W_\kappa(t, u)$. This is the first part of a bootstrapping argument that gradually decreases the power of $\log(1/t)$ in the bound for u .

Lemma 10.5. *Let u be an almost minimizer for the Signorini problem in B_1 such that $0 \in \Gamma(u)$ and $\widehat{N}(0+, u) = \kappa < \kappa_0$, $\kappa = 2m$, $m \in \mathbb{N}$. Suppose that for some $0 \leq \sigma \leq 1$*

$$\int_{\partial B_r} u^2 \leq C(\sigma) \left(\log \frac{1}{r} \right)^\sigma r^{n+2\kappa-1}, \quad 0 < r < r_0(\sigma).$$

Then,

$$0 \leq W_\kappa(t, u) \leq C(\sigma) \left(\log \frac{1}{t} \right)^{-\frac{1-\sigma\gamma}{\gamma}}, \quad 0 < t < t_0(\sigma).$$

Proof. We first observe that $W_\kappa(t, u) \geq 0$ for $0 < t < t_0$, which follows easily from the condition $\widehat{N}(0+, u) = \kappa < \kappa_0$, see the beginning of the proof of Lemma 7.1.

Next, recall that in the proof of Lemma 7.3, we have used epiperimetric inequality to show that $0 \leq W_{3/2}(t, u) \leq Ct^\delta$. This followed by obtaining a differential inequality for $W_{3/2}$. Thus, if for $0 < t < t_0$, if alternative (i) holds in Lemma 10.4, i.e., $W_\kappa^0(h) \leq (1 - \varepsilon)W_\kappa^0(w_t)$, by arguing in the same way, we can show that

$$(10.1) \quad \frac{d}{dt} W_\kappa(t, u) \geq \frac{\varepsilon/4}{t} W_\kappa(t, u) - Ct^{\alpha/2-1},$$

for $C = C(\sigma)$.

Suppose now the alternative (ii) holds in Lemma 10.4 for some $0 < t < t_0$. Then, following the computations in Lemma 7.3, we have

$$\begin{aligned} \frac{d}{dt} W_\kappa(t, u) &\geq - \frac{(n + 2\kappa - 2)(1 - t^\alpha)}{t} W_\kappa(t, u) \\ &\quad + \frac{e^{at^\alpha}(1 - bt^\alpha)}{t} \int_{\partial B_1} (\partial_\nu u_t - \kappa u_t)^2 + (\partial_\tau u_t)^2 - \kappa(n + \kappa - 2)u_t^2 \\ &\quad + (2\kappa_0 + n)t^{\alpha-1} \int_{\partial B_1} u_t^2. \end{aligned}$$

For w_t as in the statement of Lemma 10.4, by following the computations in the proof of Theorem 5.1, we have the identity

$$\int_{\partial B_1} (\partial_\tau u_t)^2 - \kappa(n + \kappa - 2)u_t^2 = (n + 2\kappa - 2)W_\kappa^0(w_t).$$

This gives

$$(10.2) \quad \begin{aligned} \frac{d}{dt} W_\kappa(t, u) &\geq - \frac{(n + 2\kappa - 2)(1 - t^\alpha)}{t} W_\kappa(t, u) \\ &\quad + \frac{e^{at^\alpha}(1 - bt^\alpha)}{t} (n + 2\kappa - 2)W_\kappa^0(w_t) + (2\kappa_0 + n)t^{\alpha-1} \int_{\partial B_1} u_t^2. \end{aligned}$$

Let now v_t be the solution of the Signorini problem in B_1 with $v_t = u_t = w_t$ on ∂B_1 . Then

$$(10.3) \quad \begin{aligned} (1 + t^\alpha)W_\kappa^0(w_t) &\geq (1 + t^\alpha)W_\kappa^0(v_t) \\ &\geq \int_{B_1} |\nabla u_t|^2 - \kappa(1 + t^\alpha) \int_{\partial B_1} u_t^2 \\ &= W_\kappa^0(u_t) - \kappa t^\alpha \int_{\partial B_1} u_t^2 \\ &= e^{-at^\alpha} W_\kappa(t, u) - \kappa(b + 1)t^\alpha \int_{\partial B_1} u_t^2. \end{aligned}$$

Now, if

$$e^{-at^\alpha} W_\kappa(t, u) - \kappa(b + 1)t^\alpha \int_{\partial B_1} u_t^2 \leq 0,$$

then by Lemma 7.1 we have

$$(10.4) \quad \begin{aligned} W_\kappa(t, u) &\leq e^{at^\alpha} \kappa(b + 1)t^\alpha \int_{\partial B_1} u_t^2 \\ &\leq Ct^\alpha \left(\log \frac{1}{t} \right) \leq Ct^{\alpha/2}. \end{aligned}$$

We then proceed under the assumption

$$e^{-at^\alpha} W_\kappa(t, u) - \kappa(b + 1)t^\alpha \int_{\partial B_1} u_t^2 > 0,$$

which also implies

$$W_\kappa^0(w_t) > 0.$$

Now, applying Lemma 10.4, we have

$$\begin{aligned}
(10.5) \quad W_\kappa^0(w_t) &\geq W_\kappa^0(v_t) + \varepsilon \left(\log \frac{1}{t} \right)^{-\sigma\gamma} W_\kappa^0(w_t)^{\gamma+1} \\
&\geq \frac{1}{1+t^\alpha} \left[e^{-at^\alpha} W_\kappa(t, u) - \kappa(b+1)t^\alpha \int_{\partial B_1} u_t^2 \right] \\
&\quad + \varepsilon \left(\log \frac{1}{t} \right)^{-\sigma\gamma} \left(\frac{1}{1+t^\alpha} \right)^{\gamma+1} \times \\
&\quad \times \left[e^{-at^\alpha} W_\kappa(t, u) - \kappa(b+1)t^\alpha \int_{\partial B_1} u_t^2 \right]^{\gamma+1} \\
&\geq (1-t^\alpha) \left[e^{-at^\alpha} W_\kappa(t, u) - \kappa(b+1)t^\alpha \int_{\partial B_1} u_t^2 \right] \\
&\quad + \varepsilon \left(\log \frac{1}{t} \right)^{-\sigma\gamma} (1-t^\alpha)^{\gamma+1} \times \\
&\quad \times \left[\frac{(e^{-at^\alpha} W_\kappa(t, u))^{\gamma+1}}{2^\gamma} - \left(\kappa(b+1)t^\alpha \int_{\partial B_1} u_t^2 \right)^{\gamma+1} \right] \\
&= (1-t^\alpha) e^{-at^\alpha} W_\kappa(t, u) \\
&\quad + \frac{\varepsilon}{2^\gamma} \left(\log \frac{1}{t} \right)^{-\sigma\gamma} (1-t^\alpha)^{\gamma+1} e^{-a(\gamma+1)t^\alpha} W_\kappa(t, u)^{\gamma+1} \\
&\quad - (1-t^\alpha) \kappa(b+1)t^\alpha \int_{\partial B_1} u_t^2 \\
&\quad - \varepsilon \left(\log \frac{1}{t} \right)^{-\sigma\gamma} (1-t^\alpha)^{\gamma+1} \kappa^{\gamma+1} (b+1)^{\gamma+1} t^{\alpha(\gamma+1)} \left(\int_{\partial B_1} u_t^2 \right)^{\gamma+1},
\end{aligned}$$

where we used (10.3) in the second inequality and the convexity of $x \mapsto x^{\gamma+1}$ on \mathbb{R}_+ in the third inequality. Now (10.2) and (10.5), together with Lemma 7.1, yield

$$\begin{aligned}
(10.6) \quad \frac{d}{dt} W_\kappa(t, u) &\geq -C_1 t^{\alpha-1} W_\kappa(t, u) \\
&\quad + C_2 t^{-1} \left(\log \frac{1}{t} \right)^{-\sigma\gamma} W_\kappa(t, u)^{\gamma+1} - C_3 t^{\alpha/2-1},
\end{aligned}$$

where $C_i = C_i(\sigma)$. Summarizing, we have that at every $0 < t < t_0(\sigma)$, either (10.1), (10.6), or the bound (10.4) holds. Further note that by the growth estimate in Lemma 7.1, the bound (10.1) implies (10.6) for sufficiently small t and thus we may assume that (10.6) holds for all $0 < t < t_0$ for which $W_\kappa(t, u) > Ct^{\alpha/2}$.

To proceed, let $0 < t < t_0$ be such that $W_\kappa(t, u) \geq t^{\alpha/8}$. Then the bound (10.6) holds and we can derive that for $C = \frac{\gamma C_2}{2(1-\sigma\gamma)}$, we have

$$\begin{aligned}
&\frac{d}{dt} \left(-W_\kappa(t, u)^{-\gamma} e^{-t^{\alpha/4}} + C \left(\log \frac{1}{t} \right)^{1-\sigma\gamma} \right) \\
&= W_\kappa(t, u)^{-\gamma-1} e^{-t^{\alpha/4}} \left(\gamma \frac{d}{dt} W_\kappa(t, u) + \frac{\alpha}{4} W_\kappa(t, u) t^{\alpha/4-1} \right)
\end{aligned}$$

$$\begin{aligned}
& -C(1-\sigma\gamma)t^{-1}\left(\log\frac{1}{t}\right)^{-\sigma\gamma} \\
& \geq W_\kappa(t,u)^{-\gamma}e^{-t^{\alpha/4}}t^{\alpha/4-1}\left(\frac{\alpha}{4}-\gamma C_1t^{3\alpha/4}-\frac{\gamma C_3t^{\alpha/4}}{W_\kappa(t,u)}\right) \\
& \quad +\left(\log\frac{1}{t}\right)^{-\sigma\gamma}t^{-1}\left(e^{-t^{\alpha/4}}\gamma C_2-C(1-\sigma\gamma)\right) \\
& \geq 0,
\end{aligned}$$

$0 < t < t_0 = t_0(\sigma)$. Since also the function $-t^{-\gamma(\alpha/8)}e^{-t^{\alpha/4}} + C\left(\log\frac{1}{t}\right)^{1-\sigma\gamma}$ is nondecreasing for small t , denoting

$$\widehat{W}_\kappa(t,u) = \max\{W_\kappa(t,u), t^{\alpha/8}\},$$

we obtain that the function

$$-\widehat{W}_\kappa(t,u)^{-\gamma}e^{-t^{\alpha/4}} + C\left(\log\frac{1}{t}\right)^{1-\sigma\gamma}$$

is nondecreasing on $(0, t_0)$. Hence,

$$\begin{aligned}
-\widehat{W}_\kappa(t,u)^{-\gamma}e^{-t^{\alpha/4}} + C\left(\log\frac{1}{t}\right)^{1-\sigma\gamma} & \leq -\widehat{W}_\kappa(t_0,u)^{-\gamma}e^{-t_0^{\alpha/4}} + C\left(\log\frac{1}{t_0}\right)^{1-\sigma\gamma} \\
& \leq C\left(\log\frac{1}{t_0}\right)^{1-\sigma\gamma}.
\end{aligned}$$

If $0 < t < t_0^2$, then $\left(\log\frac{1}{t_0}\right)^{1-\sigma\gamma} < \left(\frac{1}{2}\right)^{1-\sigma\gamma}\left(\log\frac{1}{t}\right)^{1-\sigma\gamma}$, implying that

$$-\widehat{W}_\kappa(t,u)^{-\gamma}e^{-t^{\alpha/4}} \leq C\left(\left(\frac{1}{2}\right)^{1-\sigma\gamma} - 1\right)\left(\log\frac{1}{t}\right)^{1-\sigma\gamma}$$

and hence

$$W_\kappa(t,u) \leq \widehat{W}_\kappa(t,u) \leq C\left(1 - \left(\frac{1}{2}\right)^{1-\sigma\gamma}\right)^{-\frac{1}{\gamma}}\left(\log\frac{1}{t}\right)^{-\frac{1-\sigma\gamma}{\gamma}}. \quad \square$$

Lemma 10.6. *If u is as in Lemma 10.5 with $\frac{2}{n-2} < \sigma \leq 1$, then there exist positive $C = C(\sigma)$, $t_0 = t_0(\sigma)$ such that*

$$\int_{\partial B_t} u^2 \leq C\left(\log\frac{1}{t}\right)^{\sigma-\frac{2}{n-2}}t^{n+2\kappa-1}, \quad 0 < t < t_0.$$

Proof. Going back to the proof and notations of Lemma 7.1, we have that for $0 < s < t < t_0$

$$|m(t) - m(s)| \leq C\left(\log\frac{t}{s}\right)^{1/2}(W_\kappa(t) - W_\kappa(s))^{1/2}.$$

Let now $0 \leq j \leq i$ be such that $2^{-2^{i+1}} < t \leq 2^{-2^i}$, $2^{-2^{j+1}} < t_0 \leq 2^{-2^j}$. Then

$$\begin{aligned}
& |m(t_0) - m(t)| \\
& \leq |m(t_0) - m(2^{-2^{j+1}})| + |m(2^{-2^i}) - m(t)| + \sum_{k=j+1}^{i-1} |m(2^{-2^k}) - m(2^{-2^{k+1}})| \\
& \leq \sum_{k=0}^i C \left[\log \left(2^{-2^k} \right) - \log \left(2^{-2^{k+1}} \right) \right]^{1/2} \left[W_\kappa \left(2^{-2^k} \right) - W_\kappa \left(2^{-2^{k+1}} \right) \right]^{1/2} \\
& \leq C \sum_{k=0}^i 2^{k/2} W_\kappa \left(2^{-2^k} \right)^{1/2} \\
& \leq C \sum_{k=0}^i 2^{(1 - \frac{1-\sigma\gamma}{\gamma})k/2} \\
& \leq C 2^{(\sigma - \frac{2}{n-2})i/2} \\
& \leq C \left(\log \frac{1}{t} \right)^{\frac{1}{2}(\sigma - \frac{2}{n-2})}.
\end{aligned}$$

Note that in the fifth inequality we have used that $1 - \frac{1-\sigma\gamma}{\gamma} = \sigma - \frac{2}{n-2} > 0$. Thus

$$m(t) \leq m(t_0) + C \left(\log \frac{1}{t} \right)^{\frac{1}{2}(\sigma - \frac{2}{n-2})} \leq C \left(\log \frac{1}{t} \right)^{\frac{1}{2}(\sigma - \frac{2}{n-2})}.$$

This implies the desired result. \square

Lemma 10.5 and Lemma 10.6 imply the following.

Corollary 10.7 (Bootstrapping). *Let u be an almost minimizer for the Signorini problem in B_1 such that $0 \in \Gamma(u)$ and $\widehat{N}(0+, u) = \kappa < \kappa_0$, $\kappa = 2m$, $m \in \mathbb{N}$. Suppose that for $\frac{2}{n-2} < \sigma \leq 1$*

$$\int_{\partial B_t} u^2 \leq C(\sigma) \left(\log \frac{1}{t} \right)^\sigma t^{n+2\kappa-1}, \quad 0 < t < t_0(\sigma).$$

Then

$$\int_{\partial B_t} u^2 \leq C'(\sigma) \left(\log \frac{1}{t} \right)^{\sigma - \frac{2}{n-2}} t^{n+2\kappa-1}, \quad 0 < t < t'_0(\sigma).$$

Lemma 10.8 (Optimal growth estimate at singular points). *Let u be an almost minimizer for the Signorini problem in B_1 such that $0 \in \Gamma(u)$ and $\widehat{N}(0+, u) = \kappa < \kappa_0$, $\kappa = 2m$, $m \in \mathbb{N}$. Then, for $0 < t < t_0$,*

$$\begin{aligned}
\int_{\partial B_t} u^2 & \leq C t^{n+2\kappa-1}, \\
\int_{B_t} |\nabla u|^2 & \leq C t^{n+2\kappa-2}.
\end{aligned}$$

Proof. Starting with $\sigma = 1$ in Lemma 7.1 and repeatedly applying Corollary 10.7, we find $0 < \sigma \leq \min\{\frac{2}{n-2}, 1\}$ such that

$$\int_{\partial B_t} u^2 \leq C \left(\log \frac{1}{t} \right)^\sigma t^{n+2\kappa-1}, \quad 0 < t < t_0.$$

In fact, we can make σ to be strictly less than $\frac{2}{n-2}$ by noticing that in Lemma 10.6 we can replace $\frac{2}{n-2}$ by any smaller positive number. Then by Lemma 10.5

$$0 \leq W_\kappa(t, u) \leq C \left(\log \frac{1}{t} \right)^{-\frac{1-\sigma\gamma}{\gamma}}.$$

Recall also that for $0 < s < t < t_0$

$$|m(t) - m(s)| \leq C \left(\log \frac{t}{s} \right)^{1/2} (W_\kappa(t) - W_\kappa(s))^{1/2}.$$

We then again consider the exponentially dyadic decomposition as in the proof of Lemma 10.6. Let $0 \leq j \leq i$ be such that $2^{-2^{i+1}} \leq s/t_0 < 2^{-2^i}$ and $2^{-2^{j+1}} \leq t/t_0 < 2^{-2^j}$. Then,

$$\begin{aligned} |m(t) - m(s)| &\leq C \sum_{k=j}^i 2^{k/2} W_\kappa(2^{-2^k} t_0)^{1/2} \\ &\leq C \sum_{k=j}^{\infty} 2^{(1-\frac{1-\sigma\gamma}{\gamma})k/2} \\ &\leq C 2^{(\sigma-\frac{2}{n-2})j/2} \\ &\leq C \left(\log \frac{1}{t} \right)^{(\sigma-\frac{2}{n-2})/2}. \end{aligned} \tag{10.7}$$

Particularly,

$$m(t) \leq m(t_0) + C \left(\log \frac{1}{t_0} \right)^{(\sigma-\frac{2}{n-2})/2}.$$

This gives the first bound. The second bound is obtained from the first one by arguing as at the end of Lemma 7.1. \square

Remark 10.9. The growth estimates in Lemma 10.8 enable us to consider κ -homogeneous blowups

$$u_{t_j}^\phi \rightarrow u_0^\phi \quad \text{in } C_{\text{loc}}^1(\mathbb{R}_\pm^n \cup \mathbb{R}^{n-1}).$$

for $t = t_j \rightarrow 0+$, similar to 3/2-homogeneous blowups, defined at the beginning of Section 7, see Remark 8.1.

Proposition 10.10. *Let u be an almost minimizer for the Signorini problem in B_1 such that $0 \in \Gamma(u)$ and $\widehat{N}(0+, u) = \kappa < \kappa_0$, $\kappa = 2m$, $m \in \mathbb{N}$. Then there exist $C > 0$ and $t_0 > 0$ such that*

$$\int_{\partial B_1} |u_t^\phi - u_s^\phi| \leq C \left(\log \frac{1}{t} \right)^{-\frac{1-\gamma}{2\gamma}}, \quad 0 < t < t_0.$$

In particular the blowup u_0^ϕ is unique.

Proof. Using Lemma 10.8, we apply Lemma 10.5 with $\sigma = 0$ to obtain

$$0 \leq W_\kappa(t, u) \leq C \left(\log \frac{1}{t} \right)^{-\frac{1}{\gamma}}.$$

Recall now the estimate

$$\int_{\partial B_1} |u_t^\phi - u_s^\phi| \leq C \left(\log \frac{t}{s} \right)^{1/2} (W_\kappa(t) - W_\kappa(s))^{1/2},$$

for $0 < s < t < t_0$, that we proved in Lemma 8.2 in the case $\kappa = 3/2$ – the proof actually works for any $0 < \kappa < \kappa_0$. Then, applying the exponentially dyadic argument as in the proof of Lemma 10.8, we obtain

$$\int_{\partial B_1} |u_t^\phi - u_s^\phi| \leq C \left(\log \frac{1}{t} \right)^{-\frac{1-\gamma}{2\gamma}}. \quad \square$$

Lemma 10.11 (Nondegeneracy). *Let 0 be a free boundary point of u such that $\widehat{N}(0+, u) = \kappa$, $\kappa = 2m$, $m \in \mathbb{N}$. Then*

$$\liminf_{t \rightarrow 0} \int_{\partial B_1} (u_t^\phi)^2 = \liminf_{t \rightarrow 0} \frac{1}{t^{n+2\kappa-1}} \int_{\partial B_t} u^2 > 0.$$

Proof. We use the approach of [CSV17, Lemma 7.2]. Assume to the contrary that for some $r_j \searrow 0+$

$$\lim_{j \rightarrow \infty} \frac{1}{r_j^{n+2\kappa-1}} \int_{\partial B_{r_j}} u^2 = 0.$$

Consider then the corresponding Almgren rescalings $u_{r_j}^A(x)$. By Proposition 6.1, over a subsequence, $u_{r_j}^A \rightarrow q$ for some blowup q . By a characterization of singular points in Proposition 10.2, q is κ -homogeneous and is normalized by $\|q\|_{L^2(\partial B_1)} = 1$. Next, for each Almgren rescaling $u_{r_j}^A$ consider its κ -almost homogeneous rescalings

$$[u_{r_j}^A]_t^\phi := \frac{u_{r_j}^A(tx)}{\phi(t)}.$$

Since $u_{r_j}^A$ is an almost minimizer in B_{1/r_j} with gauge function $\omega(t) = (r_j t)^\alpha$, we have

$$N(0+, u_{r_j}^A) = \lim_{s \rightarrow 0+} N(s, u_{r_j}^A) = \lim_{s \rightarrow 0+} N(r_j s, u) = N(0+, u) = \kappa.$$

Thus, by Proposition 10.10, over subsequences, $[u_{r_j}^A]_t^\phi$ converges to a unique blowup q_{r_j} and

$$\int_{\partial B_1} \left| [u_{r_j}^A]_t^\phi - q_{r_j} \right| \leq C \left(\log \frac{1}{t} \right)^{-\frac{1-\gamma}{2\gamma}}, \quad 0 < t < t_0.$$

Notice that since $\|u_{r_j}^A\|_{W^{1,2}(B_1)}$ is uniformly bounded, the constant C is independent of r_j, t . Now we fix r_j , and consider a sequence $\{\rho_i\}_{i=1}^\infty = \{r_i/r_j\}_{i=1}^\infty$. Note that up to subsequence, $[u_{r_j}^A]_{\rho_i}^\phi \rightarrow q_{r_j}$ as $\rho_i \rightarrow 0$, by the uniqueness. Then

$$\begin{aligned} \int_{\partial B_1} q_{r_j}^2 &= \lim_{\rho_i \rightarrow 0} \frac{1}{\rho_i^{n+2\kappa-1}} \int_{\partial B_{\rho_i}} (u_{r_j}^A)^2 \\ &= \frac{r_j^{n+2\kappa-1}}{\int_{\partial B_{r_j}} u^2} \lim_{i \rightarrow \infty} \frac{1}{(r_j \rho_i)^{n+2\kappa-1}} \int_{\partial B_{r_j \rho_i}} u^2 \\ &= \frac{r_j^{n+2\kappa-1}}{\int_{\partial B_{r_j}} u^2} \lim_{i \rightarrow \infty} \frac{1}{r_i^{n+2\kappa-1}} \int_{\partial B_{r_i}} u^2 \\ &= 0 \end{aligned}$$

by the contradiction assumption. Thus, $q_{r_j} = 0$ on ∂B_1 , and hence

$$\int_{\partial B_1} |[u_{r_j}^A]^\phi| \leq C \left(\log \frac{1}{t} \right)^{-\frac{1-\gamma}{2\gamma}}.$$

Now for any $\rho > 0$ and r_j ,

$$\begin{aligned} 1 &= \frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho} q^2 \\ &\leq \frac{\|q\|_{L^\infty(\partial B_\rho)}}{\rho^\kappa} \frac{1}{\rho^{n+\kappa-1}} \int_{\partial B_\rho} |q| \\ &\leq \|q\|_{L^\infty(\partial B_1)} \left[\frac{1}{\rho^{n+\kappa-1}} \int_{\partial B_\rho} |q - u_{r_j}^A| + \frac{1}{\rho^{n+\kappa-1}} \int_{\partial B_\rho} |u_{r_j}^A| \right] \\ &\leq \|q\|_{L^\infty(\partial B_1)} \left[\frac{1}{\rho^{n+\kappa-1}} C_n \rho^{\frac{n-1}{2}} \left(\int_{\partial B_\rho} |q - u_{r_j}^A|^2 \right)^{1/2} + e^{-\left(\frac{\kappa b}{\alpha}\right)\rho^\alpha} \int_{\partial B_1} |[u_{r_j}^A]^\phi| \right] \\ &\leq C \|q\|_{L^\infty(\partial B_1)} \left[\left(\frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho} |q - u_{r_j}^A|^2 \right)^{1/2} + \left(\log \frac{1}{\rho} \right)^{-\frac{1-\gamma}{2\gamma}} \right]. \end{aligned}$$

Note that $u_{r_j}^A \rightarrow q$ in $C_{\text{loc}}^1(B_1^\pm \cup B_1')$. We choose first $\rho > 0$ small and then $r_j = r_j(\rho) > 0$ small to reach a contradiction. \square

The nondegeneracy implies the following important fact, which enables the use of the Whitney Extension Theorem in the proof of the structural theorem on the singular set (Theorem 10.13 below).

For $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, we denote

$$\Sigma_\kappa(u) := \{x_0 \in \Sigma(u) : N(0+, u, x_0) = \kappa\}.$$

Lemma 10.12. *The set $\Sigma_\kappa(u)$ is of topological type F_σ ; i.e., it is a countable union of closed sets.*

Proof. For $j \in \mathbb{N}$, $j \geq 2$, let

$$E_j := \left\{ x_0 \in \Sigma_\kappa(u) \cap \overline{B_{1-1/j}} : \frac{1}{j} \leq \frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho(x_0)} u^2 \leq j \text{ for } 0 < \rho < \frac{1}{2j} \right\}.$$

Then by Lemma 10.8 and Lemma 10.11, $\Sigma_\kappa(u) = \bigcup_{j=2}^\infty E_j$. We now claim that E_j is closed for any $j \geq 2$. Indeed, take a sequence $x_i \in E_j$ such that $x_i \rightarrow x_0$ as $i \rightarrow \infty$. Then $x_0 \in \overline{B_{1-1/j}}$ and for every $0 < \rho < 1/(2j)$, by the local uniform continuity of u ,

$$(10.8) \quad \frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho(x_0)} u^2 = \lim_{i \rightarrow \infty} \frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho(x_i)} u^2 \in \left[\frac{1}{j}, j \right].$$

Next, since $\Gamma(u)$ is relatively closed in B_1' , we also know that $x_0 \in \Gamma(u)$. Moreover, since $N(0+, u, x_i) = \kappa$ and the function $x \mapsto \widehat{N}(0+, u, x)$ is upper semicontinuous, we have

$$\kappa = \limsup_{i \rightarrow \infty} \widehat{N}(0+, u, x_i) \leq \widehat{N}(0+, u, x_0).$$

If $\widehat{N}(0+, u, x_0) = \kappa' > \kappa$, then by Lemma 7.1,

$$\frac{1}{\rho^{n+2\kappa-1}} \int_{\partial B_\rho(x_0)} u^2 \leq C \rho^{2(\kappa'-\kappa)} \left(\log \frac{1}{\rho} \right) \rightarrow 0 \quad \text{as } \rho \rightarrow 0,$$

which contradicts (10.8). Therefore, $\widehat{N}(0+, u, x_0) = \kappa$ and consequently $x_0 \in E_j$. Hence, E_j is closed, $j = 2, 3, \dots$, implying that $\Sigma_\kappa(u)$ is F_σ . \square

To state the main result of this paper concerning the singular points, we need to introduce the following notations. For $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$ and $x_0 \in \Sigma_\kappa(u)$, we define

$$d_{x_0}^{(\kappa)} := \dim\{\xi \in \mathbb{R}^{n-1} : \xi \cdot \nabla_{x'} u_{x_0}^\phi(x', 0) \equiv 0 \text{ on } \mathbb{R}^{n-1}\},$$

which has the meaning of the dimension of $\Sigma_\kappa(u)$ at x_0 , and where $u_{x_0}^\phi$ is the unique κ -homogeneous blowup at x_0 . In fact, $d_{x_0}^{(\kappa)}$ is the dimension of the linear subspace $\Sigma_\kappa(u_{x_0}^\phi) \subset \mathbb{R}^{n-1}$. Since $u_{x_0}^\phi$ is a nonzero solution of the Signorini problem, it cannot vanish identically on \mathbb{R}^{n-1} (see [GP09]) and therefore $d_{x_0}^{(\kappa)} < n - 1$.

For $d = 0, 1, \dots, n - 2$, we denote

$$\Sigma_\kappa^d(u) := \{x_0 \in \Sigma_\kappa(u) : d_{x_0}^{(\kappa)} = d\}.$$

Theorem 10.13 (Structure of the singular set). *Let u be an almost minimizer for the Signorini problem in B_1 . Then for every $\kappa = 2m < \kappa_0$, $m \in \mathbb{N}$, and $d = 0, 1, \dots, n - 2$, the set $\Sigma_\kappa^d(u)$ is contained in the union of countably many submanifolds of dimension d and class $C^{1, \log}$.*

Proof. Let $\kappa = 2m$, $m \in \mathbb{N}$. For $x \in \Sigma_\kappa(u) \cap B'_{1/2}$, let $q_x \in \mathcal{Q}_\kappa$ denote the unique κ -homogeneous blowup of u . By the optimal growth (Lemma 10.8) and the nondegeneracy (Lemma 10.11), we can write

$$q_x = \lambda_x q_x^A, \quad \lambda_x > 0, \quad \|q_x^A\|_{L^2(\partial B_1)} = 1,$$

where $q_x^A \in \mathcal{Q}_\kappa$ is the corresponding Almgren blowup. We want to show that the q_x , q_x^A , λ_x depend continuously on $x \in \Sigma_\kappa$, with a logarithmic modulus of continuity.

Let $x_1, x_2 \in \Sigma_\kappa(u) \cap B'_{1/2}$. Then for $t > 0$, to be chosen below, we can write

$$(10.9) \quad \begin{aligned} \|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} &\leq \|q_{x_1} - u_{x_1, t}^\phi\|_{L^1(\partial B_1)} \\ &\quad + \|u_{x_1, t}^\phi - u_{x_2, t}^\phi\|_{L^1(\partial B_1)} + \|u_{x_2, t}^\phi - q_{x_2}\|_{L^1(\partial B_1)}. \end{aligned}$$

By Proposition 10.10, we have

$$(10.10) \quad \|q_x - u_{x, t}^\phi\|_{L^1(\partial B_1)} \leq C \left(\log \frac{1}{t} \right)^{-\frac{1}{n-2}}$$

for $x \in \Sigma_\kappa(u) \cap B'_{1/2}$. This controls the first and third term on the right hand side of (10.9). To estimate the middle term, we observe that

$$\begin{aligned} &\|u_{x_1, t}^\phi - u_{x_2, t}^\phi\|_{L^1(\partial B_1)} \\ &\leq \frac{e^{(\frac{\kappa b}{\alpha})t^\alpha}}{t^\kappa} \int_{\partial B_1} \int_0^1 |\nabla u(x_1 + tz + r(x_2 - x_1))| |x_1 - x_2| dr dS_z \end{aligned}$$

for any $0 < t < 1/2$. Recalling that $\nabla u(x_1) = 0$ and $u \in C^{1, \beta}(B_1^\pm \cup B'_1)$, we have

$$\begin{aligned} &|\nabla u(x_1 + tz + r(x_2 - x_1))| \\ &\leq C |tz + r(x_2 - x_1)|^\beta \leq C(t + |x_1 - x_2|)^\beta \leq C|x_1 - x_2|^{\frac{\beta}{2(\kappa - \beta)}} \end{aligned}$$

if we choose $t = |x_1 - x_2|^{\frac{1}{2(\kappa-\beta)}}$ and have $|x_1 - x_2| < (1/2)^{2(\kappa-\beta)}$. This gives

$$(10.11) \quad \|u_{x_1,t}^\phi - u_{x_2,t}^\phi\|_{L^1(\partial B_1)} \leq \frac{C}{t^\kappa} |x_1 - x_2|^{\frac{\beta}{2(\kappa-\beta)}} |x_1 - x_2| \leq C |x_1 - x_2|^{1/2}.$$

Combining (10.9), (10.11), and (10.10), we obtain

$$(10.12) \quad \|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} \leq C \left(\log \frac{1}{|x_1 - x_2|} \right)^{-\frac{1}{n-2}}.$$

Next, by Lemma 10.8, for any $x \in \Sigma_\kappa(u) \cap B'_{1/2}$ and small t

$$\int_{\partial B_1} (u_{x,t}^\phi)^2 \leq C$$

with C independent of x , and passing to the limit as $t \rightarrow \infty$ obtain the bound

$$\lambda_x^2 = \int_{\partial B_1} q_x^2 \leq C$$

Moreover, since q_x is a κ -homogeneous harmonic polynomial, we also have

$$(10.13) \quad \|q_x\|_{L^\infty(B_1)} \leq C(n, \kappa) \|q_x\|_{L^2(\partial B_1)} \leq C.$$

Then, by combining (10.12) and (10.13), we have

$$(10.14) \quad \begin{aligned} |\lambda_{x_1} - \lambda_{x_2}| &\leq |\lambda_{x_1}^2 - \lambda_{x_2}^2|^{1/2} \leq \left(\int_{\partial B_1} |q_{x_1}^2 - q_{x_2}^2| \right)^{1/2} \\ &\leq \|q_{x_1} + q_{x_2}\|_{L^\infty(B_1)}^{1/2} \|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)}^{1/2} \\ &\leq C \left(\log \frac{1}{|x_1 - x_2|} \right)^{-\frac{1}{2(n-2)}}. \end{aligned}$$

Finally, we want to estimate $q_{x_1}^A - q_{x_2}^A$. By writing

$$\begin{aligned} \|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} &= \int_{\partial B_1} |\lambda_{x_1} q_{x_1}^A - \lambda_{x_2} q_{x_2}^A| \\ &= \int_{\partial B_1} |\lambda_{x_1} (q_{x_1}^A - q_{x_2}^A) + (\lambda_{x_1} - \lambda_{x_2}) q_{x_2}^A| \\ &\geq \lambda_{x_1} \int_{\partial B_1} |q_{x_1}^A - q_{x_2}^A| - |\lambda_{x_1} - \lambda_{x_2}| \int_{\partial B_1} |q_{x_2}^A|, \end{aligned}$$

we estimate

$$(10.15) \quad \begin{aligned} \lambda_{x_1} \int_{\partial B_1} |q_{x_1}^A - q_{x_2}^A| &\leq \|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} + |\lambda_{x_1} - \lambda_{x_2}| \int_{\partial B_1} |q_{x_2}^A| \\ &\leq \|q_{x_1} - q_{x_2}\|_{L^1(\partial B_1)} + C(n) |\lambda_{x_1} - \lambda_{x_2}| \\ &\leq C \left(\log \frac{1}{|x_1 - x_2|} \right)^{-\frac{1}{2(n-2)}}, \end{aligned}$$

where we used $\|q_{x_2}^A\|_{L^2(\partial B_1)} = 1$ in the second inequality and (10.12) and the bound (10.14) in the third inequality. Next, using that q_x^A are κ -homogeneous harmonic polynomials, we have

$$\|q_{x_1}^A - q_{x_2}^A\|_{L^\infty(B_1)} \leq C \|q_{x_1}^A - q_{x_2}^A\|_{L^1(\partial B_1)},$$

which combined with (10.15) gives

$$(10.16) \quad \lambda_{x_1} \|q_{x_1}^A - q_{x_2}^A\|_{L^\infty(B_1)} \leq C \left(\log \frac{1}{|x_1 - x_2|} \right)^{-\frac{1}{2(n-2)}}.$$

Now we fix $x_0 \in \Sigma_\kappa(u) \cap B'_{1/4}$. Then by (10.14), there exists $\delta = \delta(x_0) \in (0, (1/2)^{2(\kappa-\beta)+1})$ such that $\lambda_x \geq 1/2\lambda_{x_0}$ if $x \in \Sigma_\kappa(u) \cap B'_\delta(x_0)$. Then by (10.16), we conclude that

$$(10.17) \quad \|q_{x_1}^A - q_{x_2}^A\|_{L^\infty(B_1)} \leq C \left(\log \frac{1}{|x_1 - x_2|} \right)^{-\frac{1}{2(n-2)}}, \quad x_1, x_2 \in \Sigma_\kappa(u) \cap B_\delta(x_0).$$

Notice that the constant C does not depend on x_1, x_2 , but both C and δ do depend on x_0 .

Once we have the estimates (10.14) and (10.17), as well as Lemma 10.12, we can apply the Whitney Extension Theorem of Fefferman's [Fef09], to complete the proof, see e.g., the proof of Theorem 5 in [CSV17]. \square

APPENDIX A. SOME EXAMPLES OF ALMOST MINIMIZERS

Example A.1. If u is a minimizer of the functional

$$\int_D a(x) |\nabla u|^2$$

over the set $\mathfrak{K}_{\psi,g}(D, \mathcal{M})$ with strictly positive $a \in C^{0,\alpha}(\overline{D})$, $0 < \alpha \leq 1$, then u is an almost minimizer for the Signorini problem with a gauge function $\omega(r) = Cr^\alpha$.

Proof. This is rather immediate. \square

Example A.2. Let u be a solution of the Signorini problem for the Laplacian with drift with the velocity field $b \in L^p(B_1)$, $p > n$:

$$\begin{aligned} -\Delta u + b(x)\nabla u &= 0 \quad \text{in } B_1^\pm \\ -\partial_{x_n} u &\geq 0, \quad u \geq 0, \quad u\partial_{x_n} u = 0 \quad \text{on } B'_1, \end{aligned}$$

even in x_n -variable. We understand this in the weak sense that u satisfies the variational inequality

$$\int_{B_1} \nabla u \nabla (w - u) + (b(x)\nabla u)(w - u) \geq 0,$$

for any competitor $w \in \mathfrak{K}_{0,u}(B_1, B'_1)$, i.e. $w \in u + W_0^{1,2}(B_1)$ such that $w \geq 0$ on B'_1 in the sense of traces. Then u is an almost minimizer for the Signorini problem with $\psi = 0$ on $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$ and a gauge function $\omega(r) = Cr^{1-n/p}$.

Proof. Let $B_r(x_0) \Subset B_1$ and $w \in \mathfrak{K}_{0,u}(B_r(x_0), B'_1)$. Extending w as equal to u in $B_1 \setminus B_r(x_0)$, and applying the variational inequality for u , we obtain

$$(A.1) \quad \int_{B_r(x_0)} \nabla u \nabla (w - u) + b(x)\nabla u (w - u) \geq 0.$$

Let v be the Signorini replacement of u on $B_r(x_0)$. Then v satisfies the variational inequality

$$(A.2) \quad \int_{B_r(x_0)} \nabla v \nabla (w - v) \geq 0,$$

for all w as above. Now, taking $w = u \pm (u - v)^+$ in (A.1) we will have

$$\int_{B_r(x_0)} \nabla u \nabla (u - v)^+ + b(x) \nabla u (u - v)^+ = 0.$$

Next, taking $w = v + (u - v)^+$ in (A.2), we have

$$\int_{B_r(x_0)} \nabla v \nabla (u - v)^+ \geq 0.$$

Taking the difference, we then obtain

$$\int_{B_r(x_0)} |\nabla (u - v)^+|^2 \leq - \int_{B_r(x_0)} b(x) \nabla u (u - v)^+.$$

Similarly, taking $w = v \pm (v - u)^+$ in (A.2) and $w = u + (v - u)^+$ in (A.1) and subtracting the resulting inequalities, we obtain

$$\int_{B_r(x_0)} |\nabla (v - u)^+|^2 \leq \int_{B_r(x_0)} b(x) \nabla u (v - u)^+.$$

Hence, combining the inequalities above, we arrive at

$$\int_{B_r(x_0)} |\nabla (v - u)|^2 \leq \int_{B_r(x_0)} |b(x)| |\nabla u| |v - u|.$$

Then, applying Hölder's inequality, we have for $p > n$

$$\int_{B_r(x_0)} |\nabla (v - u)|^2 \leq \|b\|_{L^p(B_r(x_0))} \|\nabla u\|_{L^2(B_r(x_0))} \|v - u\|_{L^{p^*}(B_r(x_0))},$$

with $p^* = 2p/(p-2)$. Next, since $v - u \in W_0^{1,2}(B_1)$, from the Sobolev's inequality we have

$$\|v - u\|_{L^{p^*}(B_r(x_0))} \leq C_{n,p} r^{1-n/p} \|\nabla (v - u)\|_{L^2(B_r(x_0))}$$

and hence we can conclude that

$$\int_{B_r(x_0)} |\nabla (v - u)|^2 \leq C r^{2(1-n/p)} \int_{B_r(x_0)} |\nabla u|^2$$

with $C = C_{n,p} \|b\|_{L^p(B_1)}^2$. This implies

$$\begin{aligned} \int_{B_r(x_0)} |\nabla u|^2 - \int_{B_r(x_0)} |\nabla v|^2 &= \int_{B_r(x_0)} (\nabla u + \nabla v)(\nabla u - \nabla v) \\ &\leq C r^\gamma \int_{B_r(x_0)} (|\nabla u|^2 + |\nabla v|^2) + C r^{-\gamma} \int_{B_r(x_0)} |\nabla (v - u)|^2 \\ &\leq C r^\gamma \int_{B_r(x_0)} (|\nabla u|^2 + |\nabla v|^2) + C r^{2(1-n/p)-\gamma} \int_{B_r(x_0)} |\nabla u|^2, \end{aligned}$$

where we have used Young's inequality in the second line. Choosing $\gamma = 1 - n/p$ we then deduce that for small enough $0 < r < r_0(n, p, \|b\|_{L^p(B_1)})$

$$\int_{B_r(x_0)} |\nabla u|^2 \leq (1 + C r^{1-n/p}) \int_{B_r(x_0)} |\nabla v|^2$$

with $C = C_{n,p} \|b\|_{L^p(B_1)}^2$. □

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