Large time behavior of solutions of the p-Laplacian equation

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Abstract

We establish the behavior of the solutions of the degenerate parabolic equation

$$u_t = \nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right), \qquad p > 2,$$

posed in the whole space with nonnegative, continuous and compactly supported initial data. We prove a nonlinear concavity estimate for the pressure $v = u^{(p-2)/(p-1)}$ away from the the maximum point of v. The estimate implies that the support of the solution becomes convex for large times and converges to a ball. In dimension one, we know also that the pressure itself eventually becomes concave.

1 Introduction

In this paper we establish the large time behavior of the solutions of the degenerate parabolic equation

$$u_t = \nabla \cdot \left(|\nabla u|^{p-2} \nabla u\right). \tag{1.1}$$

For exponent $p = 2$ this is the classical Heat Equation (HE), whose theory is well known. Among its features we find C^∞ smoothness of solutions, infinite speed of propagation of disturbances and the strong Maximum Principle. These properties generalize to a number of related evolution equations, notably those which are linear and uniformly parabolic.

A marked departure occurs in (1.1) when the exponent p is larger than 2. The equation is degenerate parabolic and finite propagation holds. It is usually called the evolution p-Laplacian equation (PLP for short). We consider the initial value problem for the PLP posed in $Q = \mathbf{R}^N \times (0, \infty)$, with initial data

$$u(x, 0) = u_0(x) \qquad \text{on} \quad \mathbf{R}^N, \tag{1.2}$$

where u_0 is a nonnegative integrable function in $\mathbf{R}^N$ whose support is contained in the ball $B(0, R)$ centered at 0 and having radius R.

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It is known that if $u_0 \in L^1(\mathbf{R}^N)$ there exists a unique nonnegative weak solution and for each t it has compact support that increases with t. Hence, there exists an *interface* or *free boundary* separating regions where $u > 0$ from regions where $u = 0$. The solution is C^∞ smooth in its positivity set, but the interface might not be a smooth surface if u_0 is topologically complicated, as the focusing solutions studied by Gil and Vázquez show, [14], see also [2]. However, the solutions are known to have locally Hölder continuous first derivatives cf [6, 9].

About the asymptotic behavior, in [18], 1988, Kamin and Vazquez studied the uniqueness and asymptotic behavior of positive solutions. They proved that the explicit solutions

$$U_M(x,t) = t^{-k}\left(C - q\left(\frac{|x|}{t^{k/N}}\right)^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{p-2}}$$

found by G. I. Barenblatt in 1952 are essentially the only positive solutions to a Cauchy problem with the initial data

$$u(x,0) = M\delta(x), \qquad M > 0.$$

Here

$$k = \left(p - 2 + \frac{p}{N}\right)^{-1}, \qquad q = \frac{p-2}{p}\left(\frac{k}{N}\right)^{\frac{1}{p-1}}$$

and C is related to the mass M by $C = cM^\alpha$, with $\alpha = p(p-2)k/N(p-1)$ and $c = c(p,N)$ determined from the condition $\int U_M(x,t)dx = M$. Using the idea of asymptotic radial symmetry, Kamin and Vazquez established that any nonnegative solution with globally integrable initial values is asymptotically equal to the Barenblatt solution as $t \to \infty$.

A consequence of the approximation to the Barenblatt profiles is the property of *asymptotic concavity* that can be best expressed in terms of the convenient variable,

$$v = \frac{p-1}{p-2}\, u^{\frac{p-2}{p-1}} \tag{1.3}$$

known as the *pressure* (in which u is the *density*). Then v satisfies the equation

$$v_t = \frac{p-2}{p-1}\, v\, \Delta_p v + |\nabla v|^p \tag{1.4}$$

The pressure variable is appropriate to study properties related to interface behavior and geometry, while u is better suited for existence and uniqueness questions. It is easy to see that for the Barenblatt solutions the formula

$$\partial_e(|\nabla v|^{p-2}\partial_e v) = -\frac{K}{t}, \qquad K = \left(\frac{p-2}{p-1}\right)^{2(p-1)}\frac{k}{N} \tag{1.5}$$

holds in the set $v > 0$ for every direction e. As one can show, this property implies the concavity of v (see Lemma 5.1 in Section 5.)

Outline of the paper:

- Section 2 contains definitions and preliminary results and in Section 3 we state our main results.

- Section 4 contains the proof of C^∞ regularity near the interface for $p > 2$. Section 5 deals with convergence to the Barenblatt solution for all p. In the next three sections we work in one dimension. Section 6 contains the proof of eventual concavity for $p < 2$ and Section 7 for $p > 2$. The study of the curve of maxima is done in Section 8.

2 Definitions and preliminary results

The Cauchy problem (1.1)–(1.2) (or problem (CP) for short) does not possess classical solutions for general data in the class: $u_0 \in L^1(\mathbf{R}^N)$, $u_0 \geq 0$ (or even in a smaller class, like the set of smooth nonnegative and rapidly decaying initial data). This is due to the fact that the equation is parabolic only where $|\nabla u| > 0$, but degenerate where $|\nabla u| = 0$. Therefore, we need to introduce a concept of generalized solution and make sure that the problem is well-posed in that class.

By a *weak solution* of the equation (1.1) we will mean a nonnegative measurable function $u(x,t)$, defined for $(x,t) \in Q$ such that: (i) viewed as a map

$$t \to u(\cdot, t) = u(t). \tag{2.1}$$

we have $u \in C((0,\infty); L^1(\mathbf{R}^N))$; (ii) the functions u and $|\nabla u|^{p-2}\nabla u$ belong to $L^1(t_1, t_2; L^1(\mathbf{R}^N))$ for all $0 < t_1 < t_2$; and (iii) the equation (1.1) is satisfied in the weak sense

$$\iint \{u\varphi_t - |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi\}\, dxdt = 0$$

for every smooth test function $\varphi \geq 0$ with compact support in Q.

By a *solution of problem* (CP) we mean a weak solution of (1.1) such that the initial data (1.2) are taken in the following sense:

$$u(t) \to u_0 \quad \text{in } L^1(\mathbf{R}^N) \quad \text{as } t \to 0. \tag{2.2}$$

In other words, $u \in C([0,\infty); L^1(\mathbf{R}^N))$ and $u(0) = u_0$.

The existence and uniqueness of solutions of problem (CP) in Q for compactly supported u_0 follows from the result of DiBenedetto and Herrero [10, 11] for general initial data $u_0 \in L^1_{\rm loc}(\mathbf{R}^N)$ with an optimal growth condition at infinity (if $p > 2$)

$$|||u_0|||_r = \sup_{\rho \geq r} \rho^{-\lambda} \int_{B_\rho(0)} u_0(x)dx < \infty, \quad \lambda = N + \frac{p}{p-2}$$

Next we list some important properties of solutions.

Property 1 *The solutions of problem* (CP) *satisfy the law of mass conservation*

$$\int_{\mathbf{R}^N} u(x,t)\,dx = \int_{\mathbf{R}^N} u_0(x)\,dx, \tag{2.3}$$

i.e., $||u(t)||_{L^1(\mathbf{R}^N)} = ||u_0||_{L^1(\mathbf{R}^N)}$ *for all* $t > 0$.

The proof of the following estimate can be found in [26]

Property 2 *The solutions are bounded for* $t \geq \tau > 0$. *More precisely,*

$$|u(x,t)| \leq U_M(0,t) = c_*(p,N) M^{pk/N} t^{-k}, \tag{2.4}$$

where $M = ||u_0||_{L^1(\mathbf{R}^N)}$ *and* $k = (p-2+p/N)^{-1}$.

Property 3 *The weak solutions* $u(x,t)$ *and their spatial gradients* $\nabla u(x,t)$ *are uniformly Hölder continuous for* $0 < \tau \leq t \leq T < \infty$.

The next semi-convexity estimate is due to Esteban-Vazquez [13]

Property 4 *For* $p > 2N/(N+1)$ *there exist a constant* $C = C(p,N)$ *such that for any nonnegative solution* u *of the Cauchy problem* (CP), *the pressure* v *satisfies the estimate*

$$\Delta_p v \geq -\frac{C}{t}. \tag{2.5}$$

in the sense of distributions.

Property 5 (Finite propagation property) *If the initial function* u_0 *is compactly supported so are the functions* $u(\cdot,t)$ *for every* $t > 0$. *Under these conditions there exists a free boundary or interface which separates the regions* $\{(x,t) \in Q : u(x,t) > 0\}$ *and* $\{(x,t) \in Q : u(x,t) = 0\}$.

This interface is usually an N*-dimensional hypersurface in* $\mathbf{R}^{N+1}$,

Property 6 (Scaling) *One of the critical properties of the p-Laplacian equation is the* scaling invariance. *Any solution* $u(x,t)$ *of (1.1) will produce a family of solutions*

$$\left(\frac{B}{A^p}\right)^{\frac{1}{p-2}} u(Ax, Bt) \tag{2.6}$$

for any $A, B > 0$. *In particular, choosing* $A = \theta^{-k/N}$, $B = \theta^{-1}$ *for* $\theta > 0$, *we obtain the scaling*

$$\frac{1}{\theta^k}\, u\left(\frac{x}{\theta^{k/N}}, \frac{t}{\theta}\right) \tag{2.7}$$

which is the one that conserves the mass for the density u.

Let us conclude this section by pointing out that the source-type solutions $U_M(x,t)$ are weak solutions of (1.1), but they are *not* solutions of problem (CP) as stated because they do not take L^1 initial data. Indeed, it is easy to check that U_M converges to a Dirac mass

$$U_M(x,t) \to M\,\delta(x) \qquad \text{as} \;\; t \to 0, \tag{2.8}$$

This is the reason for the name "source-type solutions". They are invariant under scaling for the choice $A = B^{k/N}$.

The asymptotic behavior of any solution of the Cauchy problem is described in terms of the Barenblatt solution with the same mass.

Theorem 2.1 *Let $u(x,t)$ be the unique solution of problem* (CP) *with initial data $u_0 \in L^1(\mathbf{R}^N)$, let $M = \int u_0(x)\,dx$. If U_M is the Barenblatt solution with the same mass as u_0, then as $t \to \infty$ we have*

$$\lim_{t\to\infty} t^k \|u(t) - U_M(t)\|_{L^\infty(\mathbf{R}^n)} = 0. \tag{2.9}$$

The the proof is due to [13].

Let us finally recall that the functions $U_M(x,t)$ have the self-similar form

$$U_M(x,t) = t^{-k} F(xt^{-k/N}; C), \tag{2.10}$$

where $k = (p-2+p/N)^{-1}$ and $k/N = (N(p-2)+p)^{-1}$ are the similarity exponents and

$$F(s) = (C - qs^{\frac{p}{p-1}})_+^{\frac{p-1}{p-2}}. \tag{2.11}$$

is the profile, where $C = cM^\alpha$, with $\alpha = p(p-2)k/N(p-1)$, $c = c(p,N)$ and $q = (1-2/p)(k/N)^{1/(p-1)}$. For the pressure variable we can write

$$V_M(r,t) = \frac{1}{(Lt)^{k\frac{p-2}{p-1}}}\, G_L\left(\frac{|x|}{(Lt)^{k/N}}\right),$$

with $L = M^{p-2}$ and profile

$$G_L(s) = L^{\frac{1}{p-1}}\,(c - q\,s^{\frac{p}{p-1}})_+$$

for c, q depending only on p and N. The free boundary is given by the equation $|x| = (c/q)(Lt)^{k/N}$.

Property 7 (Asymptotic error for the support.) *Using Aleaksandrov's Reflection Principle, one can prove the following sharp estimates on the size of the positivity set $\Omega(t) = \{v(\cdot,t) > 0\}$, see e.g. [29]*

$$B_{r(t)} \subset \Omega(t) \subset B_{R(t)}, \quad R(t) \le r(t) + 2R_0, \quad R(t) \sim (c/q)(Lt)^{k/N},$$

if the support of the initial data contained in the ball of radius R_0 (with center at 0).

3 Statement of main results

We are going to impose the conditions on the initial pressure v_0, which have been used to get the long-time non-degenerate Lipschitz solutions in [7, 29] and $C^{1,\alpha}$ regularity of the interface in [20].

Conditions:

1. The support of v_0, $\overline{\Omega}_0 = \overline{\{v_0 \geq 0\}}$, is contained in a ball of radius $R > 0$,
2. Regularity: $\partial\Omega_0$ is C^1 regular and $v_0 \in C^1(\overline{\Omega}_0)$,
3. Non-degeneracy: $0 < \frac{1}{K} < v_0 + |\nabla v_0| < K$ in $\overline{\Omega}_0$,
4. Semi-concavity: $\partial_{ee} v_0 \geq -K_0$ in a strip $S \subset \Omega_0$ near the boundary $\partial\Omega_0$ for any direction e.

The uniform convergence result (2.9) can be restated as

$$\lim_{t\to\infty} t^{k\frac{p-2}{p-1}} \|v(t) - V_M(t)\|_{L^\infty(\mathbf{R}^N)} = 0. \tag{3.1}$$

Our goal in this paper is improving the uniform convergence up to C^∞-convergence and then getting the convexity of the positivity set, $\Omega(t) = \{x : \ v(x,t) > 0\}$ (or the support, which is its closure) and the concavity of $v(x,t)$ in $\Omega(t)$ at all long times $t >> 1$.

Scaling will play an important role in our proof. The first use is in reducing the problem. Thus, given a solution with mass $M > 0$ we can use the scaling

$$\tilde{v}(x,t) = \frac{1}{A^{\frac{p}{p-1}}}\, v(Ax, t)$$

with $A = M^{(k/N)(p-2)}$ to get another solution $\tilde{v}$ with mass 1. Therefore, we can take $M = L = 1$ in the sequel. We will write G instead of G_1 for the Barenblatt profile, and V instead of V_1 for the corresponding solution.

On a more fundamental aspect, given a solution $v = v(x,t)$ with mass M we will define the family

$$v_\lambda(x,t) = \lambda^{k\frac{p-2}{p-1}} v(\lambda^{k/N} x, \lambda t), \qquad \lambda > 0, \tag{3.2}$$

which are again solutions of the same equation with same mass, now normalized to 1. The long-time behavior can be captured through the uniform bound for the scaled solutions. Formula (3.1) can be stated equally by

$$|v_\lambda(x,t) - V(x,t)| \to 0 \qquad \text{as } \ \lambda \to \infty,$$

uniformly in $|x| < K$, $1 < t < 2$. Therefore, we will concentrate on the convergence of v_λ towards $V(x,t) = t^{-k\frac{p-2}{p-1}} G(xt^{-k/N})$.

First we are going to show the uniform estimate of all possible derivatives of v_λ (for large $\lambda > 0$) for $t \in [1,2]$ everywhere except an arbitrary small ball B_ε centered at the origin.

Theorem 3.1 *For every $k > 0$ and $\varepsilon > 0$ there exists a value of the scaling parameter $\lambda_{k,\varepsilon}$ and a uniform constant $C_{k,\varepsilon} > 0$ such that*

$$||v_\lambda(x,t)||_{C^k_{x,t}(\overline{\Omega(v_\lambda)}\setminus B_\varepsilon(0)\times[1,2])} < C_{k,\varepsilon} \qquad \textit{for all } \lambda > \lambda_{k,\varepsilon}, \tag{3.3}$$

where

$$\Omega(v_\lambda) = \{(x,t)|\ v_\lambda(x,t) > 0,\ 1 < t < 2\}.$$

Let us translate these results into asymptotic concavity statements. We remark that we have the following identity for $G(x)$

$$\partial_e(|\nabla G|^{p-2}\partial_e G) = -q^{p-1}$$

for every direction e and at all points x such that $G(x) > 0$, and the C^2_x-convergence away from the origin implies

$$\partial_e(|\nabla v_\lambda|^{p-2}\partial_e v_\lambda) < 0 \qquad \text{on } \{v_\lambda(\cdot,t) > 0\} \setminus B_\varepsilon$$

for λ large.

Theorem 3.2 *There is $t_0 > 0$ such that $\Omega(t) = \{x:\ v(x,t) > 0\}$ is a convex subset of $\mathbf{R}^N$ for $t \geq t_0$ and its curvature converges to the constant curvature of the free boundary of the Barenblatt solution*

$$\lim_{t\to\infty} t^{k/N}\, K(x,t) = C. \tag{3.4}$$

uniformly in $x \in \partial\Omega(t)$ Moreover, for every $\varepsilon > 0$ there is $t_{0,\varepsilon}$ such that $v(x,t)$ is concave in $\Omega(t) \setminus B_{\varepsilon t^{k/N}}$ for $t \geq t_{0,\varepsilon}$. More precisely,

$$\lim_{t\to\infty} t\,\partial_e(|\nabla v|^{p-2}\partial_e v) = -q^{p-1} \tag{3.5}$$

for every direction e uniformly for $x \in \Omega(t) \setminus B_{\varepsilon t^{k/N}}$ for every $\varepsilon > 0$.

In one-dimensional case we can also establish the concavity near the origin.

Theorem 3.2′ *In the dimension $N = 1$, the convergence (3.5) is uniform for $x \in \Omega(t)$. As a consequence, all level sets $\{x : v(x,t) \geq c\}$, $c > 0$, are convex (if not empty). The function $u(\cdot,t)$ has only one maximum point $\gamma(t)$. Moreover, the curve $x = \gamma(t)$ is $C^{1,\alpha}$-regular for $t \geq t_0$.*

4 Regularity near the interface, $p > 2$

Let v be a solution of (1.4) for $p > 2$. Due Esteban and Vazquez [13], we know that

$$\Delta_p v \geq -\frac{C}{t}, \qquad \text{for } C = C(p,N) > 0 \tag{4.1}$$

$$v_t \geq -\frac{p-1}{p-2}\cdot\frac{C}{t}\cdot v \tag{4.2}$$

and also

$$(4.3) \qquad v_t,\ |\nabla v|^p = O\left(t^{-k\frac{p-2}{p-1}-1}\right),$$

which after the scaling (3.2) take the form

$$(4.4) \qquad \Delta_p v_\lambda \geq -C, \qquad v_t \geq -\frac{p-1}{p-2}\, C\, v_\lambda, \qquad v_{\lambda,t},\ |\nabla v|^p \leq C$$

with C independent of λ.

4.1 Nondegeneracy of ∇v_λ near the free boundary.

From the exact estimates on the growth of the domain $\Omega(t) = \{x : v(x,t) > 0\}$, see Property 7 in Section 2, after rescaling we can assume that

$$(4.5) \qquad B_{\rho_0} \subset \Omega_\lambda(t) = \{x : v_\lambda(x,t) > 0\} \subset B_{\rho_1}, \quad \text{for } t \in [1,2]$$

for some ρ_0, $\rho_1 > 0$, independent of λ. We claim that there is $\delta_0 > 0$ and $c_0 > 0$ independent of λ such that

$$(4.6) \qquad |\nabla v_\lambda| \geq c_0 \quad \text{in } \delta_0\text{-neigborhood of } \partial\Omega_\lambda(t)\ ,\ t \in [1,2].$$

or, equivalently,

$$(4.7) \qquad v_\lambda + |\nabla v_\lambda| \geq K_1(K, K_0) > 0 \quad \text{in } \{v_\lambda > 0\} \text{ for } t \in [1,2].$$

The proof of this statement is based on the inequality

$$(4.8) \qquad \frac{A-p}{p-1}\, v(x,t) + x \cdot \nabla v(x,t) + (At+B) v_t(x,t) \geq 0$$

in $\mathbf{R}^N \times (0,\infty)$, which can be found in [7, 29]. Here $A,\ B > 0$ depend only on K and K_0. Using straightforward computations, one can show that after rescaling (3.2) the inequality (4.8) will take the form

$$(4.9) \qquad \frac{A-p}{p-1}\, v_\lambda(x,t) + x \cdot \nabla v_\lambda(x,t) + (At + B/\lambda)\, v_{\lambda,t}(x,t) \geq 0.$$

Now, using the ideas analogous to those in the proof of Lemma 3.3 in [4], and carried out in detail for p-Laplacian equation by Ko [20], Section 3, one can prove the estimate (4.7). An important observation is that even though $B/\lambda \to 0$ as $\lambda \to \infty$, the estimates in [20] will depend actually on A and $At + B/\lambda$ but when $t \in [1,2]$ and $\lambda > 1$ we have

$$A \leq At + B/\lambda \leq 2A + B$$

which implies the uniformity of these estimates.

4.2 $C^{1,\alpha}$-regularity of the pressure

From now on, to simplify notations, we will omit the index λ and use simply v for the rescaled pressure v_λ.

From the result of Y. Ko [20], we know that the interface $\partial\Omega(t)$ will be $C^{1,\alpha}$ regular for $t \in [1,2]$. However we need also $C^{1,\alpha}$ regularity of the pressure v in order to prove C^∞ regularity of the interface. We apply the method originally due Koch [K], which was used to prove C^∞ regularity of the interface in the porous medium equation.

We know that the positivity set $\Omega(t)$ of $v(\cdot,t)$ contains a ball B_{ρ_0} for $t \in [1,2]$. Moreover, we may assume $\Omega(0)$ is contained in $B_{\rho_0/2}$ for λ large. Then, a simple reflection argument used in Proposition 2.1, [1] implies that there is a uniform cone of directions $\mathcal{C} = \{\alpha \in S^{n-1} : \text{angle}\,(\alpha, x/|x|) < \pi/2 - \eta_0\}$ such that function $v(x,t)$ is decreasing in any direction from $\mathcal{C}$ for x with $|x| > \rho_0/2$, where $\eta_0 > 0$ is a uniform constant, which can be made as small as we wish if we take λ sufficiently large. This, together with the uniform nondegeneracy of the gradient of $v(\cdot,t)$ near $\partial\Omega(t)$, allows to prove that there exist uniform positive constants δ_0 and c_0 such that

$$\partial_e v(x,t) \le -c_0$$

for $x \in B_{\delta_0}(x_0)$, where $x_0 \in \partial\Omega(t)$ and $e = x_0/|x_0|$.

Let now fix $(x_0,t_0) \in \partial\{v>0\}$ with $t_0 \in (1,2)$. Denote $e = x_0/|x_0|$ and let e_n be the direction of $\nabla v(x_0,t_0)$. Since e is the axis of the cone of monotonicity $\mathcal{C}$ with opening $\pi/2 - \eta_0$, we have angle$(e,-e_n) \le \eta_0$ and therefore if λ is sufficiently large (implying that η_0 is small), we will have

$$\partial_{e_n} v(x,t) \ge c_0$$

in $B_{\delta_0}(x_0)$ (with possibly different c_0, δ_0 then before.) Consider now the mapping $(x,t) \mapsto (y,t) = (x', v(x,t), t)$ defined in a small neighborhood

$$V_0 = B_{\delta_0}(x_0) \times (t_0 - \delta_0, t_0 + \delta_0) \cap \overline{\{v>0\}}$$

into a subset W_0 of $\{(y,t) : y_n > 0\}$ which is open in the relative topology of the halfspace and contains the point $(0,t_0)$. The Jacobian of this mapping is $\partial_{e_n} v \ge c_0 > 0$ and hence by the implicit function theorem there exist an inverse mapping $(y,t) \mapsto (x,t) = (y', w(y,t), t)$, where the functions w and v are related trough the identity

$$x_n = w(x', v(x,t), t). \tag{4.10}$$

Differentiating (4.10) we find

$$v_{x_n} = \frac{1}{w_{y_n}}, \quad v_{x_i} = -\frac{w_{y_i}}{w_{y_n}},\ i = 1,2,\ldots,n-1, \quad v_t = -\frac{w_t}{w_{y_n}}. \tag{4.11}$$

and using the differentiation rules

$$\partial_{x_n} = \frac{1}{w_{y_n}}\,\partial_{y_n}, \quad \partial_{x_i} = \partial_{y_i} - \frac{w_{y_i}}{w_{y_n}}\,\partial_{y_n} \tag{4.12}$$

one can deduce the equation for w from the equation (1.4) for v:

$$
\begin{aligned}
-\frac{w_t}{w_{y_n}} = \frac{p-1}{p-2}\, y_n \Bigg[\frac{1}{w_{y_n}} \left(a^{p-2} \frac{1}{w_{y_n}} \right)_{y_n} - \left(a^{p-2} \frac{w_{y_i}}{w_{y_n}} \right)_{y_i} + \\
+ \frac{w_{y_i}}{w_{y_n}} \left(a^{p-2} \frac{w_{y_i}}{w_{y_n}} \right)_{y_i} \Bigg] + a^p, \qquad (4.13)
\end{aligned}
$$

where

$$
a = a\left(\nabla_y w\right) = |\nabla_x v| = \frac{\sqrt{1+|\nabla_{y'} w|^2}}{w_{y_n}}. \qquad (4.14)
$$

After the simplification, the equation above can be rewritten in the form

$$
w_t = c_p\, y_n \left(a^{p-2} w_{y_i}\right)_{y_i} - c_p\, y_n^{-\sigma} \left(y_n^{1+\sigma} a^{p-2} \frac{1+|\nabla_{y'} w|^2}{w_{y_n}} \right)_{y_n}, \qquad (4.15)
$$

where

$$
c_p = \frac{p-1}{p-2} \quad \text{and} \quad \sigma = -\frac{1}{p-1} > -1. \qquad (4.16)
$$

To the equations of type (4.15) one can apply the regularity theory of Koch [K]. What follows is mainly a modification of the proof of Theorem 5.6.1 in [K]. We show first that the derivatives w_{y_i} are C^α for $i = 1, \dots, n-1$ and then we prove C^α-regularity of w_{y_n}.

Let g be a difference quotient of w in a direction tangential to the boundary. Then it satisfies

$$
g_t = y_n (\mathcal{A}^{ij}\, g_{y_j})_{y_i} + y_n^{-\sigma} \left(y_n^{1+\sigma} \mathcal{A}^{nj} g_{y_j} \right)_{y_n} \qquad (4.17)
$$

where $\mathcal{A}^{kj}$ is uniformly elliptic. Then by Theorem 4.5.5 in [K] g are uniformly C^α hence so are the derivatives w_{y_i}, $i = 1, \dots, n-1$.

Next, the derivative $g = w_{y_n}$ satisfies an equation

$$
g_t = y_n (\mathcal{B}^{ij}\, g_{y_j})_{y_i} + y_n^{-1-\sigma} \left(y_n^{2+\sigma} \mathcal{B}^{nj} g_{y_j} \right)_{y_n} + c_p (a^{p-2} w_{y_i})_{y_i}. \qquad (4.18)
$$

Applying now Theorem 4.5.6 from [K] we find constants C, $c > 0$ such that on a cube $Q_h = Q'_h(0) \times [0, 2h] \times [t^0 - h, t^0 + h]$ with closure contained in W_0 we have

$$
||g||_{C^\alpha(Q_h)} \le C + c \sum_{i=1}^{n-1} ||a^{p-2}\, w_{y_i}||_{C^\alpha(Q_h)}, \qquad (4.19)
$$

where

$$
a = a\left(\nabla_y w\right) = \frac{\sqrt{1+|\nabla_{y'} w|^2}}{w_{y_n}}. \qquad (4.20)
$$

We can rewrite

$$
a^{p-2} w_{y_i} = f^i(\nabla_{y'} w) w_{y_n}^{2-p}
$$

where $f^i(\nabla_{y'} w) = \left(1 + |\nabla_{y'} w|^2\right)^{\frac{p-2}{2}} w_{y_i}$ will be C^α and moreover

$$f^i(\nabla_{y'} w)\big|_{(0,t^0)} = 0, \qquad i = 1, \ldots, n-1. \tag{4.21}$$

Next, we can estimate

$$\begin{aligned} ||a^{p-2}\, w_{y_i}||_{C^\alpha(Q_h)} &= ||f^i(\nabla_{y'} w) w_{y_n}^{2-p}||_{C^\alpha(Q_h)} \\ &\le C_1 ||w_{y_n}||_{C^\alpha(Q_h)} ||f^i||_{L^\infty(Q_h)} + C_2 ||f^i||_{C^\alpha(Q_h)}. \end{aligned} \tag{4.22}$$

If we now take h sufficiently small, so that

$$||f^i||_{L^\infty(Q_h)} < \varepsilon_0$$

(which is possible by (4.21)) we will obtain

$$||a^{p-2}\, w_{y_i}||_{C^\alpha(Q_h)} \le C_1 \varepsilon_0 ||w_{y_n}||_{C^\alpha(Q_h)} + C_3 \tag{4.23}$$

Substituting this estimate into (4.19) we obtain

$$||w_{y_n}||_{C^\alpha(Q_h)} \le C_4 + C_5 \varepsilon_0 ||w_{y_n}||_{C^\alpha(Q_h)} \tag{4.24}$$

and taking h small enough so that $C_5 \varepsilon_0 < 1/2$ we find

$$||w_{y_n}||_{C^\alpha(Q_h)} \le C_6. \tag{4.25}$$

which proves that w and therefore v is $C^{1,\alpha}$.

4.3 C^∞-regularity of the pressure

To prove the C^∞-regularity of v we should basically iterate the argument for the $C^{1,\alpha}$-regularity. That, is we are taking successive derivatives of the equation (4.15) first in the directions e_i, $i = 1, 2, \ldots, n-1$ and then in e_n. The new terms that we will be obtaining in the equation will be of the form $f + \sum \partial_j (y_n f^j)$ with f and f^j already known to be C^α. For more details we refer to Koch's paper [K], proof of Theorem 5.6.1. The result that can be proved is as follows

Proposition 4.1 *There exist a uniform neighborhood U of $(0, t_0)$ in $\mathbf{R}^{n-1} \times [0, \infty) \times \mathbf{R}$ such that $w_\lambda \in C^\infty(U)$ and $||w_\lambda||_{C^\ell_{x,t}(U)} < C_\ell$ for $\ell > 0$ and $\lambda > \lambda_0$.*

We also find our $C^{1,\alpha}$-estimate is enough to use the Schauder-type estimates in Daskalopoulos-Hamilton [8] for higher regularity. In [8], they assumed weighted $C^{2,\alpha}_\delta$ regularity of the initial data to get a degenerate equation with Hölder coefficient in a fixed domain after a global change of coordinates. On the other hand those assumptions are not necessary in our case since we just make a local argument. In the other words, $C^{1,\alpha}$-regularity of v gives us the same type degenerate equation (4.17) with Hölder coefficient.

5 Convergence to the Barenblatt solution

In this section we prove Theorems 3.1 and 3.2.

The C^∞ estimate in Proposition 4.1, after the inverse change of variables, implies that the uniform convergence of $v_\lambda(x,t)$ to the selfsimilar $V(x,t) = t^{-k\frac{p-2}{p-1}}G(xt^{-k/N})$ for $t \in [1,2]$ as $\lambda \to \infty$ is in fact C^ℓ convergence for every $\ell > 0$ in an δ_0-neighborhood U of the interface $\partial\{G(xt^{-k/N}) > 0\}$, $t \in [1,2]$, in the sense that for large λ there exists a C^∞ function $h_\lambda(x,t)$ such that

$$v_\lambda(x,t) = t^{-k\frac{p-2}{p-1}}G\left(xt^{-k/N} + h_\lambda(x,t)x/|x|\right) \tag{5.1}$$

in U and

$$||h_\lambda(x,t)||_{C^\ell_{x,t}(U)} \to 0 \tag{5.2}$$

for every $\ell > 0$. In particular, since

$$\partial_e\left(|\nabla G|^{p-2}\partial_e G\right) = -q^{p-1}, \qquad q = q(N,p) > 0 \tag{5.3}$$

we will have that for large λ

$$\partial_e\left(|\nabla v_\lambda|^{p-2}\partial_e v_\lambda\right) < 0 \tag{5.4}$$

in $\Omega_\lambda(t) = \{v_\lambda(\cdot,t) > 0\}$ in δ-neighborhood of the interface, $t \in [1,2]$. However, we claim that given $\varepsilon > 0$ (5.4) holds true in $\Omega_\lambda(t) \setminus B_\varepsilon$ for λ sufficiently large. Indeed, if $\operatorname{dist}(x, \partial\Omega_\lambda(t)) \geq \delta$, we will have $v_\lambda(x,t) \geq \eta > 0$ and the uniform $C^{1,\alpha}$ regularity of the density $u_\lambda(\cdot,t)$ (see [9]) will imply the uniform $C^{1,\alpha}$ regularity of $v_\lambda(\cdot,t)$ in $\{v_\lambda(\cdot,t) \geq \eta\}$. In particular, $v_\lambda(\cdot,t)$ will converge to $V(\cdot,t)$ in $C^{1,\beta}$ norm. But, the gradient $|\nabla V(x,t)| > 0$ for $|x| > 0$, hence the equation (1.4) for v_λ is uniformly parabolic on $\{(x,t) : v_\lambda(x,t) \geq \eta, 1 \leq t \leq 2\} \setminus B_\varepsilon \times [1,2]$ and therefore

$$||v_\lambda(\cdot,t) - V(\cdot,t)||_{C^\ell(\{v_\lambda \geq \eta\}\setminus B_\varepsilon)} \to 0 \tag{5.5}$$

as $\lambda \to 0$ for every $\ell > 0$. As a consequence, we obtain that (5.4) holds in $\Omega(t) \setminus B_\varepsilon$ for λ sufficiently large.

Proof of Theorem 3.1. The proof follows from (5.1)–(5.2) and (5.5). $\square$

Proof of Theorem 3.2. The proof follows from (5.1)–(5.2), (5.3), (5.4) in $\Omega(t)\setminus B_\varepsilon$, and the Lemma 5.1 below by rescaling v_λ back to v. $\square$

Lemma 5.1 *Let $w(x)$ be a C^2 function in an open set U of $\mathbf{R}^n$ such that*

$$Z_e := \partial_e(|\nabla w|^{p-2}\partial_e w) < 0$$

for any spatial direction e. Then $w(x)$ is locally concave in U.

Proof. For $x_0 \in U$ define

$$Z_e = |\nabla w|^{p-2} w_{ee} + (p-2)|\nabla w|^{p-4}(\nabla w \cdot \nabla w_e) w_e$$

Choose now the spatial coordinate system so that the matrix $D^2 w(x_0)$ is diagonal and let e be directed along one of the coordinate axes. Then

$$\begin{aligned} Z_e &= |\nabla w|^{p-2} w_{ee} + (p-2)|\nabla w|^{p-4} w_e^2 w_{ee} \\ &= (|\nabla w|^{p-2} + (p-2)|\nabla w|^{p-4} w_e^2) w_{ee}. \end{aligned}$$

Now, since $|\nabla w|^{p-2} + (p-2)|\nabla w|^{p-4} w_e^2 \geq (p-1)|\nabla w|^{p-4} w_e^2$ is always nonnegative, $Z_e < 0$ implies $w_{ee} < 0$. This proves that the eigenvalues of $D^2 w(x_0)$ are nonpositive and the lemma follows. □

6 Convexity in fast diffusion, $1 < p < 2$, $N = 1$

In this section we work in dimension 1 and for $p \in (1,2)$ and we call it the fast diffusion in analogy with the porous medium equation with $m \in (0,1)$. In contrast to the case $p > 2$ the equation does not have the finite propagation property and the density becomes positive everywhere for $t > 0$.

In this case there is a problem with the definition (1.3) of the pressure v, since it becomes negative. We prefer therefore redefine it as

$$v = \frac{p-1}{2-p} u^{-\frac{2-p}{p-1}}.$$

Now it is positive and in dimension satisfies

$$v_t = c_p v\,(|v_x|^{p-2} v_x)_x - |v_x|^p, \qquad c_p = \frac{2-p}{p-1} \tag{6.1}$$

Next, what we know is that v is $C^{1,\alpha}$ and that it is close to the Barenblatt profile after we pass to the rescaled solutions v_λ. The convergence is uniform away from $x = 0$ so we assume that v is close in C^2 (hence, convex) in any compact set except a small neighborhood of 0.

To prove the convexity of v in a small neighborhood of the origin, it is enough to prove that $Z = (|v_x|^{p-2} v_x)_x > 0$, as one can see from an obvious generalization of Lemma 5.1. As a starting point we mention the following estimate by Esteban and Vazquez [13]

$$-\frac{K_1}{t} \leq Z \leq \frac{K_2}{t} \tag{6.2}$$

for some positive constants K_1 and K_2 depending only on p.

Introduce an auxiliary function $U = |v_x|^{p-2} v_x$ so that we have $Z = U_x$. We are going to derive equations for U and Z, but the problem is that these quantities are not generally smooth, so we have to use a regularization. It can be done as in [13], or as we do below.

For a given $\varepsilon > 0$ consider the solutions v^ε of the approximating equation

$$v_t = c_p v\,(f^\varepsilon(v_x))_x - g^\varepsilon(v_x), \tag{6.3}$$

where

$$f^\varepsilon(s) = (s^2+\varepsilon)^{\frac{p-2}{2}}\,s \tag{6.4}$$

$$g^\varepsilon(s) = (s^2+\varepsilon)^{\frac{p-2}{2}}\left(s^2 + (2-1/(p-1))\,\varepsilon\right) \tag{6.5}$$

Since the equation (6.3) is locally uniformly parabolic, the solutions v^ε are C^∞ and taking ε small enough we can assume that v^ε are sufficiently close to the pressure v in $C^{1,\alpha}$ norm on compact subsets of Q. Next, we introduce

$$U^\varepsilon = f^\varepsilon(v_x^\varepsilon), \qquad Z^\varepsilon = U_x^\varepsilon. \tag{6.6}$$

Differentiating (6.3) with respect to x and multiplying by $(f^\varepsilon)'(v^\varepsilon)$ we find the equation for U^ε

$$\begin{aligned} U_t^\varepsilon &= c_p v^\varepsilon (f^\varepsilon)'(v_x^\varepsilon)\,U_{xx}^\varepsilon + [c_p v_x^\varepsilon (f^\varepsilon)'(v_x^\varepsilon) - (g^\varepsilon)'(v_x^\varepsilon)]\;U_x^\varepsilon \\ &= a^\varepsilon(x,t)\,U_{xx}^\varepsilon + b^\varepsilon(x,t)\,U_x^\varepsilon \end{aligned} \tag{6.7}$$

$$a^\varepsilon(x,t) = c_p v(v_x^2+\varepsilon)^{\frac{p-4}{2}}((p-1)v_x^2+\varepsilon) \tag{6.8}$$

$$b^\varepsilon(x,t) = -2(p-1)v_x(v_x^2+\varepsilon)^{\frac{p-2}{2}}. \tag{6.9}$$

Differentiating now (6.7), we obtain the equation for Z^ε

$$Z_t^\varepsilon = a^\varepsilon\, Z_{xx}^\varepsilon + \tilde{b}^\varepsilon\, Z_x^\varepsilon - 2(p-1)(Z^\varepsilon)^2 \tag{6.10}$$

where a^ε as above, $\tilde{b}^\varepsilon = 2c_p v_x (f^\varepsilon)'(v_x) + c_p v v_{xx}(f^\varepsilon)''(v_x) - (g^\varepsilon)'(v_x)$. In computation we used the following identity

$$c_p(f^\varepsilon)''(s) - (g^\varepsilon)''(s) = C_p(f^\varepsilon)'(s), \quad C_p = 2-2p-c_p$$

Consider now Z^ε in a rectangle $\mathcal{R} = (-r,r)\times(1,2)$ and assume that $v = v_\lambda$ is the rescaled pressure. From the C^∞ convergence of v_λ to the Barenblatt solution on every compact K separated from 0, we have that $Z_\lambda \geq 2\delta_0 > 0$ on $\{-r,r\}\times[1,2]$ for large λ. But then, taking $\varepsilon < \varepsilon(\lambda)$, we can make $Z_\lambda^\varepsilon \geq \delta_0$ on $\{-r,r\}\times[1,2]$. For simplicity we will omit the indices ε and λ in what follows, if there is no ambiguity. Also, if it is not stated otherwise, the constants that appear below are uniform in ε and λ.

Lemma 6.1 *Suppose that $Z \geq \delta_0 > 0$ on the parabolic boundary of a rectangle $\mathcal{R} = (-r,r)\times(t_1,t_2)$, i.e. on $[-r,r]\times\{t_1\}\cup\{-r,r\}\times[t_1,t_2]$. Then $Z^\varepsilon \geq \delta_1$ in $\mathcal{R}$, where $\delta_1 > 0$ depends only on δ_0, t_1 and t_2.*

Proof. The proof is pretty much standard and uses the comparison with the stationary solutions of (6.10), that is functions $\zeta(t)$ satisfying

$$\zeta' = -2(p-1)\zeta^2.$$

Solutions of this ODE have the form

$$\zeta(t) = \frac{c}{t+t_0}, \qquad c = \frac{1}{2(p-1)} \tag{6.11}$$

and we can choose t_0 very large, so that $\zeta(t) < \delta_0/2$ on $[t_1, t_2]$. Then we claim $Z(x,t) > \zeta(t)$ in $\mathcal{R}$. Indeed, assuming the contrary, let t^* be the minimal $t \in [t_1, t_2]$ such that $Z(x,t) = \zeta(t)$ for some $x \in [-r, r]$. It is clear that $t^* > t_1$ since $Z(x, t_1) \geq \delta_0 > \zeta(t_1)$. Next, let $x^* \in [-r, r]$ be such that $Z(x^*, t^*) = \zeta(t^*)$. Then x^* is an interior point, since for $Z(x,t) \geq \delta_0 > \zeta(t)$ on the lateral boundary $\{-r, r\} \times [t_1, t_2]$. It is easily follows now that

$$Z_x(x^*, t^*) \geq 0, \quad Z_x(x^*, t^*) = 0, \quad Z_t(x^*, t^*) \leq \zeta'(t^*)$$

Here, we actually need to modify $\zeta(t)$ a little bit if we wish to arrive at a contradiction. Let everywhere above $\zeta(t)$ be given by (6.11) but with $c < \frac{1}{2(p-1)}$, so that we have

$$\zeta'(t) = -\frac{1}{c}\zeta^2(t) < -2(p-1)\zeta^2(t).$$

But then the contradiction is immediate:

$$-2(p-1)\zeta^2(t^*) > \zeta'(t^*) \geq Z_t(x^*, t^*) \geq -2(p-1)Z^2(x^*, t^*),$$

where in the last inequality we used the equation (6.10) for Z. Hence $Z(x,t) > \zeta(t)$ in $\mathcal{R}$ and the lemma follows. $\square$

We are thus left with the proof of strict p-convexity at some time. We make a second-order estimate for U, namely an estimate for

$$I = \iint Z_x^2 \, dxdt = \iint U_{xx}^2 \, dxdt.$$

We multiply the equation (6.7) by U_{xx} and integrate by parts in a rectangle $\mathcal{R} = (-r, r) \times (\frac{1}{,}2)$ with $r > 0$ small to get

$$\iint a\, U_{xx}^2 \, dxdt = \iint U_t \, U_{xx} \, dxdt - \iint b\, U_x U_{xx} \, dxdt = I_1 + I_2. \tag{6.12}$$

Since b is small, $b = O(r)$, we have

$$|I_2| \leq Cr \left(\iint U_x^2 \, dxdt \right)^{1/2} \left(\iint U_{xx}^2 \, dxdt \right)^{1/2} \leq Cr\, I^{1/2}, \tag{6.13}$$

since $U_x = Z$ is bounded. We estimate the other term as follows

$$\begin{aligned} I_1 = &-\iint U_x U_{xt} \, dxdt + \int_S U_t \, U_x \, dt = \\ &\frac{1}{2}\int U_x(x,1)\, dx - \frac{1}{2}\int U_x(x,2)\, dx + \int_S U_t \, U_x \, dt. \end{aligned} \tag{6.14}$$

Now, the first terms are bounded uniformly as $O(r)$ and the last is very small when $\lambda \gg 1$ because of the uniform convergence away from $x = 0$ of the rescaled solutions. Summing up, we get

$$Cr^{-(2-p)/(p-1)} \iint U_{xx}^2 \, dxdt \le \iint a\, U_{xx}^2 \, dxdt \le C + Cr\, I^{1/2} \tag{6.15}$$

which means that I is bounded and small. But as an iterated integral it means that for some $t = t_1 \in (\frac{1}{,}2)$ the integral $\int Z_x^2 \, dx$ is small. At that t we obtain

$$|Z - K_1/t| \le \varepsilon + \int_{-r}^{x} |Z_x| \, dx \le \varepsilon + r^{1/2} \left(\int Z_x^2 \, dx \right)^{1/2} \le 2\varepsilon, \tag{6.16}$$

hence $Z \ge c_0 > 0$. Observe that we may assume $t_1 \in (1, \frac{3}{2})$. But then, by Lemma 6.1 we will have that $Z \ge c_0 > 0$ on $[-r, r] \times [\frac{3}{2}, 2]$. In particular, we obtain that $v_\lambda^\varepsilon \left(\cdot, \frac{3}{2}\right)$ is convex in $\mathbf{R}$ for λ very large and $0 < \varepsilon < \varepsilon(\lambda)$, and therefore v_λ is convex everywhere in $\mathbf{R}$. But then, taking $\lambda = \frac{2}{3}t$ this precisely means $v(\cdot, t)$ is convex in $\mathbf{R}$ for large t.

7 Concavity near the origin for $p > 2$, $N = 1$

We now perform the concavity analysis in the dimension $N = 1$ for the slow diffusion case, $p > 2$, and prove the first part of Theorem 3.2′ that the rescaled solutions v_λ are concave near the origin for $\lambda \gg 1$.

As before, concavity of v will follow if we prove that the quantity $Z = (|v_x|^{p-2} v_x)_x$ is nonpositive. In this case we only have a bound from below for Z by Esteban and Vazquez [13].

The proof is similar to the convexity proof in the case of fast diffusion from the previous section. We consider an auxiliary variable $U = |v_x|^{p-2} v_x$, so that $U_x = Z$. All computations below are formal, but can be justified precisely as we did for the fast diffusion by considering regularizations v^ε, U^ε, and Z^ε.

From the pressure equation

$$v_t = c_p\, v\, (|v_x|^{p-2} v_x)_x + |v_x|^p, \qquad c_p = \frac{p-2}{p-1} \tag{7.1}$$

we obtain that U satisfies

$$\begin{aligned} U_t &= (p-2)\, v\, |v_x|^{p-2} U_{xx} + (2p-2)|v_x|^{p-2} v_x U_x \\ &= a(x,t)\, U_{xx} + b(x,t)\, U_x. \end{aligned} \tag{7.2}$$

Lemma 7.1 *There is a second-order estimate for U of the form*

$$I = \iint a\, Z_x^2 \, dxdt = \iint a\, U_{xx}^2 \, dxdt \le C. \tag{7.3}$$

Proof. We multiply by U_{xx} and integrate by parts in a rectangle $\mathcal{R} = (-r,r) \times (1,2)$ with $r > 0$ small to get

$$\iint a\,U_{xx}^2\,dxdt = \iint U_t\,U_{xx}\,dxdt - \iint b\,U_x U_{xx}\,dxdt = I_1 + I_2. \tag{7.4}$$

First we estimate I_2

$$|I_2| \le 2\iint \frac{b^2}{a} U_x^2\,dxdt + \frac{1}{2}\iint aU_{xx}^2\,dxdt. \tag{7.5}$$

The quantity b^2/a above equals $C(p)|v_x|^p$, so that $b^2/a = O(r^{\frac{p}{p-1}})$. Also we know that U_x is L^2 integrable, see Proposition 3.1, Chap. VIII in [9], which implies that

$$|I_2| \le Cr^{\frac{p}{p-1}} + \frac{1}{2}I \tag{7.6}$$

Next, to estimate I_1 we integrate by parts.

$$\begin{aligned} I_1 = -\iint U_x U_{xt}\,dxdt + \int_S U_t\,U_x\,dt = \\ \frac{1}{2}\int U_x(x,1)\,dx - \frac{1}{2}\int U_x(x,2)\,dx + \int_S U_t\,U_x\,dt. \end{aligned} \tag{7.7}$$

Again, since U_x is spatially L^2 integrable (see the reference above) the first two integrals are bounded. The last integral will be bounded since U_tU_x converges uniformly to the corresponding quantity for the Barenblatt solution on S. Hence we obtain that

$$|I_1| \le C \tag{7.8}$$

with C independent of r. Combining the estimates above we obtain that

$$\iint a\,Z_x^2\,dxdt \le C. \tag{7.9}$$

Lemma is proved. □

Proof of Theorem 3.2′. We should start with a remark that as everywhere else in this section we must work with approximations of U and Z (as well as of a and b) as in the previous section, but for simplicity of the presentation we do formal computations with U and Z. From Theorem 3.2 it follows that for a given $r > 0$ and $\varepsilon > 0$ and large λ we have

$$|Z(x,t) + K/t| < \varepsilon \qquad \text{for } x \in \operatorname{supp} v(\cdot,t) \setminus (-r,r),\ t \in [1,2],$$

where $K = K(p) > 0$. From Lemma 7.1 above it follows that for some $t = t_1 \in (1,2)$ the integral $\int aZ_x^2\,dx$ bounded. At that t we get

$$\begin{aligned} |Z(x,t) + K/t| &\le \varepsilon + \int_{-r}^{x} |Z_x|\,dx \\ &\le \varepsilon + \left(\int_{-r}^{x} \frac{1}{a}\,dx\right)^{1/2} \left(\int a\,Z_x^2\,dx\right)^{1/2} \end{aligned} \tag{7.10}$$

and the statement will follow once we show that

$$\int_{-r}^{x} \frac{1}{a}\,dx = \int_{r}^{x} |v_x|^{2-p}\,dx \to 0 \qquad \text{as } r \to 0 \tag{7.11}$$

This seems to work since $a = |v_x|^{2-p} \simeq |x|^{-(p-2)(p-1)}$ which suggests also that the above quantity should be actually $O(r^{1/(p-1)})$. We need to make this precise. We start from a small distance $x = -r$ where the difference is less than ε small enough and we integrate in the interval $[-r, x']$ where $x' \in (-r, r)$ is the first point at which $Z = -K/(2t)$ for instance. Then $U_x = Z$ will be bounded away from zero and that implies that even if U vanishes at a point $x_0 \in [-r, x']$ (in the worst case) we still have

$$|U(x,t)| \geq C|x - x_0| \quad \text{ in } I = [-r, x'].$$

and since $a = (p-2)v|U|^{(p-2)/(p-1)} < c(p)|x-x_0|^{(p-2)/(p-1)}$, the above formula (7.11) holds at $x = x'$ and leads to contradiction in the preceding estimate for Z. Indeed, we will have

$$K/(2t) = |Z(x',t) + K/t| \leq \varepsilon + Cr^{1/(p-1)}$$

with C depending on p only, which is impossible if r and ε are sufficiently small. Therefeore, Z never reaches the level $-K/2t$, and in fact stays near $-K/t$, for this particular $t = t_1 \in (1, 2)$. Observe however, that we may assume $t_1 \in (1, \frac{3}{2})$ and then apply an analogue of Lemma 6.1, which says that if $Z \leq -\delta_0 <$ on a parabolic boundary of $(-r, r) \times (t_1, t_2)$ then if fact $Z \leq -\delta_1 < 0$ everywhere in $[-r, r] \times [t_1, t_2]$. In our case we obtain $Z \leq -c_0 < 0$ in $[-r, r] \times [\frac{3}{2}, 2]$ and in particular that $v_\lambda\left(\cdot, \frac{3}{2}\right)$ is strictly concave in its positivity set. But then taking $\lambda = \frac{2}{3}t$ we find that $v(\cdot, t)$ is stricly concave in $\Omega(t) = \{v(\cdot, t) > 0\}$.

The second part of Theorem 3.2′ on the regularity of the curve of maxima is the contents of the next section, where we finish the proof of the theorem.

8 Regularity of the curve of maxima

As we have seen in the previous section, in 1-dimension, starting from some moment, the pressure $v(\cdot, t)$ will become strictly concave in its positivity set $\Omega(t)$. As a consequence, the function $v(\cdot, t)$ has only one maximum point. We will denote this point by $\gamma(t)$. Below we show that the result of M. Bertsch and D. Hilhorst [3] on the regularity of the interface in one-dimensional two-phase porous medium equation implies that the curve $x = \gamma(t)$ is $C^{1,\alpha}$ regular. The connection with the porous medium equation is as follows. It is clear that $\gamma(t)$ is also the only maximum point of $u(\cdot, t)$. Moreover, $\gamma(t)$ is the only point, where the derivative u_x crosses the value 0. In other words, the curve $x = \gamma(t)$ separates the regions $\{u_x < 0\}$ and $\{u_x > 0\}$. Finally, the function

$$w(x,t) = u_x(x,t) \tag{8.1}$$

satisfies

$$w_t = (|w|^{p-2}w)_{xx}, \tag{8.2}$$

which is precisely the two-phase the porous medium equation with the parameter $m = p - 1$.

Proposition 8.1 *Let w be a solution of* (8.2) *on $(-L, L) \times (t_0, \infty)$ with the assumptions that $w(\cdot, t_0)$ is nonincreasing on $(-L, L)$ and $w(-L, t) = a$, $w(L, t) = -b$ for $t \geq t_0$ for some positive constants a and b. Then the null-set $\mathcal{N}(t) = \{x : w(x, t) = 0\}$ can be described as follows. There exist Lipschitz functions $\gamma_-(t)$ and $\gamma_+(t)$ such that*

$$\mathcal{N}(t) = [\gamma_-(t), \gamma_+(t)], \quad \textit{for } t \geq t_0.$$

and there is $t^* \geq t_0$ such that

(i) $\gamma_-(t) = \gamma_+(t) =: \gamma(t)$ for $t \geq t^*$;

(ii) $(|w|^{p-2}w)_x = 0$ on $\mathcal{N}(t)$ for $t \in [t_0, t^*]$ and $(|w|^{p-2}w)_x < 0$ for $t > t^*$.

Moreover, $\gamma \in C^{1,\alpha}((t^*, \infty))$ for some $\alpha \in (0, 1)$.

Proof. This is a particular case of Theorem 1.3 (see also Lemma 4.1) in [3]. □

Proof of Theorem 3.2′ (continuation.) In order to use Proposition 8.1 for $w = u_x$ we must prove that $w(\cdot, t_0)$ is nonincreasing on $(-L, L)$ for small L and large t_0. We actually consider the rescaled solutions $u_\lambda(x, t)$ for on $\mathcal{R} = (-r, r) \times (1, 2)$ and respectively defined $w_\lambda = (u_\lambda)_x$. Then (omitting λ)

$$(|w|^{p-2}w)_x = C_p \left(v^{\frac{p-1}{p-2}} |v_x|^{p-2} v_x \right)_x = C_p v^{\frac{1}{p-2}} \left(c_p |v_x|^p + v(|v_x|^{p-2} v_x)_x \right)$$

and therefore for small r and large λ we have

$$(|w_\lambda|^{p-2} w_\lambda)_x < 0 \quad \text{on } \mathcal{R} = (-r, r) \times (1, 2).$$

Indeed, this simply follows from the fact that for large λ we have $(|v_x|^{p-2}v_x)_x < -C(p) < 0$ and for small $r > 0$ v_x is small and v is like a positive constant. As a consequence, we obtain also that $w_\lambda(\cdot, 1)$ is nonincreasing on $(-r, r)$. Also for $t \in [1, 2]$ $w_\lambda(-r, t) > 0$ and $w_\lambda(r, t) < 0$. Even though $w_\lambda(-r, t)$ and $w_\lambda(r, t)$ are not constants, (but separated) from 0, the conclusion of Proposition 8.1 above holds, since this condition is not essential for the proof. Moreover we can take $t^* = 1$, since we proved that $(|w_\lambda|^{p-2}w_\lambda)_x < 0$ in $\mathcal{R}$. Scaling w_λ back to w we obtain that the curve $x = \gamma(t)$ is $C^{1,\alpha}$ regular, where $\gamma(t)$ is the only maximum point of the pressure v at time t, for $t \geq t_0$. This finishes the proof of Theorem 3.2′. □

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