SOME NEW MONOTONICITY FORMULAS AND THE SINGULAR SET IN THE LOWER DIMENSIONAL OBSTACLE PROBLEM

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Abstract. We construct two new one-parameter families of monotonicity formulas to study the free boundary points in the lower dimensional obstacle problem. The first one is a family of Weiss type formulas geared for points of any given homogeneity and the second one is a family of Monneau type formulas suited for the study of singular points. We show the uniqueness and continuous dependence of the blowups at singular points of given homogeneity. This allows to prove a structural theorem for the singular set.

Our approach works both for zero and smooth non-zero lower dimensional obstacles. The study in the latter case is based on a generalization of Almgren's frequency formula, first established by Caffarelli, Salsa, and Silvestre.

Introduction

The lower dimensional obstacle problem. Let Ω be a domain in $\mathbb{R}^n$ and $\mathcal{M}$ a smooth $(n-1)$-dimensional manifold in $\mathbb{R}^n$ that divides Ω into two parts: Ω_+ and Ω_-. For given functions $\varphi : \mathcal{M} \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$ satisfying $g > \varphi$ on $\mathcal{M} \cap \partial\Omega$, consider the problem of minimizing the Dirichlet integral

$$D_\Omega(u) = \int_\Omega |\nabla u|^2 dx$$

on the closed convex set

$$\mathfrak{K} = \{u \in W^{1,2}(\Omega) \mid u = g \text{ on } \partial\Omega,\, u \geq \varphi \text{ on } \mathcal{M} \cap \Omega\}.$$

This problem is known as the lower dimensional, or *thin obstacle problem*, with φ known as the *thin obstacle*. It is akin to the *classical obstacle problem*, where u is constrained to stay above an obstacle φ which is assigned in the whole domain Ω. However, whereas the latter is by now well-understood, the thin obstacle problem still presents considerable challenges and only recently there has been some significant progress on it. While we defer a discussion of the present status of the problem and of some important open questions to the last section of this paper, for an introduction to this and related problems we refer the reader to the book by Friedman [Fri82, Chapter 1, Section 11] as well as to the survey of Ural'tseva [Ura87].

The thin obstacle problem arises in a variety of situations of interest for the applied sciences. It presents itself in elasticity (see for instance [KO88]), when an

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elastic body is at rest, partially laying on a surface $\mathcal{M}$. It also arises in financial mathematics in situations in which the random variation of an underlying asset changes discontinuously, see [CT04], [Sil07] and the references therein. It models the flow of a saline concentration through a semipermeable membrane when the flow occurs in a preferred direction (see [DL72]).

When $\mathcal{M}$ and φ are smooth, it has been proved by Caffarelli [Caf79] that the minimizer u in the thin obstacle problem is of class $C^{1,\alpha}_{\mathrm{loc}}(\Omega_\pm \cup \mathcal{M})$. Since we can make free perturbations away from $\mathcal{M}$, it is easy to see that u satisfies

$$\Delta u = 0 \quad \text{in } \Omega \setminus \mathcal{M} = \Omega_+ \cup \Omega_-,$$

but in general u does not need to be harmonic across $\mathcal{M}$. Instead, on $\mathcal{M}$, one has the following complementary conditions

$$u - \varphi \geq 0, \quad \partial_{\nu^+} u + \partial_{\nu^-} u \geq 0, \quad (u-\varphi)(\partial_{\nu^+} u + \partial_{\nu^-} u) = 0,$$

where $\nu^\pm$ are the outer unit normals to $\Omega_\pm$ on $\mathcal{M}$. One of the main objects of study in this problem is the so-called *coincidence set*

$$\Lambda(u) := \{x \in \mathcal{M} \mid u(x) = \varphi(x)\}$$

and its boundary (in the relative topology on $\mathcal{M}$)

$$\Gamma(u) := \partial_{\mathcal{M}} \Lambda(u),$$

known as the *free boundary.*

A similar problem is obtained when $\mathcal{M}$ is a part of $\partial\Omega$ and one minimizes $D_\Omega(u)$ over the convex set

$$\mathfrak{K} = \{u \in W^{1,2}(\Omega) \mid u = g \text{ on } \partial\Omega \setminus \mathcal{M},\, u \geq \varphi \text{ on } \mathcal{M}\}.$$

In this case u is harmonic in Ω and satisfies the complementary conditions

$$u - \varphi \geq 0, \quad \partial_\nu u \geq 0, \quad (u-\varphi)\partial_\nu u = 0$$

on $\mathcal{M}$, where ν is the outer unit normal on $\partial\Omega$. This problem is known as the *boundary thin obstacle problem* or the *Signorini problem.* Note that in the case when $\mathcal{M}$ is a plane and Ω and g are symmetric with respect to $\mathcal{M}$, then the thin obstacle problem in Ω is equivalent to the boundary obstacle problem in Ω_+.

Recent developments. There has been a recent surge of activity in the area of thin obstacle problems since the work of Athanasopoulos and Caffarelli [AC04], where the optimal $C^{1,\frac12}$ interior regularity has been established for the solutions of the Signorini problem with flat $\mathcal{M}$ and $\varphi = 0$. A different perspective was brought in with the paper [ACS07], where extensive use was made of the celebrated monotonicity of Almgren's *frequency function*,

$$N(r,u) := \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2},$$

see [Alm79], and also [GL86], [GL87] for generalizations to variable coefficient elliptic operators in divergence form. The name frequency comes from the fact that when u is a harmonic function in B_1 homogeneous of degree κ, then $N(r,u) \equiv \kappa$. Using such monotonicity of the frequency the authors were able to show fine regularity properties of the free boundary; namely, that the set of so-called regular free boundary points is locally a C^1-manifold of dimension $n-2$. When the obstacle is the function $\varphi \equiv 0$ and $\mathcal{M}$ is flat, then a point of the free boundary is called *regular* if at such point the frequency attains its least possible value $N(0+,u) = 2 - \frac{1}{2}$ (see Lemma 1.2.3 and Definition 1.2.9 below). For instance the origin is a regular free boundary point for $\hat{u}_{3/2}(x) = \operatorname{Re}(x_1 + i\,|x_n|)^{3/2}$. The reader is also referred to Section 1.2 for a detailed description of the main results in [AC04] and [ACS07].

Another line of developments has emerged from the fact that in the particular case $\Omega_+ = \mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0,\infty)$ and $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, the Signorini problem can be interpreted as an obstacle problem for the fractional Laplacian on $\mathbb{R}^{n-1}$

$$u - \varphi \geq 0, \quad (-\Delta_{x'})^s u \geq 0, \quad (u-\varphi)(-\Delta_{x'})^s u = 0$$

with $s = \frac{1}{2}$. A more general range of fractional powers $0 < s < 1$ has been considered by Silvestre [Sil07], who has proved the almost optimal regularity of the solutions: $u \in C^{1,\alpha}(\mathbb{R}^{n-1})$ for any $\alpha < s$. Subsequently, Caffarelli, Salsa, and Silvestre [CSS08] have established the optimal $C^{1,s}$ regularity and generalized the free boundary regularity results of [ACS07] to this setting.

One interesting aspect of [CSS08] is that the authors are able to treat the case when the thin obstacle φ is nonzero. They prove a generalization of Almgren's monotonicity of the frequency for solutions of the thin obstacle problem with nonzero thin obstacles, see Section 2.2 below. With this tool they establish the optimal interior regularity of the solution, and the regularity of the free boundary at the regular points.

Main results. In the thin obstacle problem one can subdivide the free boundary points into three categories: the set of regular points discussed above, the set $\Sigma(u)$ of the so-called singular points, and the remaining portion of those free boundary points which are neither regular, nor singular. As we have mentioned, the papers [ACS07], [Sil07] and [CSS08] study the former set.

The main objective of this paper is the study of the singular free boundary points. More specifically, using methods from geometric PDE's we study the structure of the singular set $\Sigma(u)$, which we now define.

Hereafter, the hypersurface $\mathcal{M}$ is assumed to be the hyperplane $\{x_n = 0\}$. A free boundary point $x_0 \in \Gamma(u)$ is called *singular* if the coincidence set $\Lambda(u)$ has a vanishing $(n-1)$-Hausdorff density at x_0, i.e.

$$\frac{\mathcal{H}^{n-1}(\Lambda(u) \cap B'_r(x_0))}{\mathcal{H}^{n-1}(B'_r(x_0))} \to 0.$$

We denote by $\Sigma(u)$ the set of singular points. We observe here that $\Sigma(u)$ is not necessarily a small part of the free boundary $\Gamma(u)$ in any sense. In fact, it may happen that the whole free boundary is composed exclusively of singular points. This happens for instance when u is a harmonic function, symmetric with respect to $\mathcal{M}$, touching a zero obstacle (see also Figure 2 below).

One of the main difficulties in our analysis consists in establishing the uniqueness of the blowups, which are the limits of properly defined rescalings of u, see (1.2.2) below. Proving such uniqueness is equivalent to showing that at any $x_0 \in \Sigma(u)$ one has a Taylor expansion

$$u(x', x_n) - \varphi(x') = p^{x_0}_\kappa(x - x_0) + o(|x - x_0|^\kappa),$$

where $p = p^{x_0}_\kappa$ is a nondegenerate homogeneous polynomial of a certain order κ, satisfying

$$\Delta p = 0, \quad x \cdot \nabla p - \kappa p = 0, \quad p(x', 0) \geq 0, \quad p(x', -x_n) = p(x', x_n),$$

see Theorems 1.3.6 and 2.6.3 below. The value of κ must be an even integer, and it is obtained from the above mentioned generalization of Almgren's monotonicity of frequency to the thin obstacle problem, see Theorems 1.2.1 and 2.2.1.

At this point, to put our discussion in a broader perspective, we recall that for the classical obstacle problem a related Taylor expansion was originally obtained by Caffarelli and Rivière [CR77] in dimension 2. In higher dimensions this expansion was first proved by Caffarelli [Caf98] with the use of a deep monotonicity formula of Alt, Caffarelli, and Friedman [ACF84]. Subsequently, a different proof based on a simpler monotonicity formula was discovered by Weiss [Wei99]. More recently, using the result of Weiss, Monneau [Mon03] has derived yet another monotonicity formula which is tailor made for the study of the singular free boundary points. He has then used such formula to prove the above mentioned Taylor expansion at singular points of the classical obstacle problem.

Now, in the classical obstacle problem the only frequency that appears is $\kappa = 2$. Specifically, the above mentioned monotonicity formulas in [ACF84], [Wei99], [Mon03] are only suitable for $\kappa = 2$. In the thin obstacle problem, instead, one the main complications is that at a singular free boundary point the frequency κ may be an arbitrary even integer $2m$, $m \in \mathbb{N}$.

With this observation in mind, and the objective of studying singular points, our original desire was to construct an analogue of Monneau's formula based on Almgren's frequency formula, rather than on Weiss'. This was suggested by the fact that, at least in principle, Almgren's frequency formula does not display the limitation of the specific value $\kappa = 2$. In the process, however, we have discovered a new one-parameter family of monotonicity formulas $\{W_\kappa\}$ of Weiss type (see Theorem 1.4.1) which is tailor made for studying the thin obstacle problem, and that, remarkably, is inextricably connected to Almgren's monotonicity formula, see Section 1.4 and 2.7. With these new formulas in hand, following Monneau [Mon03] we have discovered another one-parameter family $\{M_\kappa\}$ of monotonicity formulas (see Theorem 1.4.3) which are ad hoc for studying singular free boundary points with frequency $\kappa = 2m$, $m \in \mathbb{N}$. With this result, in turn, we have been able to establish the desired Taylor expansion mentioned above, thus obtaining the uniqueness of the blowups. Furthermore, the monotonicity formulas $\{M_\kappa\}$ allow to establish the nondegeneracy and continuous dependence of the polynomial $p_\kappa^{x_0}$ on the singular free boundary point x_0 with frequency κ.

We should also mention here that in the case of the nonzero thin obstacle φ there are additional technical difficulties introduced by the error terms in the computations. In fact, Almgren's monotonicity formula in its purest form will not hold in general, but if φ is assumed to be $C^{k,1}$ regular we can establish the monotonicity of the truncated versions Φ_k of the frequency functional, see Theorem 2.2.1. This kind of formula has been used first in [CSS08] in the case $k = 2$. Because of the truncation, our approach allows to effectively study only the free boundary points at which the frequency takes a value $\kappa < k$.

Finally, a standard argument based on Whitney's extension theorem implies that the set of singular points is contained in a countable union of C^1 regular manifolds of dimensions $d = 0, 1, \ldots, n-2$. In the case of a nonzero obstacle $\varphi \in C^{k,1}$ this result is limited to singular points with the frequency $\kappa < k$. For a precise formulation, see Theorems 1.3.8 and 2.6.5.

Structure of the paper. When the thin obstacle φ is identically zero our constructions and proofs are most transparent. While the majority of our results continue to hold for regular nonzero obstacles, the technicalities of the proofs are overwhelming and may easily distract from the main ideas. For this reason we have subdivided the paper into two parts: Part 1 deals exclusively with solutions with a zero thin obstacle φ, whereas Part 2 deals with a nonzero φ. The individual structure of these parts is as follows.

Part 1: $\varphi = 0$.

- In Section 1.1 we define the class $\mathfrak{S}$ of normalized solutions of the Signorini problem.
- In Section 1.2 we describe the known results, including the optimal regularity and the regularity of the free boundary.
- Section 1.3 contains the statements of our main results.
- In Section 1.4 we establish the above mentioned Weiss and Monneau type monotonicity formulas.
- Finally, in Section 1.5 we use these monotonicity formulas to establish the structure of the singular set.

Part 2: $\varphi \neq 0$.

- In Section 2.1 we describe a method based on harmonic extension of the k-th Taylor's polynomial of the thin obstacle to obtain the main class $\mathfrak{S}_k$ of normalized solutions.
- In Section 2.2 we prove a form of Almgren's monotonicity formula which generalizes a similar result in [CSS08].
- In Section 2.3 using this monotonicity of the generalized frequency we study the growth of $u \in \mathfrak{S}_k$ near the origin.
- In Section 2.4 we establish the existence of blowups.
- In Section 2.5 we give a classification of free boundary points for solutions of the Signorini problem.
- In Section 2.6 we state our main results, see Theorems 2.6.3 and 2.6.5 below.
- In Section 2.7 we prove two extended forms of the Weiss and the Monneau type monotonicity formulas obtained in the first part of the paper.
- Finally, in Section 2.8, we prove our main results.

In closing, we would like to mention that, using the extension approach developed in [CS07] and [CSS08], our technique works also for solutions of the obstacle problem for the fractional Laplacian $(-\Delta_{x'})^s$ for any $0 < s < 1$. However, for the sake of exposition, here we restrict ourselves to the case $s = 1/2$ (i.e. the Signorini problem). The consideration of the general case is deferred to a forthcoming paper.

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Part 1. Zero Thin Obstacle

1.1. Normalization

In this part of the paper we consider a solution u of the Signorini problem with *zero obstacle* on a *flat boundary*, i.e. $\varphi = 0$ and $\mathcal{M}$ a hyperplane. Since we are interested in the properties of u near a free boundary point, after possibly a translation, rotation and scaling, we can assume that u is defined in $B_1^+ \cup B_1'$, where

$$B_1^+ := B_1 \cap \mathbb{R}^n_+, \quad B_1' := B_1 \cap (\mathbb{R}^{n-1} \times \{0\}).$$

Moreover, $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1^+ \cup B_1')$, and it is such that

$$\Delta u = 0 \quad \text{in } B_1^+ \tag{1.1.1}$$

$$u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\, \partial_{x_n} u = 0 \quad \text{on } B_1' \tag{1.1.2}$$

$$0 \in \Gamma(u) := \partial\Lambda(u) := \partial\{(x', 0) \in B_1' \mid u(x', 0) = 0\}, \tag{1.1.3}$$

where, we recall, $\Lambda(u) \subset B_1'$ is the coincidence set, and the free boundary $\Gamma(u)$ is the topological boundary of $\Lambda(u)$ in the relative topology of B_1'.

Definition 1.1.1. Throughout the paper we denote by $\mathfrak{S}$ the class of solutions of the normalized Signorini problem (1.1.1)–(1.1.3).

Note that we may actually extend $u \in \mathfrak{S}$ by even symmetry to B_1

$$u(x', -x_n) := u(x', x_n). \tag{1.1.4}$$

Then the resulting function will satisfy

$$\begin{aligned} \Delta u &\leq 0 \quad \text{in } B_1 \\ \Delta u &= 0 \quad \text{in } B_1 \setminus \Lambda(u) \\ u\, \Delta u &= 0 \quad \text{in } B_1. \end{aligned}$$

It will also be useful to note the following direct relation between Δu and $\partial_{x_n} u$:

$$\Delta u = 2(\partial_{x_n} u)\, \mathcal{H}^{n-1}\big|_{\Lambda(u)} \quad \text{in } \mathcal{D}'(B_1).$$

Note that here by $\partial_{x_n} u$ on $\{x_n = 0\}$ we understand the limit $\partial_{x_n^+} u$ from inside $\{x_n > 0\}$. We keep this convention throughout the paper.

1.2. Known Results

1.2.1. **Optimal regularity.** For the solutions of the Signorini problem, finer regularity results are known. It has been proved by Athanasopoulos and Caffarelli [AC04] that in fact $u \in C^{1,\frac12}_{\mathrm{loc}}$ in $B_1^+ \cup B_1'$. This is the optimal regularity as one can see from the explicit example of a solution $\hat{u}_{3/2}$ given by

$$\hat{u}_{3/2}(x) = \mathrm{Re}(x_1 + i\,|x_n|)^{3/2}. \tag{1.2.1}$$

Below we indicate the main steps in the proof of this optimal regularity, by following the approach developed by Athanasopoulos, Caffarelli, and Salsa [ACS07] and Caffarelli, Salsa, and Silvestre [CSS08]. The main analysis is performed by considering the *rescalings*

$$u_r(x) := \frac{u(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{1/2}}, \tag{1.2.2}$$

and studying the limits as $r \to 0+$, known as the *blowups*. We emphasize that from the definition (1.2.2) one has

$$\|u_r\|_{L^2(\partial B_1)} = 1. \tag{1.2.3}$$

Note that generally the blowups might be different over different subsequences $r = r_j \to 0+$. The following monotonicity formula plays a fundamental role in controlling the rescalings.

Theorem 1.2.1 (Monotonicity of the Frequency). *Let u be a nonzero solution of (1.1.1)–(1.1.2), then the frequency of u*

$$r \mapsto N(r,u) := \frac{r \int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

is nondecreasing for $0 < r < 1$. Moreover, $N(r,u) \equiv \kappa$ for $0 < r < 1$ if and only if u is homogeneous of degree κ in B_1, i.e.

$$x \cdot \nabla u - \kappa u = 0 \quad \text{in } B_1.$$

In the case of the harmonic functions this is a classical result of Almgren [Alm79], which was subsequently generalized to divergence form elliptic operators with Lipschitz coefficients in [GL86], [GL87]. For the thin obstacle problem this formula has been first used in [ACS07]. We will provide a proof of Theorem 1.2.1 in Section 1.4. The reason for doing it is twofold. Besides an obvious consideration of completeness, more importantly we will prove that Theorem 1.2.1 is in essence equivalent to a new one-parameter family of monotonicity formulas similar to that of Weiss in [Wei99], see Theorem 1.4.1.

The following property of the frequency plays an important role: for any $0 < r, \rho < 1$ one has

$$N(\rho, u_r) = N(r\rho, u). \tag{1.2.4}$$

Suppose now $u \in \mathfrak{S}$ and $0 \in \Gamma(u)$. Consider the rescalings u_r as defined in (1.2.2). Using (1.2.3), (1.2.4) and the monotonicity of the frequency N claimed in Theorem 1.2.1, one easily has for $r \leq 1$

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) = N(r,u) \leq N(1,u).$$

Now, this implies that there exists a nonzero function $u_0 \in W^{1,2}(B_1)$, which we call a *blowup* of u at the origin, such that for a subsequence $r = r_j \to 0+$

$$\begin{aligned} u_{r_j} &\to u_0 \quad \text{in } W^{1,2}(B_1) \\ u_{r_j} &\to u_0 \quad \text{in } L^2(\partial B_1) \\ u_{r_j} &\to u_0 \quad \text{in } C^1_{\mathrm{loc}}(B_1^\pm \cup B_1'). \end{aligned} \tag{1.2.5}$$

It is easy to see the weak convergence in $W^{1,2}(B_1)$ and the strong convergence in $L^2(\partial B_1)$. The third convergence (and consequently the strong convergence in $W^{1,2}$) follows from uniform $C^{1,\alpha}_{\mathrm{loc}}$ estimates on u_r in $B_1^\pm \cup B_1'$ in terms of $W^{1,2}$-norm of u_r in B_1, see e.g. [AC04].

Proposition 1.2.2 (Homogeneity of blowups). *Let $u \in \mathfrak{S}$ and denote by u_0 any blowup of u as described above. Then u_0 satisfies (1.1.1)–(1.1.2), is homogeneous of degree $\kappa = N(0+, u)$, and $u_0 \not\equiv 0$.*

Proof. The fact that u_0 satisfies (1.1.1)–(1.1.2) follows from the above mentioned $C^{1,\alpha}_{\mathrm{loc}}$ estimates on u_r in $B_1^\pm \cup B_1'$. For the blowup u_0 over a sequence $r_j \to 0+$ we have

$$N(r, u_0) = \lim_{r_j \to 0+} N(r, u_{r_j}) = \lim_{r_j \to 0+} N(rr_j, u) = N(0+, u)$$

for any $0 < r < 1$. This implies that $N(r, u_0)$ is a constant. In view of the last part of Theorem 1.2.1 we conclude that u_0 is a homogeneous function. The fact that $u_0 \not\equiv 0$ follows from the convergence $u_{r_j} \to u_0$ in $L^2(\partial B_1)$ and that equality $\int_{\partial B_1} u_{r_j}^2 = 1$, implying that $\int_{\partial B_1} u_0^2 = 1$. □

We emphasize that although the blowups at the origin might not be unique, as a consequence of Proposition 1.2.2 they all have the same homogeneity.

Lemma 1.2.3 (Minimal homogeneity)**.** *Given* $u \in \mathfrak{S}$ *one has*

$$N(0+, u) \geq 2 - \frac{1}{2}.$$

Moreover, either

$$N(0+, u) = 2 - \frac{1}{2} \quad \textit{or} \quad N(0+, u) \geq 2. \quad \square$$

For the proof see [CSS08, Lemma 6.1]. This follows from the classification of the homogeneous solutions of the Signorini problem which are convex in the x'-variables. The lower bound is essentially contained in Silvestre's dissertation [Sil07]. The last part of the lemma was first proved in [ACS07].

The minimal homogeneity allows to establish the following maximal growth of the solution near free boundary points, see [CSS08, Theorem 6.7].

Lemma 1.2.4 (Growth estimate)**.** *Let* $u \in \mathfrak{S}$*. Then*

$$\sup_{B_r} |u| \leq C\, r^{3/2}, \quad 0 < r < 1/2,$$

where $C = C(n, \|u\|_{L^2(B_1)})$*.* □

Ultimately, this leads to the optimal regularity of u.

Theorem 1.2.5 (Optimal regularity)**.** *Let* $u \in \mathfrak{S}$ *and* $0 \in \Gamma(u)$*. Then* $u \in C^{1,\frac{1}{2}}_{\mathrm{loc}}(B_1^\pm \cup B_1')$ *with*

$$\|u\|_{C^{1,\frac{1}{2}}(B^\pm_{1/2} \cup B'_{1/2})} \leq C(n, \|u\|_{L^2(B_1)}).$$

□

1.2.2. **Regularity of the free boundary.** Another aspect of the Signorini problem is the study of the free boundary $\Gamma(u)$. In fact, the starting point in the study of the regularity of the free boundary is precisely the optimal regularity of u, which we have described in Theorem 1.2.5. First note that Almgren's frequency functional (as well as Theorems 1.2.1–1.2.5) can be defined at any point $x_0 \in \Gamma(u)$ by simply translating that point to the origin:

$$N^{x_0}(r, u) := \frac{r \int_{B_r(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)} u^2},$$

where $r > 0$ is such that $B_r(x_0) \Subset B_1$. This enables us to give the following definitions.

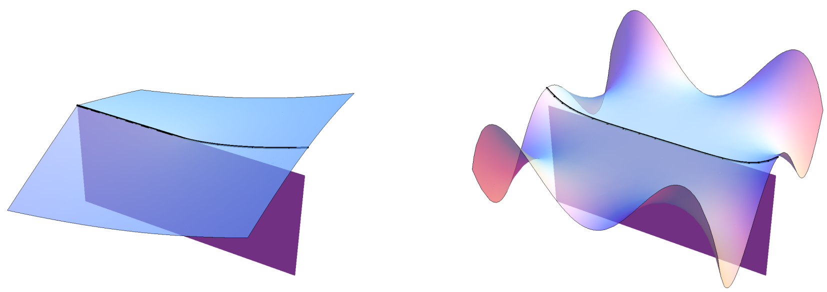

FIGURE 1. Graphs of $\mathrm{Re}(x_1 + i\,|x_2|)^{3/2}$ and $\mathrm{Re}(x_1 + i\,|x_2|)^6$

Definition 1.2.6. Given $u \in \mathfrak{S}$, for $\kappa \geq 2 - \frac{1}{2}$ we define

$$\Gamma_\kappa(u) := \{x_0 \in \Gamma(u) \mid N^{x_0}(0+, u) = \kappa\}.$$

Remark 1.2.7. One has to point out that the sets $\Gamma_\kappa(u)$ may be nonempty only for κ in a certain set of values. For instance, Lemma 1.2.3 implies that $\Gamma_k(u) = \varnothing$ whenever $2 - \frac{1}{2} < \kappa < 2$. On the other hand, if one considers the functions

$$\hat{u}_\kappa(x) = \mathrm{Re}(x_1 + i\,|x_n|)^\kappa, \quad \text{for } \kappa \in \{2m - \frac{1}{2} \mid m \in \mathbb{N}\} \cup \{2m \mid m \in \mathbb{N}\},$$

then one has $0 \in \Gamma_\kappa(\hat{u}_\kappa)$, and therefore $\Gamma_\kappa(\hat{u}_\kappa) \neq \varnothing$ for any of the above values of κ.

Remark 1.2.8. In dimension $n = 2$, a simple analysis of homogeneous harmonic functions in a halfplane shows that, up to a multiple and a mirror reflection, the only possible solutions of (1.1.1)–(1.1.2) are the functions $\hat{u}_\kappa$ above and

$$\hat{v}_\kappa(x) = \mathrm{Im}(x_1 + i\,|x_2|)^\kappa, \quad \text{for } \kappa \in \{2m + 1 \mid m \in \mathbb{N}\}.$$

However, we claim that the values $\kappa \in \{2m+1 \mid m \in \mathbb{N}\}$ cannot occur in the blowup for any $u \in \mathfrak{S}$. Indeed, since $0 \in \Gamma(u) = \partial\{u(\cdot, 0) > 0\}$, we may choose a sequence $r = r_j \to 0+$ so that $u(\frac{1}{2}r_j, 0) > 0$ (or $u(-\frac{1}{2}r_j, 0) > 0$). Then from the complementary condition (1.1.2) we will have $\partial_{x_2} u(\frac{1}{2}r_j, 0) = 0$ implying that $\partial_{x_2} u_{r_j}(\frac{1}{2}, 0) = 0$. Hence, if u_0 is a blowup over a subsequence of $\{r_j\}$ the C^1 convergence will imply that $\partial_{x_2} u_0(\frac{1}{2}, 0) = 0$. However, $\hat{v}_\kappa$ do not satisfy this condition.

Thus, the only frequencies $\kappa = N(0+, u)$ that appear in dimension $n = 2$ are $\kappa \in \{2m - \frac{1}{2} \mid m \in \mathbb{N}\} \cup \{2m \mid m \in \mathbb{N}\}$. It is plausible that a similar result hold in higher dimensions, but this is not known to the authors at the time of this writing. See also our concluding remarks in the last section of this paper.

Of special interest is the case of the smallest possible value of the frequency $\kappa = 2 - \frac{1}{2}$.

Definition 1.2.9 (Regular points)**.** For $u \in \mathfrak{S}$ we say that $x_0 \in \Gamma(u)$ is *regular* if $N^{x_0}(0+,u) = 2 - \frac{1}{2}$, i.e., if $x_0 \in \Gamma_{2-\frac{1}{2}}(u)$.

Note that from the Almgren's frequency formula it follows that the mapping $x_0 \mapsto N^{x_0}(0+,u)$ is upper semicontinuous. Moreover, since $N^{x_0}(0+,u)$ misses values in the interval $(2-\frac{1}{2},2)$, one immediately obtains that $\Gamma_{2-\frac{1}{2}}(u)$ is a relatively open subset of $\Gamma(u)$. The following regularity theorem at regular free boundary points has been proved by Athanasopoulos, Caffarelli, and Salsa [ACS07].

Theorem 1.2.10 (Regularity of the regular set)**.** *Let $u \in \mathfrak{S}$, then the free boundary $\Gamma_{2-\frac{1}{2}}(u)$ is locally a $C^{1,\alpha}$ regular $(n-2)$-dimensional surface.* □

1.3. Singular set: statement of main results

The main objective of this paper is to study the structure of the so-called singular set of the free boundary. In this section we state our main results in this direction, Theorems 1.3.6 and 1.3.8. The proofs of these results will be presented in Section 1.5.

Definition 1.3.1 (Singular points)**.** Let $u \in \mathfrak{S}$. We say that 0 is a *singular point* of the free boundary $\Gamma(u)$, if

$$\lim_{r\to 0+} \frac{\mathcal{H}^{n-1}(\Lambda(u)\cap B_r')}{\mathcal{H}^{n-1}(B_r')} = 0.$$

We denote by $\Sigma(u)$ the subset of singular points of $\Gamma(u)$. We also denote

$$\Sigma_\kappa(u) := \Sigma(u) \cap \Gamma_\kappa(u). \tag{1.3.1}$$

Note that in terms of the rescalings (1.2.2) the condition $0 \in \Sigma(u)$ is equivalent to

$$\lim_{r\to 0+} \mathcal{H}^{n-1}(\Lambda(u_r)\cap B_1') = 0.$$

The following theorem gives a complete characterization of singular points via the value $\kappa = N(0+,u)$ as well as the type of the blowups. In particular, it establishes that

$$\Sigma_\kappa(u) = \Gamma_\kappa(u) \quad \text{for } \kappa = 2m,\ m \in \mathbb{N}.$$

Theorem 1.3.2 (Characterization of singular points)**.** *Let $u \in \mathfrak{S}$ and $0 \in \Gamma_\kappa(u)$. Then the following statements are equivalent:*

(i) $0 \in \Sigma_\kappa(u)$
(ii) *any blowup of u at the origin is a nonzero homogeneous polynomial p_κ of degree κ satisfying*

$$\Delta p_\kappa = 0, \quad p_\kappa(x',0) \ge 0, \quad p_\kappa(x',-x_n) = p_\kappa(x',x_n).$$

(iii) *$\kappa = 2m$ for some $m \in \mathbb{N}$.*

Proof. (i) ⇒ (ii) The rescalings u_r satisfy

$$\Delta u_r = 2(\partial_{x_n} u_r)\mathcal{H}^{n-1}\big|_{\Lambda(u_r)} \quad \text{in } \mathcal{D}'(B_1).$$

Since $|\nabla u_r|$ are locally uniformly bounded in B_1 by (1.2.5) and $\mathcal{H}^{n-1}(\Lambda(u_r)\cap B_1) \to 0$, the formula above implies that Δu_r converges weakly to 0 in $\mathcal{D}(B_1)$ and therefore any blowup u_0 must be harmonic in B_1. On the other hand, by Proposition 1.2.2, the function u_0 is homogeneous in B_1 and therefore can be extended by homogeneity to $\mathbb{R}^n$. The resulting extension will be harmonic in $\mathbb{R}^n$ and, being homogeneous, will have at most a polynomial growth at infinity. Then by the Liouville theorem we conclude that u_0 must be a homogeneous harmonic polynomial p_κ of a certain integer degree κ. We also have that $p_\kappa \not\equiv 0$ in $\mathbb{R}^n$ by Proposition 1.2.2. The properties of u also imply that that $p_\kappa(x', 0) \geq 0$ for all $x' \in \mathbb{R}^{n-1}$ and $p_\kappa(x', -x_n) = p_\kappa(x', x_n)$ for all $x = (x', x_n) \in \mathbb{R}^n$.

(ii) ⇒ (iii) Let p_κ be a blowup of u at the origin. If κ is odd, the nonnegativity of p_κ on $\mathbb{R}^{n-1} \times \{0\}$ implies that p_κ vanishes on $\mathbb{R}^{n-1} \times \{0\}$ identically. On the other hand, from the even symmetry we also have that $\partial_{x_n} p_\kappa \equiv 0$ on $\mathbb{R}^{n-1} \times \{0\}$. Since p_κ is harmonic in $\mathbb{R}^n$, the Cauchy-Kovalevskaya theorem implies that $p_\kappa \equiv 0$ in $\mathbb{R}^n$, contrary to the assumption. Thus, $\kappa \in \{2m \mid m \in \mathbb{N}\}$.

(iii) ⇒ (ii) The proof is an immediate corollary of the following Liouville type result.

Lemma 1.3.3. *Let v be a κ-homogeneous global solution of the thin obstacle problem in $\mathbb{R}^n$ with $\kappa = 2m$ for $m \in \mathbb{N}$. Then v is a homogeneous harmonic polynomial.*

Since $\Delta v = 2(\partial_{x_n} v)\mathcal{H}^{n-1}\big|_{\Lambda(v)}$ on $\mathbb{R}^n$, with $\partial_{x_n} v \leq 0$ on $\{x_n = 0\}$, this is a particular case of the following lemma, which is essentially Lemma 7.6 in Monneau [Mon07], with an almost identical proof.

Lemma 1.3.4. *Let $v \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies $\Delta v \leq 0$ in $\mathbb{R}^n$ and $\Delta v = 0$ in $\mathbb{R}^n \setminus \{x_n = 0\}$. If v is homogeneous of degree $\kappa = 2m$, $m \in \mathbb{N}$, then $\Delta v = 0$ in $\mathbb{R}^n$.*

Proof. By assumption, $\mu := \Delta v$ is a nonpositive measure, living on $\{x_n = 0\}$. We are going to show that $\mu = 0$. To this end, let P be a $2m$-homogeneous harmonic polynomial, which is positive on $\{x_n = 0\} \setminus \{0\}$. For instance, take

$$P(x) = \sum_{j=1}^{n-1} \mathrm{Re}(x_j + i x_n)^{2m}.$$

Further, let $\psi \in C_0^\infty(0, \infty)$ with $\psi \geq 0$ and $\Psi(x) = \psi(|x|)$. Then we have

$$\begin{aligned}
-\langle \mu, \Psi P\rangle = -\langle \Delta v, \Psi P\rangle &= \int_{\mathbb{R}^n} \nabla v \cdot \nabla(\Psi P) \\
&= \int_{\mathbb{R}^n} \Psi \nabla v \cdot \nabla P + P \nabla v \cdot \nabla \Psi \\
&= \int_{\mathbb{R}^n} -\Psi v \Delta P - v \nabla \Psi \cdot \nabla P + P \nabla v \cdot \nabla \Psi \\
&= \int_{\mathbb{R}^n} -\Psi v \Delta P - \frac{\psi'(|x|)}{|x|} v (x \cdot \nabla P) + \frac{\psi'(|x|)}{|x|} P (x \cdot \nabla v) = 0,
\end{aligned}$$

where in the last step we have used that $\Delta P = 0$, $x \cdot \nabla P = 2mP$, $x \cdot \nabla v = 2mv$. This implies that the measure μ is supported at the origin. Hence $\mu = c\delta_0$, where δ_0 is the Dirac's delta. On the other hand, μ is $2(m-1)$-homogeneous and δ_0 is $(-n)$-homogeneous and therefore $\mu = 0$. □

(ii) ⇒ (i) Suppose that 0 is not a singular point and that over some sequence $r = r_j \to 0+$ we have $\mathcal{H}^{n-1}(\Lambda(u_r) \cap B_1') \geq \delta > 0$. Taking a subsequence if necessary, we may assume that u_{r_j} converges to a blowup u_0. We claim that

$$\mathcal{H}^{n-1}(\Lambda(u_0) \cap B_1') \geq \delta > 0.$$

Indeed, otherwise there exists an open set U in $\mathbb{R}^{n-1}$ with $\mathcal{H}^{n-1}(U) < \delta$ so that $\Lambda(u_0) \cap \overline{B_1'} \subset U$. Then for large j we must have $\Lambda(u_{r_j}) \cap \overline{B_1'} \subset U$, which is a contradiction, since $\mathcal{H}^{n-1}(\Lambda(u_{r_j})\cap\overline{B_1'})\geq\delta>\mathcal{H}^{n-1}(U)$. But then u_0 vanishes identically on $\mathbb{R}^{n-1}\times\{0\}$ and consequently on $\mathbb{R}^n$ by the Cauchy-Kovalevskaya theorem. This completes the proof of the theorem. □

Definition 1.3.5. Throughout the rest of the paper we denote by $\mathfrak{P}_\kappa$, $\kappa=2m$, $m\in\mathbb{N}$, the class of κ-homogeneous harmonic polynomials described in statement (ii) of Theorem 1.3.2.

Theorem 1.3.6 (κ-differentiability at singular points)**.** *Let $u\in\mathfrak{S}$ and $0\in\Sigma_\kappa(u)$ with $\kappa=2m$, $m\in\mathbb{N}$. Then there exists a* nonzero $p_\kappa\in\mathfrak{P}_\kappa$ *such that*

$$u(x)=p_\kappa(x)+o(|x|^\kappa).$$

Moreover, if for $x_0\in\Sigma_\kappa(u)$ the polynomial $p_\kappa^{x_0}\in\mathfrak{P}_\kappa$ is such that we have the Taylor expansion

$$u(x)=p_\kappa^{x_0}(x-x_0)+o(|x-x_0|^\kappa),$$

then $p_\kappa^{x_0}$ depends continuously on $x_0\in\Sigma_\kappa(u)$.

We want to point out here that the polynomials $p_\kappa\in\mathfrak{P}_\kappa$ can be recovered uniquely from their restriction to $\mathbb{R}^{n-1}\times\{0\}$. This follows from the Cauchy-Kovalevskaya theorem; see the proof of the uniqueness part of Lemma 2.1.2 in Part 2. Thus, if p_κ is not identically zero in $\mathbb{R}^n$ then its restriction to $\mathbb{R}^{n-1}\times\{0\}$ is also nonzero.

Theorem 1.3.6 can be used to prove a theorem on the structure of the singular set, similar to the one of Caffarelli [Caf98] in the classical obstacle problem. In order to state the result we define the dimension $d=d_\kappa^{x_0}$ of $\Sigma_\kappa(u)$ at a given point x_0 based on the polynomial $p_\kappa^{x_0}$. Roughly speaking, we expect $\Sigma_\kappa(u)$ to be contained in a d-dimensional manifold near x_0.

Definition 1.3.7 (Dimension at the singular point)**.** For a singular point $x_0\in\Sigma_\kappa(u)$ we denote

$$d_\kappa^{x_0}:=\dim\{\xi\in\mathbb{R}^{n-1}\mid\xi\cdot\nabla_{x'}p_\kappa^{x_0}(x',0)=0\text{ for all }x'\in\mathbb{R}^{n-1}\},$$

which we call the *dimension* of $\Sigma_\kappa(u)$ at x_0. Note that since $p_\kappa^{x_0}\not\equiv0$ on $\mathbb{R}^{n-1}\times\{0\}$ one has

$$0\leq d_\kappa^{x_0}\leq n-2.$$

For $d=0,1,\dots,n-2$ we define

$$\Sigma_\kappa^d(u):=\{x_0\in\Sigma_\kappa(u)\mid d_\kappa^{x_0}=d\}.$$

Theorem 1.3.8 (Structure of the singular set)**.** *Let $u\in\mathfrak{S}$. Then $\Sigma_\kappa(u)=\Gamma_\kappa(u)$ for $\kappa=2m$, $m\in\mathbb{N}$, and every set $\Sigma_\kappa^d(u)$, $d=0,1,\dots,n-2$ is contained in a countable union of d-dimensional C^1 manifolds.*

The following example provides a small illustration of Theorem 1.3.8. Consider the harmonic polynomial $u(x)=x_1^2x_2^2-\left(x_1^2+x_2^2\right)x_3^2+\frac{1}{3}x_3^4$ in $\mathbb{R}^3$. Note that $u\in\mathfrak{P}_4\subset\mathfrak{S}$. On $\mathbb{R}^2\times\{0\}$, we have $u(x_1,x_2,0)=x_1^2x_2^2$ and therefore the coincidence set $\Lambda(u)$ as well as the free boundary $\Gamma(u)$ consist of the union of the lines $\mathbb{R}\times\{0\}\times\{0\}$ and $\{0\}\times\mathbb{R}\times\{0\}$. Thus, all free boundary points are singular. It is straightforward to check that $0\in\Sigma_4^0(u)$ and that the rest of the free boundary points are in $\Sigma_2^1(u)$, see Figure 2.

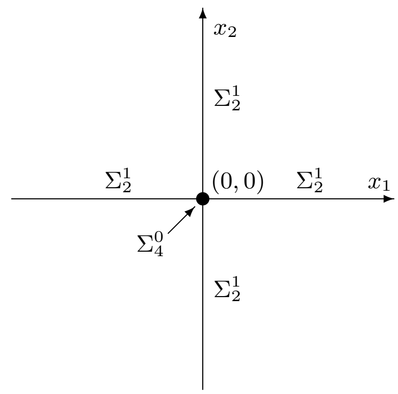

FIGURE 2. Free boundary for $u(x) = x_1^2x_2^2 - \left(x_1^2 + x_2^2\right) x_3^2 + \frac{1}{3}x_3^4$ in $\mathbb{R}^3$ with zero thin obstacle on $\mathbb{R}^2 \times \{0\}$.

1.4. WEISS AND MONNEAU TYPE MONOTONICITY FORMULAS

In this section we introduce two new one-parameter families of monotonicity formulas that will play a key role in our analysis. Before doing so, however, we give a proof of Almgren's frequency formula since the latter has served as one of our main sources of inspiration. We refer the reader to the original paper by Almgren [Alm79] for the case of harmonic functions, to [GL86], [GL87] for solutions to divergence form elliptic equations, and to Lemma 1 in [ACS07] for the thin obstacle problem.

Proof of Theorem 1.2.1. Let $u \in \mathfrak{S}$ and consider the quantities

$$D(r) := \int_{B_r} |\nabla u|^2, \qquad H(r) := \int_{\partial B_r} u^2. \tag{1.4.1}$$

Denoting by $u_\nu = \partial_\nu u$, where ν is the outer unit normal on ∂B_r, we have

$$H'(r) = \frac{n-1}{r}\, H(r) + 2\int_{\partial B_r} u u_\nu. \tag{1.4.2}$$

On the other hand, using that $\Delta(u^2/2) = u\Delta u + |\nabla u|^2 = |\nabla u|^2$ and integrating by parts, we obtain

$$\int_{\partial B_r} u u_\nu = \int_{B_r} |\nabla u|^2 = D(r). \tag{1.4.3}$$

Further, to compute $D'(r)$ we use Rellich's formula

$$\int_{\partial B_r} |\nabla u|^2 = \frac{n-2}{r}\int_{B_r} |\nabla u|^2 + 2\int_{\partial B_r} u_\nu^2 - \frac{2}{r}\int_{B_r} (x\cdot\nabla u)\Delta u.$$

Notice that in view of the fact $(x\cdot\nabla u)u_{x_n} = 0$ on B_1' the last integral in the right-hand side vanishes. Hence,

$$D'(r) = \frac{n-2}{r}\, D(r) + 2\int_{\partial B_r} u_\nu^2. \tag{1.4.4}$$

Thus, as in the classical case of harmonic functions we have

$$\begin{aligned}\frac{N'(r)}{N(r)} &= \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}\\ &= \frac{1}{r} + \frac{n-2}{r} - \frac{n-1}{r} + 2\left\{\frac{\int_{\partial B_r} u_\nu^2}{\int_{\partial B_r} u u_\nu} - \frac{\int_{\partial B_r} u u_\nu}{\int_{\partial B_r} u^2}\right\} \geq 0,\end{aligned}$$

where we have let $N(r) = N(r,u)$. The last inequality is obtained form the Cauchy-Schwarz inequality and implies the monotonicity statement in the theorem. Analyzing the case of equality in Cauchy-Schwarz, we obtain the second part of the theorem. For details, see the end of the proof of [ACS07, Lemma 1]. □

1.4.1. **Weiss type monotonicity formulas.** Here we introduce a new one-parameter family of monotonicity formulas inspired by that introduced by Weiss [Wei99] in the study of the classical obstacle problem. Given $\kappa \geq 0$, we define a functional $W_\kappa(r,u)$, which is suited for the study of the blowups at free boundary points where $N(0+,u) = \kappa$.

Theorem 1.4.1 (Weiss type Monotonicity Formula)**.** *Given $u \in \mathfrak{S}$, for any $\kappa \geq 0$ we introduce the function*

$$W_\kappa(r,u) := \frac{1}{r^{n-2+2\kappa}} \int_{B_r} |\nabla u|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r} u^2.$$

For $0 < r < 1$ one has

$$\frac{d}{dr} W_\kappa(r,u) = \frac{2}{r^{n+2\kappa}} \int_{\partial B_r} (x \cdot \nabla u - \kappa\, u)^2.$$

As a consequence, $r \mapsto W_\kappa(r,u)$ is nondecreasing on $(0,1)$. Furthermore, $W_\kappa(\cdot,u)$ is constant if and only if u is homogeneous of degree κ.

Proof. Using the same notations (1.4.1) as in the proof of Theorem 1.2.1, we have

$$W_\kappa(r,u) = \frac{1}{r^{n-2+2\kappa}}\, D(r) - \frac{\kappa}{r^{n-1+2\kappa}}\, H(r).$$

Using the identities (1.4.2)–(1.4.4), we obtain

$$\begin{aligned}\frac{d}{dr} W_\kappa(r,u) &= \frac{1}{r^{n-2+2\kappa}} \left\{ D'(r) - \frac{n-2+2\kappa}{r}\, D(r) - \frac{\kappa}{r}\, H'(r) + \frac{\kappa(n-1+2\kappa)}{r^2}\, H(r)\right\}\\ &= \frac{1}{r^{n-2+2\kappa}} \left\{ 2\int_{\partial B_r} u_\nu^2 - \frac{2\kappa}{r}\int_{\partial B_r} u u_\nu - \frac{2\kappa}{r}\int_{\partial B_r} u u_\nu + \frac{2\kappa^2}{r^2}\int_{\partial B_r} u^2 \right\}\\ &= \frac{2}{r^{n+2\kappa}} \int_{\partial B_r} (x \cdot \nabla u - \kappa u)^2.\end{aligned}$$

□

Remark 1.4.2. We note that in the statement of Theorem 1.4.1 it is not necessary to assume $0 \in \Gamma_\kappa(u)$. However, the monotonicity formula is most useful under such assumption. The original formula by Weiss [Wei99] for the classical obstacle problem is stated only for $\kappa = 2$. While such limitation is natural in the classical obstacle problem, in the lower dimensional obstacle problem we need the full range of κ's, or at least $\kappa = 2m - \frac{1}{2},\ 2m,\ m \in \mathbb{N}$.

1.4.2. **Monneau type monotonicity formulas.** The next family of monotonicity formulas is related to a formula first used by Monneau [Mon03] in the study of singular points in the classical obstacle problem. Here we derive a κ-homogeneous analogue of his formula, suited for the study of singular points in the thin obstacle problem. Recall that for $\kappa = 2m$, $m \in \mathbb{N}$ we denote by $\mathfrak{P}_\kappa$ the family of harmonic homogeneous polynomial p_κ of degree κ, positive on $x_n = 0$; i.e.

$$\mathfrak{P}_\kappa = \{p_\kappa(x) \mid \Delta p_\kappa = 0,\ x \cdot \nabla p_\kappa - \kappa p_\kappa = 0,\ p_\kappa(x', 0) \geq 0.\}$$

Theorem 1.4.3 (Monneau type Monotonicity Formula)**.** *Let $u \in \mathfrak{S}$ with $0 \in \Sigma_\kappa(u)$, $\kappa = 2m$, $m \in \mathbb{N}$. Then for arbitrary $p_\kappa \in \mathfrak{P}_\kappa$*

$$r \mapsto M_\kappa(r, u, p_\kappa) := \frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} (u - p_\kappa)^2$$

is nondecreasing for $0 < r < 1$.

Proof. We note that

$$N(r, u) \geq \kappa, \quad W_\kappa(r, u) \geq 0, \quad \text{and} \quad W_\kappa(r, p_\kappa) = 0.$$

The first inequality follows from Almgren's monotonicity formula

$$N(r, u) \geq N(0+, u) = \kappa.$$

The second inequality follows from the identity

$$W_\kappa(r, u) = \frac{H(r)}{r^{n-1+2\kappa}}(N(r, u) - \kappa) \geq 0.$$

Finally, the third equality follows from the identity $N(r, p_\kappa) = \kappa$.

We then follow the proof of [Mon03, Theorem 1.8]. Let $w = u - p_\kappa$ and write

$$\begin{aligned}
W_\kappa(r, u) &= W_\kappa(r, u) - W_\kappa(r, p_\kappa) \\
&= \frac{1}{r^{n-2+2\kappa}} \int_{B_r} (|\nabla w|^2 + 2\nabla w \cdot \nabla p_\kappa) - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r} (w^2 + 2wp_\kappa) \\
&= \frac{1}{r^{n-2+2\kappa}} \int_{B_r} |\nabla w|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r} w^2 + \frac{2}{r^{n-1+2\kappa}} \int_{\partial B_r} w(x \cdot \nabla p_\kappa - \kappa p_\kappa) \\
&= \frac{1}{r^{n-2+2\kappa}} \int_{B_r} |\nabla w|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r} w^2 \\
&= \frac{1}{r^{n-2+2\kappa}} \int_{B_r} (-w\Delta w) + \frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} w(x \cdot \nabla w - \kappa w).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\frac{d}{dr}\left(\frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} w^2(x)\right) &= \frac{d}{dr} \int_{\partial B_1} \frac{w^2(ry)}{r^{2\kappa}} \\
&= \int_{\partial B_1} \frac{2w(ry)(ry \cdot \nabla w(ry) - \kappa w(ry))}{r^{2\kappa+1}} \\
&= \frac{2}{r^{n+2\kappa}} \int_{\partial B_r} w(x \cdot \nabla w - \kappa w)
\end{aligned}$$

and

$$w\Delta w = (u - p_\kappa)(\Delta u - \Delta p_\kappa) = -p_\kappa \Delta u \geq 0$$

as $\Delta u = 0$ off $\{x_n = 0\}$ and $\Delta u \leq 0$ and $p_\kappa \geq 0$ on $\{x_n = 0\}$. Combining, we obtain

$$\frac{d}{dr} M_\kappa(r, u, p_\kappa) \geq \frac{2}{r} W_\kappa(r, u) \geq 0.$$

□

1.5. Singular set: proofs

We now apply the monotonicity formulas in the previous sections to study the singular points; in particular, we give the proofs of Theorems 1.3.6 and 1.3.8. We start with establishing the correct growth rate at such points.

Lemma 1.5.1 (Growth estimate)**.** *Let* $u \in \mathfrak{S}$ *and* $0 \in \Gamma_\kappa(u)$*. There exists* $C > 0$ *such that*

$$|u(x)| \leq C|x|^\kappa \quad \text{in } B_1.$$

Proof. This is already proved in Caffarelli, Salsa, Silvestre [CSS08, Lemma 6.6] but for the reader's convenience we provide a proof. From (1.4.2), (1.4.3) we have

$$\frac{H'(r)}{H(r)} = \frac{n - 1 + 2N(r)}{r} \geq \frac{n - 1 + 2\kappa}{r}.$$

Hence

$$\log \frac{H(1)}{H(r)} \geq (n - 1 + 2\kappa) \log \frac{1}{r}$$

which implies

$$H(r) \leq H(1) r^{n-1+2\kappa}.$$

Finally, the L^∞ bound follows from the fact that u^+ and u^- are actually subharmonic, see e.g. [AC04, Lemma 1]. □

Lemma 1.5.2 (Nondegeneracy at singular points)**.** *Let* $u \in \mathfrak{S}$ *and* $0 \in \Sigma_\kappa(u)$*. There exists* $c > 0$*, possibly depending on* u*, such that*

$$\sup_{\partial B_r} |u(x)| \geq c\, r^\kappa, \quad \text{for } 0 < r < 1.$$

Proof. Assume the contrary. Then for a sequence $r = r_j \to 0$ one has

$$h_r := \left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2 \right)^{1/2} = o(r^\kappa).$$

Passing to a subsequence if necessary we may assume that

$$u_r(x) = \frac{u(rx)}{h_r} \to q_\kappa(x) \quad \text{uniformly on } \partial B_1$$

for some $q_\kappa \in \mathfrak{P}_\kappa$, see Theorem 1.3.2. Note that q_κ is nonzero as it must satisfy $\int_{\partial B_1} q_\kappa^2 = 1$. Now consider the functional $M_\kappa(r, u, q_\kappa)$ with q_κ as above. From the assumption on the growth of u it is easy to realize that

$$M_\kappa(0+, u, q_\kappa) = \int_{\partial B_1} q_\kappa^2 = \frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} q_\kappa^2$$

Hence, we have that

$$\frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} (u - q_\kappa)^2 \geq \frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} q_\kappa^2$$

or equivalently

$$\int_{\partial B_r} u^2 - 2uq_\kappa \geq 0.$$

On the other hand, rescaling, we obtain

$$\int_{\partial B_1} h_r^2 u_r^2 - 2h_r r^\kappa u_r q_\kappa \geq 0.$$

Factoring out $h_r r^\kappa$, we have

$$\int_{\partial B_1} \frac{h_r}{r^\kappa} u_r^2 - 2u_r q_\kappa \geq 0,$$

and passing to the limit over $r = r_j \to 0$

$$-\int_{\partial B_1} q_\kappa^2 \geq 0.$$

Since $q_\kappa \neq 0$, we have thus reached a contradiction. □

Lemma 1.5.3 ($\Sigma_k(u)$ is F_σ)**.** *For any $u \in \mathfrak{S}$, the set $\Sigma_\kappa(u)$ is of type F_σ, i.e., it is a union of countably many closed sets.*

Proof. Let E_j be the set of points $x_0 \in \Sigma_\kappa(u) \cap \overline{B_{1-1/j}}$ such that

$$\tfrac{1}{j}\,\rho^\kappa \leq \sup_{|x-x_0|=\rho} |u(x)| < j\rho^\kappa \tag{1.5.1}$$

for $0 < \rho < 1 - |x_0|$. Note that from Lemmas 1.5.1 and 1.5.2 we have that

$$\Sigma_\kappa(u) = \bigcup_{j=1}^\infty E_j.$$

The lemma will follow once we show that E_j is a closed set. Indeed, if $x_0 \in \overline{E}_j$ then x_0 satisfies (1.5.1) and we need to show only that $x_0 \in \Sigma_\kappa(u)$, or equivalently that $N^{x_0}(0+, u) = \kappa$ by Theorem 1.3.2. Since the function $x \mapsto N^x(0+, u)$ is upper semicontinous, we readily have that $N^{x_0}(0+, u) \geq \kappa$. On the other hand, if $N^{x_0}(0+, u) = \kappa' > \kappa$, we would have

$$|u(x)| < C|x - x_0|^{\kappa'} \quad \text{in } B_{1-|x_0|}(x_0),$$

which would contradict the estimate from below in (1.5.1). Thus $N^{x_0}(0+, u) = \kappa$ and therefore $x_0 \in E_j$. □

Theorem 1.5.4 (Uniqueness of the homogeneous blowup at singular points)**.** *Let $u \in \mathfrak{S}$ and $0 \in \Sigma_\kappa(u)$. Then there exists a unique nonzero $p_\kappa \in \mathfrak{P}_\kappa$ such that*

$$u_r^{(\kappa)}(x) := \frac{u(rx)}{r^\kappa} \to p_\kappa(x).$$

Proof. Let $u_r^{(\kappa)}(x) \to u_0(x)$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ over a certain subsequence $r = r_j \to 0+$. The existence of such limit follows from the growth estimate $|u(x)| \leq C|x|^\kappa$. We call such u_0 a *homogeneous blowup*, in contrast to the blowups that were based on the scaling (1.2.2). Note that Lemma 1.5.2 implies that u_0 in not identically zero. Next, we have for any $r > 0$

$$W_\kappa(r, u_0) = \lim_{r_j \to 0+} W_\kappa(r, u_{r_j}^{(\kappa)}) = \lim_{r_j \to 0+} W_\kappa(rr_j, u) = W_\kappa(0+, u) = 0.$$

In view of Theorem 1.4.1 this implies that the harmonic function u_0 is homogeneous of degree κ. Repeating the arguments in Theorem 1.3.2, we see that u_0 must be a polynomial in $\mathfrak{P}_\kappa$. (The same could be achieved by looking at $N(r, u_0)$).

We now apply Monneau's monotonicity formula to the pair u, u_0. By Theorem 1.4.3 the limit $M_\kappa(0+, u, u_0)$ exists and can be computed by

$$M_\kappa(0+, u, u_0) = \lim_{r_j \to 0+} M_\kappa(r_j, u, u_0) = \lim_{j\to\infty} \int_{\partial B_1} (u_{r_j}^{(\kappa)} - u_0)^2 = 0.$$

In particular, we obtain that

$$\int_{\partial B_1} (u_r^{(\kappa)}(x) - u_0)^2 = M_\kappa(r, u, u_0) \to 0$$

as $r \to 0+$ (not just over $r = r_j \to 0+$!). Thus, if u_0' is a limit of $u_r^{(\kappa)}$ over another sequence $r = r_j' \to 0$, we obtain that

$$\int_{\partial B_1} (u_0' - u_0)^2 = 0.$$

Since both u_0 and u_0' are homogeneous of degree κ, they must coincide in $\mathbb{R}^n$. □

We note explicitly that the conclusion of Theorem 1.5.4 is equivalent to the Taylor expansion

$$u(x) = p_\kappa(x) + o(|x|^\kappa)$$

and therefore it proves the first part of Theorem 1.3.6. The next result is essentially the second part of Theorem 1.3.6.

Theorem 1.5.5 (Continuous dependence of the blowups)**.** *Let $u \in \mathfrak{S}$. For $x_0 \in \Sigma_\kappa(u)$ denote by $p_\kappa^{x_0}$ the blowup of u at x_0 as in Theorem 1.5.4, so that*

$$u(x) = p_\kappa^{x_0}(x - x_0) + o(|x - x_0|^\kappa).$$

Then the mapping $x_0 \mapsto p_\kappa^{x_0}$ from $\Sigma_\kappa(u)$ to $\mathfrak{P}_\kappa$ is continuous. Moreover, for any compact $K \subset \Sigma_\kappa(u) \cap B_1$ there exists a modulus of continuity σ_K, $\sigma_K(0+) = 0$ such that

$$|u(x) - p_\kappa^{x_0}(x - x_0)| \le \sigma_K(|x - x_0|)|x - x_0|^\kappa$$

for any $x_0 \in K$.

Proof. Note that since $\mathfrak{P}_\kappa$ is a convex subset of a finite-dimensional vector space, namely the space of all κ-homogeneous polynomials, all the norms on such space are equivalent. We can then endow $\mathfrak{P}_\kappa$ with the norm of $L^2(\partial B_1)$.

This being said, the proof is similar to that of the last part of the previous theorem. Given $x_0 \in \Sigma_\kappa(u)$ and $\varepsilon > 0$ fix $r_\varepsilon = r_\varepsilon(x_0)$ such that

$$M_\kappa^{x_0}(r_\varepsilon, u, p_\kappa^{x_0}) := \frac{1}{r_\varepsilon^{n-1+2\kappa}} \int_{\partial B_{r_\varepsilon}} (u(x + x_0) - p_\kappa^{x_0})^2 < \varepsilon.$$

There exists $\delta_\varepsilon = \delta_\varepsilon(x_0)$ such that if $x_0' \in \Sigma_\kappa(u)$ and $|x_0' - x_0| < \delta_\varepsilon$, then

$$M_\kappa^{x_0'}(r_\varepsilon, u, p_\kappa^{x_0}) = \frac{1}{r_\varepsilon^{n-1+2\kappa}} \int_{\partial B_{r_\varepsilon}} (u(x + x_0') - p_\kappa^{x_0})^2 < 2\varepsilon.$$

From the monotonicity of the Monneau's functional, we will have that

$$M_\kappa^{x_0'}(r, u, p_\kappa^{x_0}) < 2\varepsilon, \quad 0 < r < r_\varepsilon.$$

Letting $r \to 0$, we will therefore obtain

$$M^{x_0'}(0+, u, p_\kappa^{x_0}) = \int_{\partial B_1} (p_\kappa^{x_0'} - p_\kappa^{x_0})^2 \leq 2\varepsilon.$$

This shows the first part of the theorem.

To show the second part, we notice that we have

$$\begin{aligned}\|u(\cdot + x_0') - p_\kappa^{x_0'}\|_{L^2(\partial B_r)} &\leq \|u(\cdot + x_0') - p_\kappa^{x_0}\|_{L^2(\partial B_r)} + \|p_\kappa^{x_0} - p_\kappa^{x_0'}\|_{L^2(\partial B_r)}\\ &\leq 2(2\varepsilon)^{\frac{1}{2}} r^{\frac{n-1}{2}+\kappa},\end{aligned}$$

for $|x_0' - x_0| < \delta_\varepsilon$, $0 < r < r_\varepsilon$, or equivalently

$$\|w_r^{x_0'} - p_\kappa^{x_0'}\|_{L^2(\partial B_1)} \leq 2(2\varepsilon)^{\frac{1}{2}}, \tag{1.5.2}$$

where

$$w_r^{x_0'}(x) := \frac{u(rx + x_0')}{r^\kappa}.$$

Now, covering the compact $K \subset \Sigma_\kappa(u) \cap B_1$ with finitely many balls $B_{\delta_\varepsilon(x_0^i)}(x_0^i)$ for some $x_0^i \in K$, $i = 1, \ldots, N$, we obtain that (1.5.2) is satisfied for all $x_0' \in K$ with $r < r_\varepsilon^K := \min\{r_\varepsilon(x_0^i) \mid i = 1, \ldots, N\}$. Now notice that

$$w_r^{x_0'} \in \mathfrak{S}, \quad p_\kappa^{x_0'} \in \mathfrak{S}$$

with uniformly bounded $C^{1,\alpha}(\overline{B_1})$ norms. The solutions of the Signorini problem (1.1.1)–(1.1.3) enjoy the uniqueness property in the sense that they coincide if they have the same trace on ∂B_1. Thus, arguing by contradiction and using a compactness argument, we can establish the estimate

$$\|w_r^{x_0'} - p_\kappa^{x_0'}(x)\|_{L^\infty(B_{1/2})} \leq C_\varepsilon$$

for $x_0' \in K$, $0 < r < r_\varepsilon^K$ with $C_\varepsilon \to 0$ as $\varepsilon \to 0$. Clearly, this implies the second part of the theorem. □

We are now ready to prove Theorem 1.3.8 on the structure of the singular set.

Proof of Theorem 1.3.8. First, recall that the equality $\Sigma_\kappa(u) = \Gamma_\kappa(u)$ for $\kappa = 2m$, $m \in \mathbb{N}$, is proved in Theorem 1.3.2.

The proof of the structure of $\Sigma_\kappa^d(u)$ is based on two classical results in analysis: Whitney's extension theorem [Whi34] and the implicit function theorem. This parallels the approach in [Caf98] for the classical obstacle problem.

Step 1: Whitney's extension. Let $K = E_j$ be the compact subset of $\Sigma_\kappa(u)$ defined in the proof of Lemma 1.5.3. Write the polynomials $p_\kappa^{x_0}$ in the expanded form

$$p_\kappa^{x_0}(x) = \sum_{|\alpha|=\kappa} \frac{a_\alpha(x_0)}{\alpha!} x^\alpha.$$

Then the coefficients $a_\alpha(x)$ are continuous on $\Sigma_\kappa(u)$ by Theorem 1.5.5. Moreover, since $u(x) = 0$ on $\Sigma_\kappa(u)$, we have

$$|p_\kappa^{x_0}(x - x_0)| \leq \sigma(|x - x_0|)|x - x_0|^\kappa, \quad \text{for } x, x_0 \in K.$$

For any multi-index α, $|\alpha| \leq \kappa$, define

$$f_\alpha(x) = \begin{cases} 0 & |\alpha| < \kappa \\ a_\alpha(x) & |\alpha| = \kappa, \end{cases} \qquad x \in \Sigma_k(u).$$

We claim that the following compatibility condition is satisfied, which will enable us to apply the Whitney's extension theorem.

Lemma 1.5.6. *For any* $x_0, x \in K$

$$f_\alpha(x) = \sum_{|\beta|\leq\kappa-|\alpha|} \frac{f_{\alpha+\beta}(x_0)}{\beta!}(x-x_0)^\beta + R_\alpha(x,x_0) \tag{1.5.3}$$

with

$$|R_\alpha(x,x_0)| \leq \sigma_\alpha(|x-x_0|)|x-x_0|^{\kappa-|\alpha|}, \tag{1.5.4}$$

where $\sigma_\alpha = \sigma_\alpha^K$ *is a certain modulus of continuity.*

Proof. 1) Consider first the case $|\alpha| = \kappa$. Then we have

$$R_\alpha(x,x_0) = a_\alpha(x) - a_\alpha(x_0)$$

and therefore $|R_\alpha(x,x_0)| \leq \sigma_\alpha(|x-x_0|)$ from continuity of the mapping $x \mapsto p^x$ on K.

2) For $0 \leq |\alpha| < \kappa$ we have

$$R_\alpha(x,x_0) = -\sum_{\gamma>\alpha,|\gamma|=\kappa} \frac{a_\gamma(x_0)}{(\gamma-\alpha)!}(x-x_0)^{\gamma-\alpha} = -\partial^\alpha p_\kappa^{x_0}(x-x_0).$$

Now suppose that there exists no modulus of continuity σ_α such that (1.5.4) is satisfied for all $x_0, x \in K$. Then there exists $\delta > 0$ and a sequence $x_0^i, x^i \in K$ with

$$|x^i - x_0^i| =: \rho_i \to 0$$

such that

$$\Big|\sum_{\gamma>\alpha,|\gamma|=\kappa} \frac{a_\gamma(x_0^i)}{(\gamma-\alpha)!}(x^i-x_0^i)^{\gamma-\alpha}\Big| \geq \delta|x^i-x_0^i|^{\kappa-|\alpha|}. \tag{1.5.5}$$

Consider the rescalings

$$w^i(x) = \frac{u(x_0^i+\rho_i x)}{\rho_i^\kappa}, \quad \xi^i = (x^i-x_0^i)/\rho_i.$$

Without loss of generality we may assume that $x_0^i \to x_0 \in K$ and $\xi^i \to \xi_0 \in \partial B_1$. From Theorem 1.5.5 we have that

$$|w^i(x) - p_\kappa^{x_0^i}(x)| \leq \sigma(\rho_i|x|)|x|^\kappa$$

and therefore $w^i(x)$ converges locally uniformly in $\mathbb{R}^n$ to $p_\kappa^{x_0}(x)$. Further, note that since x^i and x_0^i are from the set $K = E_j$, the inequalities (1.5.1) are satisfied there. Moreover, we also have that similar inequalities are satisfied for the rescaled function w^i at 0 and ξ^i. Therefore, passing to the limit, we obtain that

$$\tfrac{1}{j}\rho^\kappa \leq \sup_{|x-\xi_0|=\rho} p_\kappa^{x_0}(x) \leq j\rho^\kappa, \quad 0<\rho<\infty.$$

This implies that ξ_0 is a point of frequency $\kappa = 2m$ for the polynomial $p_\kappa^{x_0}$ and by Theorem 1.3.2 we have that $\xi_0 \in \Sigma_\kappa(p_\kappa^{x_0})$. In particular,

$$\partial^\alpha p_\kappa^{x_0}(\xi_0) = 0, \quad \text{for } |\alpha| < \kappa.$$

However, dividing both parts of (1.5.5) by $\rho_i^{\kappa-|\alpha|}$ and passing to the limit, we obtain that

$$|\partial^\alpha p_\kappa^{x_0}(\xi_0)| = \Big| \sum_{\gamma>\alpha, |\gamma|=\kappa} \frac{a_\gamma(x_0)}{(\gamma-\alpha)!}(\xi_0)^{\gamma-\alpha} \Big| \geq \delta,$$

a contradiction. □

So in all cases, the compatibility conditions (1.5.4) are satisfied and we can apply Whitney's extension theorem. Thus, there exists a function $F \in C^\kappa(\mathbb{R}^n)$ such that

$$\partial^\alpha F = f_\alpha \quad \text{on } E_j$$

for any $|\alpha| \leq \kappa$.

Step 2: Implicit function theorem. Suppose now $x_0 \in \Sigma_\kappa^d(u) \cap E_j$. Recalling Definition 1.3.7 this means that

$$d = \dim\{\xi \in \mathbb{R}^{n-1} \mid \xi \cdot \nabla_{x'} p_\kappa^{x_0} \equiv 0\}.$$

Then there are $n-1-d$ linearly independent unit vectors $\nu_i \in \mathbb{R}^{n-1}$, $i = 1, \ldots,$ $n-1-d$, such that

$$\nu_i \cdot \nabla_{x'} p_\kappa^{x_0} \neq 0 \quad \text{on } \mathbb{R}^n.$$

This implies that there exist multi-indices β^i of order $|\beta^i| = \kappa - 1$ such that

$$\partial_{\nu_i}(\partial^{\beta^i} p_\kappa^{x_0})(0) \neq 0.$$

This can be written as

$$\partial_{\nu_i}\partial^{\beta^i} F(x_0) \neq 0, \quad i = 1, \ldots, n-1-d. \tag{1.5.6}$$

On the other hand,

$$\Sigma_\kappa^d(u) \cap E_j \subset \bigcap_{i=1}^{n-1-d} \{\partial^{\beta^i} F = 0\}.$$

Therefore, in view of the implicit function theorem, the condition (1.5.6) implies that $\Sigma_\kappa^d(u) \cap E_j$ is contained in a d-dimensional manifold in a neighborhood of x_0. Finally, since $\Sigma_k(u) = \bigcup_{j=1}^\infty E_j$ this implies the statement of the theorem. □

Part 2. Nonzero Thin Obstacle

2.1. Normalization

We now want to study the Signorini problem with a not necessarily zero thin obstacle φ defined on a flat portion of the boundary. More precisely, given a function $\varphi \in C^{k,1}(B_1')$, for some $k \in \mathbb{N}$, we consider the unique minimizer in the Signorini problem in B_1^+, with thin obstacle φ. Such v satisfies

$$\Delta v = 0 \quad \text{in } B_1^+ \tag{2.1.1}$$

$$v - \varphi \geq 0, \quad -\partial_{x_n} v \geq 0, \quad (v-\varphi)\,\partial_{x_n} v = 0 \quad \text{on } B_1' \tag{2.1.2}$$

$$0 \in \Gamma(v) := \partial\{v(\cdot, 0) - \varphi = 0\}, \tag{2.1.3}$$

where (2.1.1) is to be interpreted in the weak sense.

Definition 2.1.1. We say that $v \in C^{1,\alpha}(B_1^+ \cup B_1')$ belongs to the class $\mathfrak{S}^\varphi$ if it satisfies (2.1.1)–(2.1.3).

The basic idea now is considering the difference $u(x', x_n) = v(x', x_n) - \varphi(x')$. The complication is that u is no longer harmonic in B_1^+, but instead satisfies

$$\Delta u = -\Delta_{x'}\varphi.$$

This introduces a certain error in the computations that potentially could prevent us from successfully studying nonregular points. Thus we need a slightly refined argument that will enable us to control the error.

2.1.1. **Subtracting the Taylor polynomial.**

Lemma 2.1.2 (Harmonic extension of homogeneous polynomials)**.** *Let $q_k(x')$ be a homogeneous polynomial of degree k on $\mathbb{R}^{n-1}$. There exists a unique homogeneous polynomial $\tilde{q}_k$ of degree k on $\mathbb{R}^n$ such that*

$$\begin{aligned} \Delta\tilde{q}_k &= 0 \quad \textit{in } \mathbb{R}^n,\\ \tilde{q}_k(x',0) &= q_k(x'), \quad \textit{for any } x'\in\mathbb{R}^{n-1},\\ \tilde{q}_k(x',-x_n) &= \tilde{q}_k(x',x_n) \quad \textit{for any } x'\in\mathbb{R}^{n-1},\ x\in\mathbb{R}. \end{aligned}$$

Proof. 1) *Existence.* In the simplest case when $q_k(x') = x_j^k, \quad j = 1, \dots, n-1$, one can take $\tilde{q}_k(x) = \mathrm{Re}(x_j + i\,x_n)^k$. Arguing in analogy with this situation, for

$$q_k(x') = (e'\cdot x')^k, \quad \text{where } e'\in\mathbb{R}^{n-1},\ |e'| = 1 \tag{2.1.4}$$

one can take

$$\tilde{q}_k(x) = \mathrm{Re}(e'\cdot x' + i\,x_n)^k.$$

Now the existence for an arbitrary polynomial of order k follows from the fact that any homogeneous polynomial of degree k is a linear combination of those of the form (2.1.4).

2) *Uniqueness.* By the linearity of the Laplacian, it is sufficient to show that the only extension of $q_k = 0$ is $\tilde{q}_k = 0$. Note that for any such extension both $\tilde{q}_k$ and $\partial_{x_n}\tilde{q}_\kappa$ will vanish on $\mathbb{R}^{n-1}\times\{0\}$ (the latter following from even symmetry in x_n). Since $\tilde{q}_k$ is also harmonic, by the Cauchy-Kovalevskaya theorem it must vanish identically. □

Assume now that the lower dimensional obstacle is given by $\varphi \in C^{k,1}(B_1')$. Let $Q_k(x')$ be the Taylor polynomial of degree k of φ at the origin , i.e.,

$$\varphi(x') = Q_k(x') + O(|x'|^{k+1}).$$

Moreover, we will also have

$$\Delta_{x'}\varphi(x') = \Delta_{x'}Q_k(x') + O(|x'|^{k-1}).$$

Representing $Q_k = \sum_{m=0}^k q_m$, where q_m are homogeneous polynomials of degree m, by Lemma 2.1.2 we can find a harmonic extension $\tilde{Q}_k$ of Q_k into $\mathbb{R}^n$. For the solution v of (2.1.1)–(2.1.3) consider the difference

$$u(x', x_n) := v(x) - \tilde{Q}_k(x', x_n) - (\varphi(x') - Q_k(x')).$$

It is easy to see that u satisfies

$$|\Delta u| = |\Delta_{x'}(\varphi - Q_k)| \le M|x'|^{k-1} \quad \text{in } B_1^+ \tag{2.1.5}$$

$$u \ge 0, \quad -\partial_{x_n}u \ge 0, \quad u\,\partial_{x_n}u = 0 \quad \text{on } B_1' \tag{2.1.6}$$

$$0 \in \Gamma(u) := \partial\{u(\cdot, 0) = 0\}. \tag{2.1.7}$$

Definition 2.1.3. We say that $u \in C^{1,\alpha}(B_1^+ \cup B_1')$ belongs to the class $\mathfrak{S}_k(M)$ if it satisfies (2.1.5)–(2.1.7) and moreover

$$\|u\|_{C^1(B_1)} \le M.$$

We will use the full notation $\mathfrak{S}_k(M)$ if the value of the constant M is important. In all other cases we will denote this class simply by $\mathfrak{S}_k$.

As before, we may assume that $u \in \mathfrak{S}_k$ is automatically extended to B_1 by even symmetry

$$u(x', -x_n) = u(x', x_n).$$

For this extension, the distributional Laplacian Δu is a sum of a nonpositive measure supported in B_1' and an L^∞ function in B_1. More precisely, integrating by parts in $B_1^\pm$ and using (2.1.5)–(2.1.6), we have

$$|\Delta u| \le M|x'|^{k-1} + 2|\partial_{x_n} u| \mathcal{H}^{n-1}\big|_{B_1'} \quad \text{in } \mathcal{D}'(B_1).$$

2.2. Generalized frequency formula

By allowing nonzero obstacles one sacrifices Almgren's frequency formula in its purest form. However, the following modified version holds. In the case $k = 2$ Theorem 2.2.1 below has first been established by Caffarelli, Salsa, Silvestre [CSS08]. For their purposes they only needed to consider the class $\mathfrak{S}_2$ since it allows to capture the slowest growth rate of the solution at a regular free boundary point and thus establish the optimal regularity. For singular free boundary points, instead, we need to consider the full range of values of k.

Theorem 2.2.1 (Generalized Frequency Formula)**.** *Let $u \in \mathfrak{S}_k(M)$. With $H(r)$ as in* (1.4.1) *there exist $r_M > 0$ and $C_M > 0$ such that*

$$r \mapsto \Phi_k(r, u) := (r + C_M r^2) \frac{d}{dr} \log \max \left\{ H(r), r^{n-1+2k} \right\},$$

is nondecreasing for $0 < r < r_M$.

The rest of this section is devoted to proving Theorem 2.2.1 and is rather technical. The reader might want to skip it, at least in the first reading, and proceed directly to the next section.

2.2.1. **Proof of Theorem 2.2.1.**

In order to prove Theorem 2.2.1 we first establish two auxiliary lemmas. For $u \in \mathfrak{S}_k(M)$, with $D(r)$ and $H(r)$ as in (1.4.1), we also consider the following quantities

$$G(r) := \int_{B_r} u^2, \quad I(r) := \int_{\partial B_r} u u_\nu = \int_{B_r} |\nabla u|^2 + \int_{B_r} u \Delta u. \tag{2.2.1}$$

Lemma 2.2.2. *For $u \in \mathfrak{S}_k$ we have the following identities*

$$\begin{aligned} G'(r) &= H(r),\\ H'(r) &= \frac{n-1}{r}\, H(r) + 2\int_{\partial B_r} u u_\nu,\\ D'(r) &= \frac{n-2}{r}\, D(r) + 2\int_{\partial B_r} u_\nu^2 - \frac{2}{r}\int_{B_r}(x\cdot\nabla u)\Delta u,\\ I'(r) &= \frac{n-2}{r}\, I(r) + 2\int_{\partial B_r} u_\nu^2 - \frac{2}{r}\int_{B_r}(x\cdot\nabla u)\Delta u\\ &\quad - \frac{n-2}{r}\int_{B_r} u\Delta u + \int_{\partial B_r} u\Delta u. \end{aligned}$$

Proof. The proof of these statements is rather standard and is therefore omitted. The reader may consult [GL86], [GL87] for similar computations. □

Remark 2.2.3. Note that both u and $x\cdot\nabla u$ vanish continuously on $\operatorname{supp}\partial_{x_n} u\big|_{B_1'} = \operatorname{supp}\Delta u\big|_{B_1'}$, hence we can discard the integrals of $(x\cdot\nabla u)\Delta u$ and $u\Delta u$ over B_r' and $\partial B_r'$.

Lemma 2.2.4. *For any $u \in \mathfrak{S}_k(M)$ we have the following estimates*

$$\int_{\partial B_r} u^2 \le C_M r\int_{B_r}|\nabla u|^2 + C_M r^{n+2k+1} \tag{2.2.2}$$

$$\int_{B_r} u^2 \le C_M r^2\int_{B_r}|\nabla u|^2 + C_M r^{n+2k+2} \tag{2.2.3}$$

Equivalently, we may write these inequalities as

$$H(r) \le C_M r D(r) + C_M r^{n+2k+1} \tag{2.2.4}$$

$$G(r) \le C_M r^2 D(r) + C_M r^{n+2k+2} \tag{2.2.5}$$

Proof. These inequalities are essentially established in [CSS08]. Below we outline the main steps with references to the corresponding lemmas in [CSS08].

One starts with the well-known trace inequality

$$\int_{\partial B_r}|u(x)-\bar u_r|^2 \le Cr\int_{B_r}|\nabla u|^2, \tag{2.2.6}$$

where

$$\bar u_r = ⨍_{\partial B_r} u.$$

Next, since $u \in \mathfrak{S}_k$, by [CSS08, Lemma 2.9] one has

$$u(0) \ge ⨍_{\partial B_r} u - Cr^{k+1}. \tag{2.2.7}$$

Now, (2.2.6) gives

$$\int_{\partial B_r} u^2 \le Cr\int_{B_r}|\nabla u|^2 + 2\bar u_r\int_{\partial B_r} u.$$

Further, (2.2.7) gives (since $u(0)=0$)

$$⨍_{\partial B_r} u \le Cr^{k+1},$$

which implies

$$\int_{\partial B_r} u^+ \leq \int_{\partial B_r} u^- + Cr^{n+k}.$$

On the other hand,

$$\int_{\partial B_r} u^- \leq Cr^{\frac{n}{2}} \left(\int_{B_r} |\nabla u|^2 \right)^{\frac{1}{2}},$$

see [CSS08, Lemma 2.13]. Hence

$$\int_{\partial B_r} |u| \leq Cr^{\frac{n}{2}} \left(\int_{B_r} |\nabla u|^2 \right)^{\frac{1}{2}} + Cr^{n+k}.$$

Therefore

$$\begin{aligned}\int_{\partial B_r} u^2 &\leq Cr \int_{B_r} |\nabla u|^2 + \frac{C}{r^{n-1}} \left(\int_{\partial B_r} |u| \right)^2 \\ &\leq Cr \int_{B_r} |\nabla u|^2 + Cr^{n+2k+1}.\end{aligned}$$

This proves (2.2.2). Integrating in r, we obtain (2.2.3). □

We are now ready to prove the Generalized Frequency Formula.

Proof of Theorem 2.2.1. 1) First we want to make a remark on the definition of $\Phi_k(r,u)$. The functions $H(r)$ and r^{n-1+2k} are continuously differentiable and therefore the function $\max\{H(r), r^{n-1+2k}\}$ is absolutely continuous or, equivalently, belongs to the Sobolev space $W^{1,1}_{\text{loc}}((0,1))$. It follows that Φ_k is uniquely identified only up to a set of measure zero. The monotonicity of Φ_k should be understood in the sense that there exists a monotone increasing function which equals Φ_k almost everywhere. Therefore, without loss of generality we may assume that

$$\Phi_k(r,u) = (r + C_M r^2) \frac{d}{dr} \log r^{n-1+2k} = (n-1+2k)(1 + C_M r)$$

on $F := \{r \in (0,1) \mid H(r) \leq r^{n-1+2k}\}$ and

$$\Phi_k(r,u) = (r + C_M r^2) \frac{d}{dr} \log H(r) = (r + C_M r^2) \frac{H'(r)}{H(r)}$$

on $U := \{r \in (0,1) \mid H(r) > r^{n-1+2k}\}$. Following an idea introduced in [GL86] we now note that it will be enough to check that $\Phi_k'(r,u) > 0$ in U. Indeed, it is clear that Φ_k is monotone on F and if (r_0, r_1) is a maximal open interval in U, then $r_0, r_1 \in F$ and we will have that

$$\Phi_k(r_0, u) \leq \Phi_k(r_0+, u) \leq \Phi_k(r_1-, u) \leq \Phi_k(r_1, u).$$

Therefore, we will concentrate only on the set U.

2) Now suppose $r \in (0,1)$ is such that $H(r) > r^{n-1+2k}$. Using (2.2.1) and the second identity in Lemma 2.2.2 we find

$$\begin{aligned}\Phi_k(r,u) &= (r + C_M r^2)\frac{H'(r)}{H(r)}\\ &= (r + C_M r^2)\left(\frac{n-1}{r} + 2\frac{I(r)}{H(r)}\right)\\ &= (n-1)(1+C_M r) + 2r(1+C_M r)\frac{I(r)}{H(r)}.\end{aligned}$$

Since $(n-1)(1+C_M r)$ is clearly nondecreasing, it will be enough to show the monotonicity of $r(1+C_M r)\frac{I(r)}{H(r)}$.

From Lemma 2.2.2 we now have

$$\begin{aligned}&\frac{d}{dr}\log\left(r(1+C_M r)\frac{I(r)}{H(r)}\right) = \frac{1}{r} + \frac{C_M}{1+C_M r} + \frac{I'(r)}{I(r)} - \frac{H'(r)}{H(r)}\\ &= \frac{C_M}{1+C_M r} + 2\left\{\frac{\int_{\partial B_r} u_\nu^2}{\int_{\partial B_r} u u_\nu} - \frac{\int_{\partial B_r} u u_\nu}{\int_{\partial B_r} u^2}\right\}\\ &\quad + \frac{-\frac{2}{r}\int_{B_r}(x\cdot\nabla u)\Delta u - \frac{n-2}{r}\int_{B_r} u\Delta u + \int_{\partial B_r} u\Delta u}{\int_{\partial B_r} u u_\nu}.\end{aligned}$$

The expression in curly brackets in the right-hand side of the latter equation is the same as that in the proof of Theorem 1.2.1 and is nonnegative by the Cauchy-Schwarz inequality. We thus obtain

$$\frac{d}{dr}\log\left(r(1+C_M r)\frac{I(r)}{H(r)}\right) \geq \frac{C_M}{1+C_M r} + E(r),$$

where we have let

$$E(r) := \frac{-\dfrac{2}{r}\displaystyle\int_{B_r}(x\cdot\nabla u)\Delta u - \dfrac{n-2}{r}\displaystyle\int_{B_r} u\Delta u + \displaystyle\int_{\partial B_r} u\Delta u}{\displaystyle\int_{\partial B_r} u u_\nu}.$$

This is the error term that derives from the non-vanishing of Δu. Since the first term $C_M/(1+rC_M)$ is greater than a positive constant for small r, to complete the proof of the theorem it will be enough to show that $E(r)$ is bounded below.

3) *Estimating* $E(r)$. We estimate the denominator and the numerator of E separately.

Denominator: Using Cauchy-Schwarz and the inequalities (2.2.4)–(2.2.5), we have

$$\begin{aligned}
\int_{\partial B_r} u\partial_\nu u &= \int_{B_r} |\nabla u|^2 + \int_{B_r} u\Delta u \\
&\geq D(r) - 2\left(\int_{B_r^+} u^2\right)^{\frac{1}{2}} \left(\int_{B_r^+} |\Delta u|^2\right)^{\frac{1}{2}} \\
&\geq D(r) - CG(r)^{\frac{1}{2}} r^{\frac{n}{2}+k-1} \\
&\geq D(r) - C\left(rD(r)^{\frac{1}{2}} + r^{\frac{n}{2}+k+1}\right) r^{\frac{n}{2}+k-1} \\
&\geq D(r) - CD(r)^{\frac{1}{2}} r^{\frac{n}{2}+k} - Cr^{n+2k}.
\end{aligned}$$

Numerator: Again using Cauchy-Schwarz and the inequalities (2.2.4)–(2.2.5), we have

$$\begin{aligned}
\left|\frac{1}{r}\int_{B_r} u\Delta u\right| &\leq \frac{2}{r}\left(\int_{B_r^+} u^2\right)^{\frac{1}{2}} \left(\int_{B_r^+} |\Delta u|^2\right)^{\frac{1}{2}} \\
&\leq CD(r)^{\frac{1}{2}} r^{\frac{n}{2}+k-1} + Cr^{n+2k-1}
\end{aligned}$$

$$\begin{aligned}
\left|\frac{1}{r}\int_{B_r} \Delta u(x\cdot\nabla u)\right| &\leq \frac{2}{r}\left(\int_{B_r^+} |\nabla u|^2|x|^2\right)^{\frac{1}{2}} \left(\int_{B_r^+} |\Delta u|^2\right)^{\frac{1}{2}} \\
&\leq CD(r)^{\frac{1}{2}} r^{\frac{n}{2}+k-1}
\end{aligned}$$

$$\begin{aligned}
\left|\int_{\partial B_r} u\Delta u\right| &\leq 2\left(\int_{\partial B_r^+} u^2\right)^{\frac{1}{2}} \left(\int_{\partial B_r^+} |\Delta u|^2\right)^{\frac{1}{2}} \\
&\leq CH(r)^{\frac{1}{2}} r^{\frac{n-1}{2}+k-1} \\
&\leq C\left(r^{\frac{1}{2}}D(r)^{\frac{1}{2}} + r^{\frac{n}{2}+k+\frac{1}{2}}\right) r^{\frac{n-1}{2}+k-1} \\
&\leq CD(r)^{\frac{1}{2}} r^{\frac{n}{2}+k-1} + Cr^{n+2k-1}.
\end{aligned}$$

Now, collecting the estimates on the denominator and the numerator of $E(r)$ we obtain

$$|E(r)| \leq C\frac{D(r)^{\frac{1}{2}} r^{\frac{n}{2}+k-1} + r^{n+2k-1}}{D(r) - CD(r)^{\frac{1}{2}} r^{\frac{n}{2}+k} - Cr^{n+2k}}$$

Finally, recall that we assume

$$H(r) > r^{n-1+2k}.$$

Then by Lemma 2.2.4 we also have

$$D(r) > c\, r^{n-2+2k}.$$

The latter inequality now implies that $|E(r)|$ is uniformly bounded for sufficiently small r. This completes the proof of the theorem. □

2.3. Growth near the free boundary

To get a better sense of the relation between the frequency function in Theorem 1.2.1, and the generalized frequency in Theorem 2.2.1 above, we note that for $u \in \mathfrak{S}_k$, we have

$$\Phi_k(r,u) = (1 + C_M r)(n - 1 + 2N(r,u)) \quad \text{if } H(r) > r^{n-1+2k}.$$

So one expects $\Phi_k(0+,u)$ to behave similarly to $n-1+2N(0+,u)$.

Lemma 2.3.1 (Consistency of $\Phi_k(0+,u)$)**.** *Let $u \in \mathfrak{S}_k$. For any m such that $2 \le m \le k$ one has*

$$\Phi_m(0+,u) = \min\{\Phi_k(0+,u), n-1+2m\}.$$

If $m = k$ we obtain in particular,

$$\Phi_k(0+,u) \le n-1+2k.$$

Proof. 1) We start with the latter inequality. Let κ be such that

$$\Phi_k(0+,u) = n-1+2\kappa.$$

We want to show that $\kappa \le k$. Observe that, in general, if $H(r) < r^{n-1+2k}$ along a sequence $r = r_j \to 0+$, then we must have $\Phi_k(0+,u) = n-1+2k$. Therefore if $\kappa \ne k$, we must have for small r

$$H(r) \ge r^{n-1+2k}, \quad \Phi_k(r,u) = (r + C\,r^2)\frac{H'(r)}{H(r)}.$$

Assume now $\kappa > k$. Fix some $\kappa' \in (k,\kappa)$. Then for small enough $0 < r \le r_0$

$$r\frac{H'(r)}{H(r)} \ge n-1+2\kappa'.$$

Dividing by r and integrating from r to r_0, we obtain

$$\log\frac{H(r_0)}{H(r)} \ge (n-1+2\kappa')\log\frac{r_0}{r},$$

which gives

$$H(r) \le C r^{n-1+2\kappa'}.$$

This, however, contradicts the lower bound $H(r) \ge r^{n-1+2k}$. We must therefore have $\kappa \le k$, and this establishes the second part of the lemma.

2) For the first part of the lemma we need to show that if

$$\Phi_m(0+,u) = n-1+2\mu$$

then

$$\mu = \min\{\kappa, m\}.$$

We consider two possibilities:

a) $\kappa < m$. Fix $\mu' \in (\kappa, m)$. Since $\kappa < k$, we must have $H(r) \ge r^{n-1+2k}$ for small r and since $\kappa < \mu'$, we must have

$$r\frac{H'(r)}{H(r)} \le n-1+2\mu'$$

for $0 < r < r_0$. Integrating, we obtain that

$$H(r) \ge H(r_0)\left(\frac{r}{r_0}\right)^{n-1+2\mu'} = c\, r^{n-1+2\mu'}$$

for $0 < r < r_0$. In particular, $H(r) > r^{n-1+2m}$ for sufficiently small r and therefore

$$\Phi_m(r,u) = (r + C_m r^2)\frac{H'(r)}{H(r)} = \frac{1 + C_m r}{1 + C_k r}\,\Phi_k(r,u).$$

Hence $\mu = \kappa$ in this case.

b) $\kappa \geq m$. We need to show that $\mu = m$ in this case. In general, we know that $\mu \leq m$ from part 1) above, so arguing by contradiction, assume $\mu < m$. Fix $\mu' \in (\mu, m)$. Then similarly to the arguments above, we will have

$$r\frac{H'(r)}{H(r)} < n - 1 + 2\mu'$$

and consequently there exists $c > 0$ such that

$$H(r) > c\, r^{n-1+2\mu'},$$

for small $0 < r < r_0$. But then again $H(r) > r^{n-1+2m} \geq r^{n-1+2k}$ and therefore

$$\Phi_m(r,u) = \frac{1 + C_m r}{1 + C_k r}\,\Phi_k(r,u)$$

which again implies $\mu = \kappa$. However, as $\mu < m \leq \kappa$ this is not possible. This contradiction proves that $\mu = m$ in this case. □

Lemma 2.3.2 (Minimal and maximal frequency)**.** *Let $u \in \mathfrak{S}_k$ with $\Phi_k(0+,u) = n - 1 + 2\kappa$, then one has*

$$2 - \frac{1}{2} \leq \kappa \leq k.$$

Moreover, one has that either

$$\kappa = 2 - \frac{1}{2} \quad \text{or} \quad 2 \leq \kappa \leq k.$$

Proof. For $k = 2$, it has been proved in [CSS08] that either $\kappa = 2 - \frac{1}{2}$ or $\kappa \geq 2$. The same statement is also true for all $k \geq 2$ from the identity

$$\Phi_2(0+,u) = \min\{\Phi_k(0+,u), 2\},$$

which is a particular case of Lemma 2.3.1 relating values of $\Phi_k(0+,u)$ for different k. The upper bound is also contained in Lemma 2.3.1. □

Lemma 2.3.3 (Growth near the free boundary)**.** *Let $u \in \mathfrak{S}_k(M)$ and suppose that $\Phi_k(0+,u) = n - 1 + 2\kappa$ with $\kappa \leq k$. Then*

$$H(r) = \int_{\partial B_r} u^2 \leq C_M r^{n-1+2\kappa}$$

$$G(r) = \int_{B_r} u^2 \leq C_M r^{n+2\kappa}$$

$$D(r) = \int_{B_r} |\nabla u|^2 \leq C_M r^{n-2+2\kappa}$$

for $0 < r < 1/2$.

Proof. We first prove the estimate for $H(r)$. The estimate is automatically satisfied for values r such that $H(r) \leq r^{n-1+2k}$. Consider now a maximal open interval (r_1, r_0) in $(0, 1/2)$ where $H(r) > r^{n-1+2k}$. Then either $H(r_0) = r_0^{n-1+2k}$ or $r_0 = 1/2$. In both cases $H(r_0) \leq M r_0^{n-1+2k}$.

Further, we have

$$(r+Cr^2)\frac{H'(r)}{H(r)} \geq n-1+2\kappa, \quad r_1<r<r_0.$$

Dividing both sides by $r+Cr^2$ and integrating from r to r_0, we obtain

$$\begin{aligned}\log\frac{H(r_0)}{H(r)} &\geq (n-1+2\kappa)\int_r^{r_0}\frac{ds}{s(1+Cs)}\\ &= (n-1+2\kappa)\log\frac{r_0/(1+Cr_0)}{r/(1+Cr)}.\end{aligned}$$

Exponentiation gives

$$H(r) \leq C_M r^{n-1+2\kappa}, \quad r_1<r<r_0.$$

This proves the growth estimate for $H(r)$ and, after integration, for $G(r)$.

To estimate $D(r)$ we note that u satisfies the energy inequality

$$\int_{B_{r/2}}|\nabla u|^2 \leq \frac{C}{r^2}\int_{B_r^+}u^2 + Cr^2\int_{B_r^+}(\Delta u)^2,$$

which is proved exactly as the standard energy inequality in the full ball B_r (the boundary term on B'_r vanishes since $u\,\partial_{x_n}u=0$ there). This gives

$$D(r/2) \leq Cr^{n+2\kappa-2} + Cr^{n+2k} \leq Cr^{n+2\kappa-2}.$$

□

2.3.1. **Optimal regularity.** Combining Lemmas 2.3.2 and 2.3.3 with the results of Caffarelli, Salsa, and Silvestre [CSS08], we obtain the following information about the optimal (slowest possible) growth near the origin for a function $u \in \mathfrak{S}_k$.

Proposition 2.3.4 (Optimal growth)**.** *Let $u \in \mathfrak{S}_k$, then*

$$\sup_{B_r}|u| \leq C\,r^{3/2}. \quad \square$$

Proposition 2.3.4 in turn leads to the optimal regularity of the solutions of the Signorini problem (2.1.1)–(2.1.3).

Theorem 2.3.5 (Optimal regularity)**.** *Let $v \in \mathfrak{S}^\varphi$ with $\varphi \in C^{2,1}(B_1)$, then $v \in C^{1,\frac12}_{\mathrm{loc}}(B_1^\pm \cup B'_1)$.* □

2.4. Blowups

The generalized frequency formula in Theorem 2.2.1 above allows to study the blowups. However, the situation is much subtler than in the case of the zero obstacle. For $u \in \mathfrak{S}_k$ and $r>0$ we consider the rescalings u_r introduced in (1.2.2). If it happens that $H(r)$ converges to 0 faster than r^{n-1+2k}, then $\Phi_k(r,u)$ simply equals $(1+C_Mr)(n-1+2k)$ for small $r>0$, which does not help to control the Dirichlet integral of u_r on B_1. Thus, we don't know if the blowups exist in that case. On the other hand, the functional $\Phi_k(r,u)$ does contain enough information to establish the uniform estimates for u_r if $\Phi_k(0+,u)<n-1+2k$.

Lemma 2.4.1 (Uniform bounds of rescalings)**.** *Let $u \in \mathfrak{S}_k$ and suppose that $\Phi_k(0+,u)=n-1+2\kappa$ with $\kappa<k$. Then there exists a sufficiently small $r_0(u)>0$ such that the family $\{u_r\}_{0<r<r_0(u)}$ is uniformly bounded in $W^{1,2}(B_1)\cap C^1_{\mathrm{loc}}(B_1)$.*

Proof. We may assume that $H(r) > r^{n-1+2k}$ for $0 < r < r_0$ and therefore the inequality $\Phi_k(r,u) \le \Phi_k(r_0,u)$ will imply that

$$r\,\frac{H'(r)}{H(r)} \le C, \qquad 0<r<r_0.$$

Using the formula for $H'(r)$ in Lemma 2.2.2, we have

$$(n-1) + 2r\,\frac{\int_{B_r}|\nabla u|^2 + \int_{B_r} u\Delta u}{\int_{\partial B_r} u^2} \le C, \qquad 0<r<r_0.$$

From this inequality we obtain for the rescalings u_r

$$\int_{B_1}|\nabla u_r|^2 \le C - r\frac{\int_{B_r} u\Delta u}{\int_{\partial B_r} u^2}, \qquad 0<r<r_0.$$

The second term in the right hand side can be controlled as follows.

$$\left|\int_{B_r} u\Delta u\right| \le \left(\int_{B_r} u^2\right)^{1/2}\left(\int_{B_r\setminus B_r'}|\Delta u|^2\right)^{1/2} \le C\,G(r)^{\frac12} r^{\frac n2+k-1}$$
$$\le Cr^{\frac n2+\kappa} r^{\frac n2+k-1} = Cr^{n-1+\kappa+k}$$

On the other hand, for any $\kappa' > \kappa$, we have that for sufficiently small r

$$\int_{\partial B_r} u^2 > c\,r^{n-1+2\kappa'},$$

for some $c>0$, see the arguments in the proof of Lemma 2.3.1. This implies that

$$\left|\frac{\int_{B_r} u\Delta u}{\int_{\partial B_r} u^2}\right| \le Cr^{\kappa+k-2\kappa'} \le C,$$

provided we choose

$$\kappa < \kappa' < \frac{\kappa+k}{2}.$$

We thus obtain that

$$\int_{B_1}|\nabla u_r|^2 \le C, \qquad 0<r<r_0.$$

Together with

$$\int_{\partial B_1} u_r^2 = 1,$$

this gives the uniform boundedness of $\{u_r\}$ in $W^{1,2}(B_1)$.

Next, to see the boundedness in C^1_{loc}, notice that

$$|\Delta u_r(x)| = |\Delta u(rx)|\,\frac{r^2 r^{\frac{n-1}{2}}}{H(r)^{\frac12}} \le M|x'|^{k-1}\frac{r^{k+1} r^{\frac{n-1}{2}}}{H(r)^{\frac12}} \le M\,r|x'|^{k-1} \quad \text{in } B_1\setminus B_1'$$

if one uses the bound $H(r) > r^{n-1+2k}$. Thus, we obtain that u_r is bounded in $C^{1,\alpha}_{\text{loc}}(B_1^\pm \cup B_1')$, see e.g. [Caf79]. □

Remark 2.4.2. Although the extremal case $\kappa = k$ is not covered by Lemma 2.4.1, it should be considered as a natural limitation that comes from having assumed that the thin obstacle is in the class $C^{k,1}$. Indeed, if more regularity of the thin obstacle were assumed, then the consistency Lemma 2.3.1 would allow to study the blowups in the case $\kappa = k$ as well.

At this point, using Lemma 2.4.1 we see that, under its assumptions, there exists a subsequence $r_j \to 0+$ such that

$$\begin{aligned} u_{r_j} &\to u_0 \quad \text{in } W^{1,2}(B_1)\\ u_{r_j} &\to u_0 \quad \text{in } L^2(\partial B_1)\\ u_{r_j} &\to u_0 \quad \text{in } C^1_{\mathrm{loc}}(B_1^\pm \cup B_1'). \end{aligned} \tag{2.4.1}$$

We call such u_0 a *blowup* of u at the origin.

Proposition 2.4.3 (Homogeneity of blowups)**.** *Let $u \in \mathfrak{S}_k$ and $\Phi_k(0+, u) = n - 1 + 2\kappa$ with $\kappa < k$. Then every blowup u_0 is homogeneous of degree κ, $u_0 \not\equiv 0$, and satisfies* (1.1.1)–(1.1.2) (*i.e. u_0 solves the Signorini problem with zero obstacle*)*.*

Proof. First notice that u_0 satisfies (1.1.1)–(1.1.2). Indeed, this follows from from the C^1_{loc} convergence of $u_{r_j} \to u_0$ on $B_1^\pm \cup B_1'$ and the estimate

$$|\Delta u_r(x)| \le Mr|x'|^{k-1}, \quad \text{in } B_1 \setminus B_1'$$

that we established in the proof of Lemma 2.4.1.

Next, since $\kappa < k$, the fact that $\Phi_k(0+, u) = n - 1 + 2\kappa$ is equivalent to

$$\lim_{r\to 0+} r\,\frac{H'(r)}{H(r)} = n - 1 + 2\kappa.$$

Arguing as in the proof of Lemma 2.4.1, this relation can be reduced to

$$\lim_{r\to 0+} \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2} = \kappa,$$

or in other words

$$N(0+, u) = \kappa.$$

But then we obtain for any $0 < \rho < 1$

$$N(\rho, u_0) = \lim_{r_j\to 0+} N(\rho, u_{r_j}) = \lim_{r_j\to 0+} N(\rho r_j, u) = \kappa.$$

This implies that u_0 is homogeneous of degree κ. Finally, u_0 does not vanish identically since the convergence $u_{r_j} \to u_0$ in $L^2(\partial B_1)$ and the equality $\int_{\partial B_1} u_{r_j}^2 = 1$ imply $\int_{\partial B_1} u_0^2 = 1$. □

2.5. The free boundary

Suppose now we have a solution v of the Signorini problem (2.1.1)–(2.1.3) with $\varphi \in C^{k,1}(B')$. If $Q_k(x')$ and $\tilde{Q}_k(x)$ are the k-th Taylor polynomial of φ and its symmetric harmonic extension to $\mathbb{R}^n$, then

$$u_k(x) := v(x) - \tilde{Q}_k(x) - (\varphi(x') - Q_k(x')) \in \mathfrak{S}_k. \tag{2.5.1}$$

More generally, for $x_0 \in \Gamma(v)$ we define

$$u_k^{x_0}(x) := v(x + x_0) - \tilde{Q}_k^{x_0}(x) - (\varphi(x' + x_0) - Q_k^{x_0}(x')) \tag{2.5.2}$$

where $Q_k^{x_0}$ is the k-th Taylor polynomial of $\varphi(\cdot + x_0)$. The functions $u_k^{x_0}$ will satisfy the conditions (2.1.5)–(2.1.7) but only in a smaller ball $B_{1-|x_0|}$ instead of the full ball B_1. So technically speaking $u_k^{x_0}$ are not in $\mathfrak{S}_k$, however, most of the time the results for class $\mathfrak{S}_k$ can be used with insignificant or no modification for functions $u_k^{x_0}$. In particular, $\Phi_k(r, u_k^{x_0})$ will be monotone for $0 < r < 1 - |x_0|$ and we can use the value $\Phi_k(0+, u_k)$ to classify the free boundary points.

Definition 2.5.1. For $v \in \mathfrak{S}^\varphi$ we say that $0 \in \Gamma^{(k)}_\kappa(v)$ for $2 - \frac{1}{2} \le \kappa \le k$ if and only if $\Phi_k(0+, u_k) = n - 1 + 2\kappa$. More generally, we define

$$\Gamma^{(k)}_\kappa(v) := \{x_0 \in \Gamma(v) \mid \Phi_k(0+, u_k^{x_0}) = n - 1 + 2\kappa\}.$$

The next lemma shows how this classification of points changes with k.

Lemma 2.5.2. *Let $v \in \mathfrak{S}^\varphi$ with $\varphi \in C^{k,1}(B'_1)$. Then for any $2 \le m \le k$ one has*

$$\Gamma^{(m)}_\kappa(v) = \Gamma^{(k)}_\kappa(v) \quad \text{for } \kappa < m$$
$$\Gamma^{(m)}_m(v) = \bigcup_{\kappa \ge m} \Gamma^{(k)}_\kappa(v).$$

Proof. This lemma is a direct consequence of (in fact, it is equivalent to) the consistency Lemma 2.3.1. Let $u_k \in \mathfrak{S}_k$ be as in (2.5.1) and $u_m \in \mathfrak{S}_m$ be the function corresponding to an integer $2 \le m \le k$. Then $u_m - u_k = o(|x|^m)$ and therefore

$$\Phi_m(0+, u_m) = \Phi_m(0+, u_k) = \min\{\Phi_k(0+, u_k), n - 1 + 2m\}.$$

In fact, to establish the former equality, one has to argue similarly to the proof of Lemma 2.3.1 and just notice that

$$\int_{\partial B_r} u_m^2 > C\, r^{n-1+2\mu'} \iff \int_{\partial B_r} u_k^2 > C\, r^{n-1+2\mu'}$$

whenever $\mu' < m$. We leave the details to the reader. □

Remark 2.5.3. Thanks to Lemma 2.5.2 one can define the sets

$$\Gamma_\kappa(v) := \Gamma^{(m)}_\kappa(v), \quad \text{for } \kappa < m \le k,$$

and the latter definition will not depend on the choice of m. On the other hand, the classification obtained by using the functional Φ_k could be viewed, loosely speaking, as a "truncation" of a possibly finer classification of points. In particular, the set $\Gamma^{(k)}_k$ can be considered as the bulk of points of frequencies $\kappa \ge k$. More precisely, if one knows higher $C^{k',1}$ regularity of the thin obstacle φ with $k' > k$, then $\Gamma^{(k)}_k$ is refined into the union of $\Gamma^{(k')}_\kappa$ with $k \le \kappa \le k'$.

As we have already mentioned in the case of the zero thin obstacle, the sets $\Gamma_\kappa(v)$ are nonempty only for specific values of κ, see Remarks 1.2.7 and 1.2.8. In higher dimensions the only information known is the one contained in Lemma 2.3.2, i.e.

$$\kappa = 2 - \frac{1}{2}, \quad \text{or} \quad \kappa \ge 2.$$

Definition 2.5.4 (Regular points)**.** For $v \in \mathfrak{S}^\varphi$ with $\varphi \in C^{2,1}(B'_1)$ the free boundary point $x_0 \in \Gamma(v)$ is called *regular* if $x_0 \in \Gamma_{2-\frac{1}{2}}(v)$.

It is easy to see that the mapping $x_0 \mapsto \Phi_k(0+, u_k^{x_0})$ is upper semicontinuous, and since the value $\kappa = 2 - \frac{1}{2}$ is isolated, one immediately obtains that $\Gamma_{2-\frac{1}{2}}(v)$ is a relatively open subset of $\Gamma(v)$. Furthermore, the following theorem has been established by Caffarelli, Salsa, and Silvestre [CSS08].

Theorem 2.5.5 (Regularity of the regular set)**.** *Let $v \in \mathfrak{S}^\varphi$ with $\varphi \in C^{2,1}$. Then $\Gamma_{2-\frac{1}{2}}(v)$ is locally a $C^{1,\alpha}$-regular $(n-2)$-dimensional surface.*

2.6. Singular set: statement of main results

Similarly to the case of the zero obstacle, in this section we state our main results on the structure of the singular set of the free boundary. The proofs will be given in Section 2.8. We begin with the relevant definition.

Definition 2.6.1 (Singular points)**.** Let $v \in \mathfrak{S}^\varphi$. We say that $x_0 \in \Gamma(v)$ is a *singular free boundary point* if

$$\lim_{r\to 0+} \frac{\mathcal{H}^{n-1}(\Lambda(v)\cap B_r')}{\mathcal{H}^{n-1}(B_r')} = 0.$$

We denote the set of singular points by $\Sigma(v)$. If the thin obstacle φ is $C^{k,1}$ regular and $\kappa < k$ then we also define

$$\Sigma_\kappa(v) := \Gamma_\kappa(v) \cap \Sigma(v).$$

It will be convenient to abuse the notation and write $0 \in \Sigma_\kappa(u)$ for $u \in \mathfrak{S}_k$, whenever $\Lambda(u)$ satisfies a vanishing condition similar to that for $\Lambda(v)$ in Definition 2.6.1 above.

We start with a characterization of singular points in terms of blowups at the generalized frequency. In particular, we show that

$$\Sigma_\kappa(v) = \Gamma_\kappa(v), \quad \text{for } \kappa = 2m < k,\ m \in \mathbb{N}.$$

Theorem 2.6.2 (Characterization of singular points)**.** *Let $u \in \mathfrak{S}_k$ and $0 \in \Gamma_\kappa(u)$ for $\kappa < k$. Then the following statements are equivalent:*

(i) $0 \in \Sigma_\kappa(u)$
(ii) *any blowup of u at the origin is a nonzero homogeneous polynomial p_κ of degree κ from the class $\mathfrak{P}_\kappa$, i.e.*
$$\Delta p_\kappa = 0, \quad p_\kappa(x',0) \geq 0, \quad p_\kappa(x',-x_n) = p_\kappa(x',x_n)$$
(iii) *$\kappa = 2m$ for some $m \in \mathbb{N}$.*

Proof. The proof is a minor modification of that of Theorem 1.3.2 and is therefore omitted. □

The next result is the key step in the study of the singular set.

Theorem 2.6.3 (κ-differentiability at singular points)**.** *Let $u \in \mathfrak{S}_k$ and $0 \in \Sigma_\kappa(u)$ for $\kappa = 2m < k$, $m \in \mathbb{N}$. There exists a nonzero $p_\kappa \in \mathfrak{P}_\kappa$ such that*

$$u(x) = p_\kappa(x) + o(|x|^\kappa).$$

Moreover, if $v \in \mathfrak{S}^\varphi$ with $\varphi \in C^{k,1}(B_1')$, $x_0 \in \Sigma_\kappa(v)$ and $u_k^{x_0}$ is obtained as in (2.5.2)*, then in the Taylor expansion*

$$u_k^{x_0}(x) = p_\kappa^{x_0}(x) + o(|x|^\kappa)$$

the mapping $x_0 \mapsto p_\kappa^{x_0}$ from $\Sigma_\kappa(v)$ to $\mathfrak{P}_\kappa$ is continuous.

Definition 2.6.4 (Dimension at the singular point)**.** For $v \in \mathfrak{S}^\varphi$ and a singular point $x_0 \in \Sigma_k(v)$ we denote

$$d_\kappa^{x_0} := \dim\{\xi \in \mathbb{R}^{n-1} \mid \xi \cdot \nabla_{x'} p_\kappa^{x_0}(x',0) = 0 \text{ for every } x' \in \mathbb{R}^{n-1}\}$$

the degree of degeneracy of the polynomial $p_\kappa^{x_0}$, which we call the dimension of $\Sigma_\kappa(v)$ at x_0. Note that since $p_\kappa^{x_0} \not\equiv 0$ on $\mathbb{R}^{n-1}\times\{0\}$ one has

$$0 \leq d_\kappa^{x_0} \leq n-2.$$

Then for $d = 0, 1, \ldots, n-2$ define

$$\Sigma_\kappa^d(v) := \{x_0 \in \Sigma_\kappa(u) \mid d_\kappa^{x_0} = d\}.$$

Theorem 2.6.5 (Structure of the singular set)**.** *Let $v \in \mathfrak{S}^\varphi$ with $\varphi \in C^{k,1}(B_1')$. Then $\Sigma_\kappa(v) = \Gamma_\kappa(v)$ for $\kappa = 2m < k$, $m \in \mathbb{N}$, and every set $\Sigma_\kappa^d(v)$ for $d = 0, 1, \ldots, n-2$, is contained in a countable union of d-dimensional C^1 manifolds.*

2.7. Weiss and Monneau type monotonicity formulas

The main tools in the proof of Theorems 2.6.3 and 2.6.5 are Weiss and Monneau type monotonicity formulas, similar to those in Section 1.4.

2.7.1. **Weiss type monotonicity formulas.**

Theorem 2.7.1 (Weiss type Monotonicity Formula)**.** *Let $u \in \mathfrak{S}_k(M)$ and suppose that $0 \in \Gamma_\kappa^{(k)}(u)$ for $\kappa \le k$. There exist $r_M > 0$ and $C_M \ge 0$ such that*

$$\begin{aligned} W_\kappa(r,u) &:= \frac{1}{r^{n-2+2\kappa}} \int_{B_r} |\nabla u|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r} u^2 \\ &= \frac{1}{r^{n-2+2\kappa}} D(r) - \frac{\kappa}{r^{n-1+2\kappa}} H(r). \end{aligned}$$

satisfies

$$\frac{d}{dr} W_\kappa(r) \ge -C_M \quad \text{for } 0 < r < r_M.$$

Proof. Proof by a direct computation. Using Lemmas 2.2.2 and 2.3.3, we obtain

$$\begin{aligned} \frac{d}{dr} W_\kappa(u,r) &= \frac{1}{r^{n-2+2\kappa}} \left\{ D'(r) - \frac{n-2+2\kappa}{r} D(r) - \frac{\kappa}{r} H'(r) + \frac{\kappa(n-1+2\kappa)}{r^2} H(r) \right\} \\ &= \frac{2}{r^{n-2+2\kappa}} \left\{ \int_{\partial B_r} (\partial_\nu u)^2 - \frac{\kappa}{r} D(r) - \frac{\kappa}{r} \int_{\partial B_r} u \partial_\nu u + \frac{\kappa^2}{r^2} \int_{\partial B_r} u^2 \right. \\ &\qquad \left. - \frac{1}{r} \int_{B_r} \Delta u (\nabla u \cdot x) \right\} \\ &= \frac{2}{r^{n-2+2\kappa}} \left\{ \int_{\partial B_r} (\partial_\nu u)^2 - \frac{2\kappa}{r} \int_{\partial B_r} u \partial_\nu u + \frac{\kappa^2}{r^2} \int_{\partial B_r} u^2 \right. \\ &\qquad \left. + \frac{\kappa}{r} \int_{B_r} u \Delta u - \frac{1}{r} \int_{B_r} \Delta u (\nabla u \cdot x) \right\} \\ &= \frac{2}{r^{n-2+2\kappa}} \left\{ \frac{1}{r^2} \int_{\partial B_r} (x \cdot \nabla u - \kappa u)^2 - \frac{1}{r} \int_{B_r} \Delta u (x \cdot \nabla u - \kappa u) \right\} \\ &\ge -\frac{2}{r^{n-1+2\kappa}} \int_{B_r} \Delta u (x \cdot \nabla u - \kappa u) \\ &\ge -\frac{C}{r^{n-1+2\kappa}} \left(\int_{B_r^+} (\Delta u)^2 \right)^{\frac{1}{2}} \left(\int_{B_r^+} (x \cdot \nabla u - \kappa u)^2 \right)^{\frac{1}{2}} \\ &\ge -C \frac{r^{\frac{n}{2}+k-1} r^{\frac{n}{2}+\kappa}}{r^{n-1+2\kappa}} = -C r^{k-\kappa} \ge -C, \end{aligned}$$

where in the step next to the last we have used the estimates in Lemma 2.3.3. □

2.7.2. **Monneau type monotonicity formulas.**

Theorem 2.7.2 (Monneau type Monotonicity Formula)**.** *Let $u \in \mathfrak{S}_k(M)$ and suppose that $0 \in \Sigma_\kappa(u)$ with $\kappa = 2m < k$, $m \in \mathbb{N}$. For any $p_\kappa \in \mathfrak{P}_\kappa$ there exist $r_M > 0$ and $C_M \geq 0$ such that*

$$M_\kappa(r, u, p_\kappa) = \frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} (u - p_\kappa)^2$$

satisfies

$$\frac{d}{dr} M_\kappa(r, u, p_\kappa) \geq -C_M \left(1 + \|p_\kappa\|_{L^2(B_1)}\right) \quad \text{for } 0 < r < r_M.$$

Proof. First note that if $0 \in \Gamma_\kappa^{(k)}(u)$ for $\kappa < k$, then

$$W_\kappa(0+, u) = 0.$$

Indeed, using Lemma 2.2.2, we represent

$$\begin{aligned} W_\kappa(r, u) &= \frac{1}{2\, r^{n-1+2\kappa}} \left(rH'(r) - (n-1+2\kappa)H(r) - 2r \int_{B_r} u\Delta u \right) \\ &= \frac{H(r)}{2\, r^{n-1+2\kappa}} \left(r\frac{H'(r)}{H(r)} - (n-1+2\kappa) \right) - \frac{\int_{B_r} u\Delta u}{r^{n-2+2\kappa}}. \end{aligned}$$

Now the identity $W_\kappa(0+, u) = 0$ follows from the following facts:

(i) $r\dfrac{H'(r)}{H(r)} \to n-1+2\kappa$, since $\kappa < k$,

(ii) $\dfrac{H(r)}{r^{n-2+2\kappa}}$ is bounded, in view of the growth estimate in Lemma 2.3.3,

(iii) $\dfrac{\left|\int_{B_r} u\Delta u\right|}{r^{n-1+2\kappa}} \leq C\, r^{k+1-\kappa}$, by the arguments in the proof of Lemma 2.4.1.

Next, we also observe that $W_\kappa(r, p_\kappa) = 0$ for any $p_\kappa \in \mathfrak{P}_\kappa$. Setting $w = u - p_\kappa$, and repeating the the computations in the proof of Theorem 1.4.3, we then obtain

$$W_\kappa(r, u) = \frac{1}{r^{n-2+2\kappa}} \int_{B_r} (-w\Delta w) + \frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} w(x\nabla w - \kappa w)$$

and

$$\frac{d}{dr} M_\kappa(r, u, p_\kappa) = \frac{2}{r^{n+2\kappa}} \int_{\partial B_r} w(x \cdot \nabla w - \kappa w).$$

This gives

$$\frac{d}{dr} M_\kappa(r, u, p_\kappa) = \frac{2}{r}\, W_\kappa(r, u) + \frac{2}{r^{n-1+2\kappa}} \int_{B_r} w\Delta w.$$

For the first term in the right-hand side note that by Theorem 2.7.1 we obtain

$$W_\kappa(r, u) = W_\kappa(r, u) - W_\kappa(0+, u) \geq -C_M r.$$

For the second term, note that $w\Delta w$ is a nonnegative measure on B'_r. Off B'_r, we can control $w\Delta w = w\Delta u$ by the Cauchy-Schwarz inequality. Hence,

$$\begin{aligned} \frac{d}{dr} M_\kappa(r, u, p_\kappa) &\geq -C_M - \frac{2}{r^{n-1+2\kappa}} \left(\int_{B_r^+} w^2 \right)^{\frac{1}{2}} \left(\int_{B_r^+} (\Delta u)^2 \right)^{\frac{1}{2}} \\ &\geq -C_M - C_M \left(C_M + \|p_\kappa\|_{L^2(B_1)} \right) r^{k-\kappa} \\ &\geq -C_M \left(1 + \|p_\kappa\|_{L^2(B_1)} \right). \end{aligned}$$

□

2.8. Singular set: proofs

In this section we prove Theorems 2.6.3 and 2.6.5. The structure of the proofs is essentially the same as for their counterparts in the zero obstacle case, see Theorems 1.3.6 and 1.3.8. The only difference is that they based on the results that have been so far developed for the case of a nonzero obstacle. We start with the nondegeneracy lemma, similar to Lemma 1.5.2 in the zero obstacle case.

Lemma 2.8.1 (Nondegeneracy at singular points)**.** *Let $u \in \mathfrak{S}_k$ and $0 \in \Sigma_\kappa(u)$ for $\kappa < k$. There exists $c > 0$ such that*

$$\sup_{\partial B_r} |u(x)| \geq c\, r^\kappa.$$

Proof. Assume the contrary. Then for a sequence $r = r_j \to 0$ one has

$$h_r := \left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{1/2} = o(r^\kappa).$$

Passing to a subsequence if necessary we may assume that

$$u_r(x) = \frac{u(rx)}{h_r} \to q_\kappa(x) \quad \text{uniformly on } \partial B_1$$

for some nonzero $q_\kappa \in \mathfrak{P}_\kappa$. Now for such q_κ we apply Theorem 2.7.2 to $M_\kappa(r, u, q_\kappa)$. From the assumption on the growth of u is is easy to recognize that

$$M_\kappa(0+, u, q_\kappa) = \int_{\partial B_1} q_\kappa^2 = \frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} q_\kappa^2$$

Therefore, using the monotonicity of $M(r, u, q_\kappa) + C\, r$ (see Theorem 2.7.2) for appropriately chosen $C > 0$, we will have that

$$C\, r + \frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} (u - q_\kappa)^2 \geq \frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} q_\kappa^2$$

or equivalently

$$\frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} u^2 - 2u q_\kappa \geq -C\, r.$$

After rescaling, we obtain

$$\frac{1}{r^{2\kappa}} \int_{\partial B_1} h_r^2 u_r^2 - 2 h_r r^\kappa u_r q_\kappa \geq -C\, r,$$

which we can rewrite as

$$\int_{\partial B_1} \frac{h_r}{r^\kappa} u_r^2 - 2 u_r q_\kappa \geq -C\, \frac{r^{\kappa+1}}{h_r}.$$

Now from the arguments in the proof of Lemma 2.3.1 we have $H(r) > c\, r^{n-1+2\kappa'}$ for any $\kappa' > \kappa$ and if we choose $\kappa' < \kappa + 1$ we will have that $r^{\kappa+1}/h_r \to 0$. Thus, passing to the limit over $r = r_j \to 0$, we arrive at

$$-\int_{\partial B_1} q_\kappa^2 \geq 0,$$

which is a contradiction as $q_\kappa \neq 0$. □

Lemma 2.8.2 ($\Sigma_k(v)$ is F_σ). *For any $v \in \mathfrak{S}^\varphi$ with $\varphi \in C^{k,1}(B_1')$, the set $\Sigma_\kappa(v)$ with $\kappa = 2m < k$, $m \in \mathbb{N}$, is of type F_σ, i.e., it is a union of countably many closed sets.*

Proof. As in the zero-obstacle case (see Lemma 1.5.3) we show that $\Sigma_\kappa(v)$ is the union of sets E_j of points $x_0 \in \Sigma_\kappa(v) \cap \overline{B_{1-1/j}}$ satisfying

$$\frac{1}{j}\,\rho^\kappa \le \sup_{|x|=\rho} |u_k^{x_0}(x)| < j\rho^\kappa \tag{2.8.1}$$

for $0 < \rho < 1 - |x_0|$. The proof that E_j are closed is almost identical to that in Lemma 1.5.3. □

Theorem 2.8.3 (Uniqueness of the homogeneous blowup at singular points). *Let $u \in \mathfrak{S}_k$ and $0 \in \Sigma_\kappa(u)$ with $\kappa < k$. Then there exists a unique nonzero $p_\kappa \in \mathfrak{P}_\kappa$ such that*

$$u_r^{(\kappa)}(x) := \frac{u(rx)}{r^\kappa} \to p_\kappa(x).$$

Proof. Let $u_r^{(\kappa)}(x) \to u_0(x)$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ over a certain subsequence $r = r_j \to 0+$. The existence of such limit follows from the growth estimate $|u(x)| \le C|x|^\kappa$. Moreover, the nondegeneracy implies that u_0 in not identically zero. Next, we have that

$$W_\kappa(r, u_0) = \lim_{r_j \to 0+} W_\kappa(r, u_{r_j}^{(\kappa)}) = \lim_{r_j \to 0+} W_\kappa(r r_j, u) = W_\kappa(0+, u) = 0,$$

for any $r > 0$, implying that u_0 is homogeneous of degree κ solution of (1.1.1)–(1.1.2). Repeating the arguments in Theorem 2.6.2, we see that u_0 must be a polynomial in $\mathfrak{P}_\kappa$.

We now apply Theorem 2.7.2 to the pair u, u_0. We obtain that the limit $M_\kappa(0+, u, u_0)$ exists and can be computed by

$$M_\kappa(0+, u, u_0) = \lim_{r_j \to 0+} M_\kappa(r_j, u, u_0) = \lim_{j \to \infty} \int_{\partial B_1} (u_{r_j}^{(\kappa)} - u_0)^2 = 0.$$

In particular, we obtain that

$$\int_{\partial B_1} (u_r^{(\kappa)}(x) - u_0)^2 = M_\kappa(r, u, u_0) \to 0$$

as $r \to 0+$ (not just over $r = r_j \to 0+$!). Thus, if u_0' is a limit of $u_r^{(\kappa)}$ over another sequence $r = r_j' \to 0$, we conclude that

$$\int_{\partial B_1} (u_0' - u_0)^2 = 0.$$

Since both u_0 and u_0' are homogeneous of degree κ, they must coincide. □

Theorem 2.8.4 (Continuous dependence of blowup). *Let $v \in \mathfrak{S}^\varphi$ with $\varphi \in C^{k,1}(B_1')$. For $x_0 \in \Sigma_\kappa(v)$ let $u_k^{x_0}$ be as in (2.5.2) and denote by $p_\kappa^{x_0}$ the blowup of $u_k^{x_0}$ at x_0 as in Theorem 2.8.3 so that*

$$u_k^{x_0}(x) = p_\kappa^{x_0}(x) + o(|x|^\kappa).$$

Then the mapping $x_0 \mapsto p_\kappa^{x_0}$ from $\Sigma_\kappa(v)$ to $\mathfrak{P}_\kappa$ is continuous. Moreover, for any compact subset K of $\Sigma_\kappa(v) \cap B_1$ there exists a modulus of continuity $\sigma = \sigma^K$, $\sigma(0+) = 0$ such that

$$|u_k^{x_0}(x) - p_\kappa^{x_0}(x)| \le \sigma(|x|)|x|^\kappa$$

for any $x_0 \in K$.

Proof. The proof is similar to that of Theorem 1.5.5. Given x_0 and $\varepsilon > 0$ fix $r_\varepsilon = r_\varepsilon(x_0) > 0$ such that

$$M_\kappa(r_\varepsilon, u_k^{x_0}, p_\kappa^{x_0}) := \frac{1}{r_\varepsilon^{n-1+2\kappa}} \int_{\partial B_{r_\varepsilon}} (u_k^{x_0}(x) - p_\kappa^{x_0})^2 < \varepsilon.$$

Then there exists $\delta_\varepsilon = \delta_\varepsilon(x_0)$ such that if $x_0' \in \Sigma_\kappa(u)$ and $|x_0' - x_0| < \delta_\varepsilon$ then

$$M_\kappa(r_\varepsilon, u_k^{x_0'}, p_\kappa^{x_0}) = \frac{1}{r_\varepsilon^{n-1+2\kappa}} \int_{\partial B_{r_\varepsilon}} (u_k^{x_0'} - p_\kappa^{x_0})^2 < 2\varepsilon.$$

This follows from the continuous dependence of $u_k^{x_0}$ on $x_0 \in \Gamma(v)$, which in turn is a consequence of C^k differentiability of the thin obstacle φ.

From Theorem 2.7.2, we will have that

$$M_\kappa(r, u_k^{x_0'}, p_\kappa^{x_0}) < 2\varepsilon + C\, r_\varepsilon, \quad 0 < r < r_\varepsilon$$

for a constant $C = C(x_0)$ depending on L^2 norms of $u_k^{x_0'}$ and $p_\kappa^{x_0}$, which can be made uniform for x_0' in a small neighborhood of x_0 as $u_k^{x_0'}$ depends continuously on x_0'. Passing $r \to 0$ we will therefore obtain

$$M(0+, u_k^{x_0'}, p_\kappa^{x_0}) = \int_{\partial B_1} (p_\kappa^{x_0'} - p_\kappa^{x_0})^2 \le 2\varepsilon + C\, r_\varepsilon.$$

This shows the first part of the theorem.

To show the second part, we notice that we have

$$\begin{aligned}\|u_k^{x_0'} - p_\kappa^{x_0'}\|_{L^2(\partial B_r)} &\le \|u_k^{x_0'} - p_\kappa^{x_0}\|_{L^2(\partial B_r)} + \|p_\kappa^{x_0'} - p_\kappa^{x_0'}\|_{L^2(\partial B_r)} \\ &\le 2(2\varepsilon + C\, r_\varepsilon)^{\frac12} r^{\frac{n-1}{2}+\kappa},\end{aligned}$$

for $|x_0' - x_0| < \delta_\varepsilon$, $0 < r < r_\varepsilon$, or equivalently

$$\|w_r^{x_0'} - p_\kappa^{x_0'}\|_{L^2(\partial B_1)} \le 2(2\varepsilon + C r_\varepsilon)^{\frac12}, \tag{2.8.2}$$

where

$$w_r^{x_0'}(x) := \frac{u_k^{x_0'}(rx)}{r^\kappa}.$$

Making a finite cover of the compact K with balls $B_{\delta_\varepsilon(x_0^i)}(x_0^i)$ for some $x_0^i \in K$, $i = 1, \dots, N$, we see that (2.8.2) is satisfied for all $x_0' \in K$, $r < r_\varepsilon^K := \min\{r_\varepsilon(x_0^i) \mid i = 1, \dots, N\}$ and $C = C^K := \max\{C(x_0^i) \mid i = 1, \dots, N\}$.

Now notice that

$$w_r^{x_0'} \in \mathfrak{S}_k, \quad p_\kappa^{x_0'} \in \mathfrak{S}_k$$

with $C^{1,\alpha}(\overline{B_1})$ norms of $w_r^{x_0'}$ and $p_\kappa^{x_0'}$ uniformly bounded and the estimate

$$|\Delta w_r^{x_0'}| \le M r^{k-\kappa} \to 0, \quad \text{a.e. in } B_1^\pm.$$

Thus, using a compactness argument, we can show that

$$\|w_r^{x_0'} - p_\kappa^{x_0'}\|_{L^\infty(B_{1/2})} \le C_\varepsilon,$$

for all $x_0' \in K$, $r < r_\varepsilon^K$ and $C_\varepsilon \to 0$ as $\varepsilon \to 0$. It is now easy to see that this implies the second part of the theorem. □

At this point we are ready to present the proofs of the main results.

Proof of Theorem 2.6.3. It is obtained by combining Theorems 2.8.3 and 2.8.4 above. □

Proof of Theorem 2.6.5. The proof is almost the same as for Theorem 1.3.8, but is now based on Theorem 2.6.3 instead of Theorem 1.3.6.

We give few details. For any $x_0 \in \Sigma_\kappa(v)$ let the polynomial $p_\kappa^{x_0} \in \mathfrak{P}_\kappa$ be as in Theorem 2.8.4. Write it in the expanded form

$$p_\kappa^{x_0}(x) = \sum_{|\alpha|=\kappa} \frac{a_\alpha(x_0)}{\alpha!} x^\alpha.$$

Then the coefficients $a_\alpha(x)$ are continuous on $\Sigma_\kappa(v)$. Moreover, if $x \in \Sigma_\kappa(v)$, we have $x \in B_1'$ and therefore $\tilde{Q}_k^{x_0}(x) = Q_k^{x_0}(x)$ in the definition (2.5.2), which implies that

$$u_k^{x_0}(x - x_0) = v(x) - \varphi(x) = 0.$$

Hence, from Theorem 2.8.4 we obtain

$$|p_\kappa^{x_0}(x - x_0)| \leq \sigma(|x - x_0|)|x - x_0|^\kappa, \quad x, x_0 \in K.$$

for a compact subset $K \subset \Sigma_\kappa(v)$. Furthermore, if we define

$$f_\alpha(x) = \begin{cases} 0 & |\alpha| < \kappa \\ a_\alpha(x) & |\alpha| = \kappa, \end{cases} \qquad x \in \Sigma_k(v),$$

as in the zero-obstacle case, then we have the following compatibility lemma .

Lemma 2.8.5. *Let $K = E_j$ as in Lemma 2.8.2. Then for any $x_0, x \in K$*

$$f_\alpha(x) = \sum_{|\beta| \leq \kappa - |\alpha|} \frac{f_{\alpha+\beta}(x_0)}{\beta!}(x - x_0)^\beta + R_\alpha(x, x_0) \tag{2.8.3}$$

with

$$|R_\alpha(x, x_0)| \leq \sigma_\alpha(|x - x_0|)|x - x_0|^{\kappa - |\alpha|}, \tag{2.8.4}$$

where $\sigma_\alpha = \sigma_\alpha^K$ is a certain modulus of continuity.

Proof. There are few additional details compared to Lemma 1.5.6, so we give a complete proof.

1) In the case $|\alpha| = \kappa$ we have

$$R_\alpha(x, x_0) = a_\alpha(x) - a_\alpha(x_0)$$

and the statement follows from the continuity of $a_\alpha(x)$ on $\Sigma_k(v)$.

2) For $0 \leq |\alpha| < \kappa$ we have

$$R_\alpha(x, x_0) = -\sum_{\gamma > \alpha, |\gamma| = \kappa} \frac{a_\gamma(x_0)}{(\gamma - \alpha)!}(x - x_0)^{\gamma - \alpha} = -\partial^\alpha p_\kappa^{x_0}(x - x_0).$$

Now suppose that there exists no modulus of continuity σ_α such that (2.8.4) is satisfied for all $x_0, x \in K$. Then there exists $\delta > 0$ and a sequence $x_0^i, x^i \in K$ with

$$|x^i - x_0^i| =: \rho_i \to 0$$

such that

$$\Big| \sum_{\gamma > \alpha, |\gamma| = \kappa} \frac{a_\gamma(x_0^i)}{(\gamma - \alpha)!}(x^i - x_0^i)^{\gamma - \alpha} \Big| \geq \delta |x^i - x_0^i|^{\kappa - |\alpha|}. \tag{2.8.5}$$

Consider the rescalings

$$w^i(x) = \frac{u_k^{x_0^i}(\rho_i x)}{\rho_i^\kappa}, \quad \xi^i = (x^i - x_0^i)/\rho_i.$$

Without loss of generality we may assume that $x_0^i \to x_0 \in K$ and $\xi^i \to \xi_0 \in \partial B_1$. From Theorem 2.8.4 we have that

$$|w^i(x) - p_\kappa^{x_0^i}(x)| \le \sigma(\rho_i|x|)|x|^\kappa$$

and therefore

$$w^i(x) \to p_\kappa^{x_0}(x) \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}^n). \tag{2.8.6}$$

We also consider the rescalings at x^i instead of x_0^i

$$\tilde{w}^i(x) = \frac{u_k^{x^i}(\rho_i x)}{\rho_i^\kappa}.$$

We then claim that the $C^{k,1}$ regularity of the thin obstacle φ implies that

$$w^i(x + \xi^i) - \tilde{w}^i(x) \to 0 \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}^n). \tag{2.8.7}$$

Indeed, if $Q_k^{x_0}(x')$ denotes the k-th Taylor polynomial of $\varphi(x')$ at x_0, then

$$\begin{aligned}
&\frac{Q_k^{x_0^i}(\rho_i(x' + \xi_i)) - Q_k^{x^i}(\rho_i x')}{\rho_i^\kappa}\\
&= \frac{\varphi(x_0^i + \rho_i(x' + \xi_i)) + o(\rho_i^k|x' + \xi_i|^k) - \varphi(x^i + \rho_i x') - o(\rho_i^k|x'|^k)}{\rho_i^\kappa}\\
&= o(\rho_i^{k-\kappa}) \to 0
\end{aligned}$$

and this implies the convergence (2.8.7), if we write the explicit definition of w^i using (2.5.2). Further, note that since $x^i \in E_j$, we have

$$\tfrac{1}{j}\rho^\kappa \le \sup_{|x|=\rho} |u_k^{x^i}(x)| \le j\rho^\kappa$$

and therefore

$$\tfrac{1}{j}\rho^\kappa \le \sup_{|x|=\rho} |\tilde{w}^i(x)| \le j\rho^\kappa.$$

Passing to the limit in (2.8.6)–(2.8.7) we obtain that

$$\tfrac{1}{j}\rho^\kappa \le \sup_{|x|=\rho} p_\kappa^{x_0}(x + \xi_0) \le j\rho^\kappa, \quad 0 < \rho < \infty.$$

This implies that ξ_0 is a point of frequency $\kappa = 2m$ for the polynomial $p_\kappa^{x_0}$ and by Theorem 1.3.2 we have that $\xi_0 \in \Sigma_\kappa(p_\kappa^{x_0})$. In particular,

$$\partial^\alpha p_\kappa^{x_0}(\xi_0) = 0, \quad \text{for } |\alpha| < \kappa.$$

However, dividing both parts of (2.8.5) by $\rho_i^{\kappa-|\alpha|}$ and passing to the limit, we obtain that

$$|\partial^\alpha p_\kappa^{x_0}(\xi_0)| = \Big| \sum_{\gamma>\alpha, |\gamma|=\kappa} \frac{a_\gamma(x_0)}{(\gamma-\alpha)!}(\xi_0)^{\gamma-\alpha} \Big| \ge \delta,$$

a contradiction. □

We then apply Whitney's extension theorem and the implicit function theorem as in the proof of Theorem 1.3.8 to complete the proof. □

Concluding remarks and open problems

The intent of this closing section is to provide a summarizing overview of the state of our knowledge on the lower dimensional obstacle problem. We also point to some open problems in the study of the free boundary $\Gamma(u)$ for solutions of (1.1.1)–(1.1.3), and more generally for (2.1.1)–(2.1.3).

The regularity of $\Gamma_{2-\frac{1}{2}}(u)$, i.e. of that portion of the free boundary which is composed of regular points has been proved in [ACS07] for the zero obstacle and in [CSS08] for a nonzero one. The present paper studies the singular set $\Sigma(u)$, which is the collection of those free boundary points where the coincidence set $\Lambda(u)=\{u=\varphi\}$ has a vanishing $\mathcal{H}^{n-1}$-density.

What remains to study is the set of nonregular nonsingular points, i.e. the set

$$\Gamma(u)\setminus(\Gamma_{2-\frac{1}{2}}(u)\cup\Sigma(u))=\bigcup_{\substack{\kappa>2-\frac{1}{2},\\ \kappa\neq 2m, m\in\mathbb{N}}}\Gamma_\kappa(u).$$

In the case of the nonzero thin obstacle $\varphi\in C^{k,1}$ it is reasonable to limit ourselves to $\kappa<k$, since at the free boundary points in $\Gamma_k^{(k)}(u)$ even the blowups are not properly defined.

Possible values of κ. First, one must identify the possible values of κ which can be reduced to a classification of homogeneous global solutions of the Signorini problem with zero obstacle by Propositions 1.2.2 and 2.4.3. A partial classification of global solutions (convex in x') has been given in [CSS08] and [ACS07], which excluded the interval $(2-\frac{1}{2},2)$ from the range of possible values of κ. However, a full classification is needed for obtaining the full range of possible values of κ. As we mentioned earlier in the text it is plausible that the only possible values are

$$\kappa\in\{2m-\frac{1}{2}\mid m\in\mathbb{N}\}\cup\{2m\mid m\in\mathbb{N}\}.$$

This fact is easy to establish when the dimension $n=2$, see Remark 1.2.8, but is unknown in higher dimensions.

Free boundary. We know that the only possible global solutions for $\kappa\in\{2m\mid m\in\mathbb{N}\}$ are the polynomials $p_\kappa\in\mathfrak{P}_\kappa$, see Lemma 1.3.3. The other known global solutions are the rotations in x'-variable of

$$\hat{u}_\kappa(x)=\mathrm{Re}(x_1+i\,|x_n|)^\kappa,\quad\text{for}\quad\kappa\in\{2m-\frac{1}{2}\mid m\in\mathbb{N}\}.$$

If these are the only global solutions for this range of κ's then it is plausible that

$$\Gamma_\kappa(u)\text{ is locally a }(n-2)\text{-dimensional }C^1\text{-manifold for }\kappa\in\{2m-\frac{1}{2}\mid m\in\mathbb{N}\},$$

or at least Lipschitz regular. However, the true picture may be more complicated than that.

Degenerate points. One complication that may occur in dimensions $n\geq 3$ is that a blowup u_0 may vanish identically on $\mathbb{R}^{n-1}\times\{0\}$ for some $u\in\mathfrak{S}$. This cannot happen in dimension $n=2$, see Remark 1.2.8, however, the possibility in higher dimensions is unknown to the authors. If a blowup u_0 for $u\in\mathfrak{S}$ vanishes on $\mathbb{R}^{n-1}\times\{0\}$, we call the origin a *degenerate free boundary point* of u. Such points are characterized by the property that the coincidence set $\Lambda(u)$ has a $\mathcal{H}^{n-1}$-density 1 there. At degenerate points, it is easy to see that one must have $\kappa\in\{2m+1\mid$

$m \in \mathbb{N}\}$ and that u_0 must coincide in $\mathbb{R}^{n-1} \times [0, \infty)$ with a harmonic polynomial q_κ from the class

$$\mathfrak{Q}_\kappa = \{q_\kappa \mid \Delta q_\kappa = 0,\ x \cdot \nabla q_\kappa - \kappa q_\kappa = 0,\ q_\kappa(x', 0) = 0,\ -\partial_{x_n} q_\kappa(x', 0) \geq 0\}.$$

It would be interesting to study the set of such points, provided they exist.

References

[Alm79] Frederick J. Almgren Jr., *Dirichlet's problem for multiple valued functions and the regularity of mass minimizing integral currents*, Minimal submanifolds and geodesics (Proc. Japan-United States Sem., Tokyo, 1977), North-Holland, Amsterdam, 1979, pp. 1–6.

[ACF84] Hans Wilhelm Alt, Luis A. Caffarelli, and Avner Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. **282** (1984), no. 2, 431–461.

[AC04] I. Athanasopoulos and L. A. Caffarelli, *Optimal regularity of lower dimensional obstacle problems*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **310** (2004), no. Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 35 [34], 49–66, 226 (English, with English and Russian summaries); English transl., J. Math. Sci. (N. Y.) **132** (2006), no. 3, 274–284.

[ACS07] I. Athanasopoulos, L. A. Caffarelli, and S. Salsa, *The structure of the free boundary for lower dimensional obstacle problems*, Amer. J. Math. (2007), to appear.

[Caf79] L. A. Caffarelli, *Further regularity for the Signorini problem*, Comm. Partial Differential Equations **4** (1979), no. 9, 1067–1075.

[Caf98] ———, *The obstacle problem revisited*, J. Fourier Anal. Appl. **4** (1998), no. 4-5, 383–402.

[CR77] L. A. Caffarelli and N. M. Rivière, *Asymptotic behaviour of free boundaries at their singular points*, Ann. Math. (2) **106** (1977), no. 2, 309–317.

[CSS08] Luis Caffarelli, Sandro Salsa, and Luis Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*, Invent. Math. **171** (2008), no. 2, 425–461.

[CS07] Luis Caffarelli and Luis Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260.

[CT04] Rama Cont and Peter Tankov, *Financial modelling with jump processes*, Chapman and Hall, CRC Financial Mathematics Series, Chapman and Hall, CRC, Boca Raton, FL, 2004.

[DL72] G. Duvat and J. L. Lions, *Les inequations en mechanique et en physique*, Dunod, Paris, 1972.

[Fri82] Avner Friedman, *Variational principles and free-boundary problems*, Pure and Applied Mathematics, John Wiley & Sons Inc., New York, 1982. A Wiley-Interscience Publication.

[GL86] Nicola Garofalo and Fang-Hua Lin, *Monotonicity properties of variational integrals, A_p weights and unique continuation*, Indiana Univ. Math. J. **35** (1986), no. 2, 245–268.

[GL87] ———, *Unique continuation for elliptic operators: a geometric-variational approach*, Comm. Pure Appl. Math. **40** (1987), no. 3, 347–366.

[KO88] N. Kikuchi and J. T. Oden, *Contact problems in elasticity: a study of variational inequalities and finite element methods*, SIAM Studies in Applied Mathematics, 8, Society for Industrial and Applied Mathematics, Philadelphia, PA., 1988.

[Mon03] R. Monneau, *On the number of singularities for the obstacle problem in two dimensions*, J. Geom. Anal. **13** (2003), no. 2, 359–389.

[Mon07] ———, *Pointwise estimates for Laplace equation. Applications to the free boundary of the obstacle problem with Dini coefficients* (2007), preprint.

[Sil07] Luis Silvestre, *Regularity of the obstacle problem for a fractional power of the Laplace operator*, Comm. Pure Appl. Math. **60** (2007), no. 1, 67–112.

[Ura87] N. N. Ural′tseva, *On the regularity of solutions of variational inequalities*, Uspekhi Mat. Nauk **42** (1987), no. 6(258), 151–174, 248 (Russian); English transl., Russian Math. Surveys **42** (1987), no. 6, 191–219.

[Wei99] Georg S. Weiss, *A homogeneity improvement approach to the obstacle problem*, Invent. Math. **138** (1999), no. 1, 23–50.

[Whi34] Hassler Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), no. 1, 63–89.

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