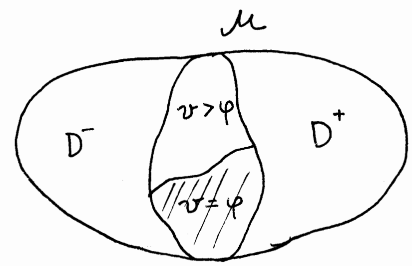
Lecture 1

Arshak Petrosyan
Lecture notes for a mini-course given at IMPA
July 6–17, 2015

Thin obstacle problem

1. (Interior) thin obstacle problem



$D\subset \mathbb{R}^n$, $\mathcal{M}$ hypersurface

$D\setminus\mathcal{M} = D^+\cup D^-$

$J_0(v) = \int_D |\nabla v|^2 \longrightarrow \min$ among $v\geq\varphi$ on $\mathcal{M}$ (thin obstacle)

$v = g$ on ∂D (boundary values)

$\varphi : \mathcal{M}\longrightarrow \mathbb{R}$ "thin" obstacle

Coincidence set $\Lambda(v) = \{x\in\mathcal{M} : v=\varphi\}$

Free boundary $\Gamma(v) = \partial_{\mathcal{M}}\Lambda(v)$ "thin" free boundary
expected to be of codimension 2

Euler–Lagrange:

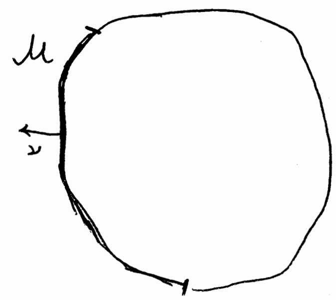
$$\begin{cases} \Delta v = 0 & \text{in } D^\pm \\ v\geq\varphi,\quad \partial_{\nu^+}v + \partial_{\nu^-}v \geq 0,\quad (v-\varphi)(\partial_{\nu^+}v + \partial_{\nu^-}v) = 0 & \text{on } \mathcal{M}\end{cases}$$

(Need to be understood properly) complementarity cond.

2. Boundary thin obstacle (Signorini) problem

$\mathcal{M}\subset\partial D$

E-L



$$\begin{cases} \Delta v = 0 \text{ in } D \\ v\geq\varphi,\quad \partial_\nu v\geq 0,\quad (v-\varphi)\partial_\nu v = 0 \text{ on } \mathcal{M}\end{cases}$$

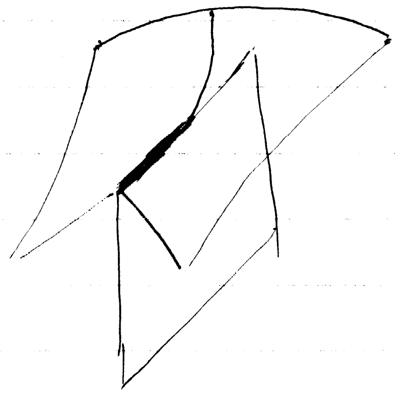
Signorini cond.

Note: When $\mathcal{M}$ flat and v is even w.r.t $\mathcal{M}$ then interior thin obst. problem is equivalent to Signorini problem in D^+

3. Example to keep in mind

$\mathcal{M} = \{x_n = 0\}$, $\varphi = 0$, $g(x', -x_n) = g(x', x_n)$

$u = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$



$u \in C^{1,1/2}(B_1^{\pm} \cup B_1')$

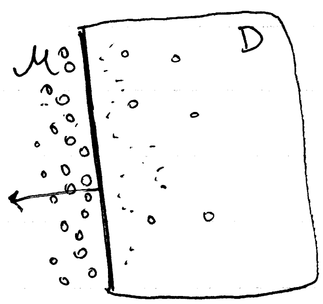
$[\text{but } u \in C^{0,1}(B_1)]$

Other solutions:

$\hat{u}_\kappa = \mathrm{Re}(x_{n-1} + i|x_n|)^\kappa$

$\kappa = 3/2, 2, 7/2, 4, \ldots, 2m - \frac{1}{2}, 2m, \ldots, \quad m \in \mathbb{N}$

4) Application: Semi-permeable membranes



- D biological cell
- $\mathcal{M}$ semi-permeable membrane
  - lets smaller molecules to flow (solvents)
  - blocks larger molecules (solutes)
- u pressure of chemical solution

$\Delta u = 0$ in D

- flow occurs in one direction (smaller concentr. $\to$ larger conc)

  osmosis

- flow stops when pressure becomes too high ($u > \varphi$)

Mathematically on $\mathcal{M}$

$$\begin{cases} u > \varphi \Rightarrow \dfrac{\partial u}{\partial \nu} = 0 \quad (\text{no flow}) \\ u \leq \varphi \Rightarrow \dfrac{\partial u}{\partial \nu} = -\dfrac{1}{\varepsilon}(u - \varphi) \end{cases}$$

$\frac{1}{\varepsilon}$ permeability constant

Letting $\varepsilon \to 0$ (formally)

$$u \geq \varphi, \quad \frac{\partial u}{\partial \nu} \geq 0, \quad (u - \varphi)\frac{\partial u}{\partial \nu} = 0 \text{ on } \mathcal{M}$$

5. Regularity of minimizers

5.1 Proposition Let v be a solution of Signorini problem with smooth $\mathcal{M}$ and φ. Then $v \in C^{1,\alpha}(D \cup \mathcal{M})$ for some $\alpha > 0$

Model case $\mathcal{M} = \{x_n = 0\} \cap B_1 = B_1'$, $D = B_1$, $\varphi = 0$, $g(x', -x_n) = g(x', x_n)$

$\Delta u = 0$ in $B_1^\pm$, $u \geq 0, -u_{x_n} \geq 0$, $u u_{x_n} = 0$ on B_1'

Sketch of the proof in model case:

5.2. Lemma (Filling holes) For any $x^\circ \in B_{1/2}'$, $0 < \rho < \frac{1}{4}$

$$\int_{B_\rho(x^\circ)} \frac{|\nabla u_{x_k}|^2}{|x - x^\circ|^{n-2}} dx \leq \frac{C_n}{\rho^n} \int_{B_{2\rho}(x^\circ) \setminus B_\rho(x^\circ)} u_{x_k}^2 , \quad k = 1, 2, \dots, n$$

- Like energy inequality
- Based on observation that $u_{x_k}^\pm$ are subharmonic
- Rigorous proof throug $\frac{1}{\varepsilon}$ permeability problem

Immediate corollary $u \in W^{2,2}(B_{3/4}^\pm)$

Proof of Prop 5.1. We will show $u \in C^{1,\alpha}(B_{1/2}^\pm \cup B_{1/2})$

1) Enough to show $u_{x_n} \in C^\alpha(B_{1/2}')$
2) Enough to show

$$\int_{B_\rho(x_0)} \frac{|\nabla u_{x_n}|^2}{|x - x^\circ|^{n-2}} \leq C \rho^\alpha , \quad 0 < \rho < \frac{1}{4} , \quad x^\circ \in B_{1/2}'$$

Let

$$I^{(k)}(\rho) = \int_{B_\rho} \frac{|\nabla u_{x_k}|^2}{|x|^{n-2}} , \quad k = 1, 2, \dots, n$$

5.3 Lemma $\exists\, \theta \in (0,1)$ s.t. for $0 < \rho < \frac{1}{4}$

either $I^{(n)}(\rho) \leq \theta\, I^{(n)}(2\rho)$

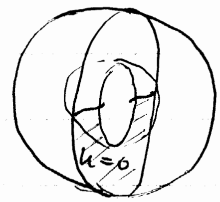
or $I^{(j)}(\rho) \leq \theta\, I^{(j)}(2\rho)$ for all $j = 1, 2, \dots, n-1$

Proof of Lemma 5.3 We will strongly use the complementarity

$u u_{x_n} = 0$ on B_1'

Case 1 Portion of $u=0$ in $B_{2\rho}' \setminus B_\rho'$ is large

$$|\{u=0\} \cap (B_{2\rho}' \setminus B_\rho')| \geq \frac{1}{2} |B_{2\rho}' \setminus B_\rho'|$$



By Poincare Ineq

$$\frac{1}{\rho^n} \int_{B_{2\rho} \setminus B_\rho} u_{x_j}^2 \leq \frac{C_n}{\rho^{n-2}} \int_{B_{2\rho} \setminus B_\rho} |\nabla u_{x_j}|^2 \qquad j = 1, \dots, n-1$$

With Lemma 5.2 this gives

$$I^{(j)}(\rho) \leq C_n \left(I^{(j)}(2\rho) - I^{(j)}(\rho) \right)$$

$$I^{(j)}(\rho) \leq \frac{C_n}{C_n + 1} I^{(j)}(2\rho)$$

Case 2 Portion of $u_{x_n} = 0$ is large

$$I^{(n)}(\rho) \leq \frac{C_n}{C_n+1} I^{(n)}(2\rho)$$

Iterating we obtain for any $0 < \rho < \frac{1}{4}$

either

$$I^{(n)}(\rho) \leq C \rho^\alpha \quad \text{or}$$

$$I^{(j)}(\rho) \leq C \rho^\alpha \quad \text{for all } j = 1, \dots, n-1$$

$$\Delta u = 0 \quad \Rightarrow \quad u_{x_n x_n} = - \sum u_{x_i x_i} \quad \Rightarrow \quad I^{(n)}(\rho) \leq \sum_{j=1}^{n-1} I^{(j)}(\rho)$$

$$\Rightarrow \quad I^{(n)}(\rho) \leq C \rho^\alpha$$

6. Optimal regularity

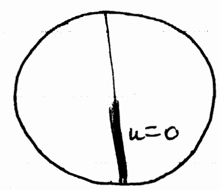
6.1 Theorem Let u be a solution of Signorini problem with smooth $\mathcal{M}$ and φ. Then $u \in C^{1,1/2}(D \cup \mathcal{M})$

Model case:

$$\begin{cases} \Delta u = 0 \text{ in } B_1^{\pm} \\ u \geq 0, \ -u_{x_n} \geq 0, \ u u_{x_n} = 0 \text{ on } B_1' \end{cases}$$

$$u(x', -x_n) = u(x', x_n)$$

$$\Delta u = 2 u_{x_n} \mathcal{H}^{n-1}|_{\Lambda}$$



6.2 Lemma (Almgren's frequency formula)

$$N(r) = \frac{r \int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2} \nearrow \quad r \in (0,1) \qquad \left(N^{x_0}(r) = \frac{r \int_{B_r(x^0)} |\nabla u|^2}{\int_{\partial B_r(x^0)} u^2} \nearrow \right)$$

Sketch of the proof

$$H(r) = \int_{\partial B_r} u^2, \qquad D(r) = \int_{B_r} |\nabla u|^2$$

- $H'(r) = \frac{n-1}{r} H(r) + 2 \int_{\partial B_r} u u_\nu \qquad D(r) = \int_{\partial B_r} u u_\nu$ (use $u \Delta u = 0$)
- $D'(r) = \frac{n-2}{r} D(r) + 2 \int_{\partial B_r} u_\nu^2$ (use Rellich's identity and $(x \cdot \nabla u) \Delta u = 0$)

$$\Rightarrow \frac{d}{dr} \log N(r) = \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} = \frac{1}{r} + \frac{n-2}{r} - \frac{n-1}{r} + $$

$$+ 2 \left(\frac{\int_{\partial B_r} u_\nu^2}{\int_{\partial B_r} u u_\nu} - \frac{\int_{\partial B_r} u u_\nu}{\int_{\partial B_r} u^2} \right) \geq 0$$

Cauchy-Schwarz

6.3 **Lemma** If $x^\circ \in \Gamma$ then

$$N^{x^\circ}(r) \geq 3/2$$

Proof: Next time

Proof of Theorem 6.1 in model case

1) Observe

$$r\frac{H'(r)}{H(r)} = n-1+2N(r) \quad , \quad N(r) \geq \frac{3}{2} \Rightarrow$$

$$\frac{d}{dr}\log H(r) \geq \frac{n+2}{r}$$

Integrate from r= to $\frac{1}{2}$, exponentiate

$$H(r) \leq C\, r^{n+2} \quad , \quad C = \frac{H(1/2)}{(1/2)^{n+2}}$$

$$\fint_{\partial B_r} u^2 \leq C r^3$$

2) Now, $u^\pm$ are subharmonic ($u^\pm \geq 0$, $\Delta u^\pm = 0$ if $u^\pm > 0$)

$L^\infty - L^2$ estimate

$$\sup_{B_{r/2}} |u| \leq C_n \left(\fint_{\partial B_r} u^2\right)^{1/2} \leq C r^{3/2}$$

3) Interior estimates (using even or odd reflections in x_n)

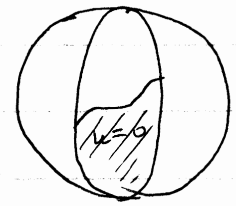
give

$$u \in C^{1,1/2}(B^\pm_{1/2} \cup B'_{1/2})$$

Lecture 2

1. Structure of the free boundary

Signorini problem



$$\begin{cases} \Delta u = 0 & \text{in } B_1^{\pm} \\ u \geq 0, \quad -u_{x_n} \geq 0, \quad u u_{x_n} = 0 & \text{on } B_1' \end{cases}$$

1.1 (Almgren) Rescalings $\quad x_0 \in \Gamma, \; r > 0$

$$u_r = u_{r,x_0} = \frac{u(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{1/2}} \qquad N(r,u) = \frac{r \int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

- Normalization: $\|u_r\|_{L^2(\partial B_1)} = 1$
- Scaling property for N: $\quad N(\rho, u_r) = N(\rho r, u)$

1.2 Existence of blowups

$$\int_{B_1} |\nabla u_r|^2 = N(1,u_r) = N(r,u) \leq N(1,u)$$

$\exists\, r_j \to 0$ s.t.

$$u_{r_j} \rightharpoonup u_0 \text{ in } W^{1,2}(B_1)$$
$$u_{r_j} \to u_0 \text{ in } L^2(\partial B_1)$$
$$u_{r_j} \to u_0 \text{ in } C^{1,\alpha}(B_1^{\pm} \cup B_1')$$

u_0 solves $\Delta u_0 = 0$ in $\mathbb{R}^n_{\pm}$, $u_0 \geq 0$, $-(u_0)_{x_n} \geq 0$, $u_0 (u_0)_{x_n} = 0$ on $\mathbb{R}^{n-1}$

1.3 Proposition

$u_0 \not\equiv 0$, u_0 homogeneous of degree $\kappa = N(0+,u)$

$$u_0(\lambda x) = \lambda^{\kappa} u_0(x), \quad \lambda > 0$$

Proof

$$N(\rho, u_0) = \lim_{j\to\infty} N(\rho, u_{r_j}) = \lim_{j\to\infty} N(\rho r_j, u) = N(0+,u) = \kappa$$

Case of equality in Almgren's freq. formula

$$N(\rho,u) \equiv \kappa \iff u(\lambda x) = \lambda^{\kappa} u(x)$$

1.4. Classification of free boundary points

$$\kappa(x_0) = N^{x_0}(0+, u), \quad x_0 \in \Gamma \qquad \underline{\text{Frequency}} \text{ at } x_0$$

1.4.1 Proposition $x_0 \mapsto \kappa(x_0)$ is upper semicontinuous

Proof Notice that

$$x_0 \mapsto N^{x_0}(r, u) \quad \text{is continuous on } \Gamma$$

Then use that $N^{x_0}(r,u) \searrow \kappa(x_0)$ as $r \searrow 0$

Define $\Gamma_\kappa = \{x_0 \in \Gamma : \kappa(x_0) = \kappa\}$, $\quad \Gamma = \bigcup_{\kappa \geq 0} \Gamma_\kappa$ foliation of the free boundary

$x_0 \in \Gamma_\kappa \iff$ blowups u_0 at x_0 are homogeneous of degree κ.

Question What are possible values of κ?

- If $n = 2$, then

$$\kappa = 3/2, 2, 7/2, 4, \dots, 2m - 1/2, 2m, \dots \qquad m \in \mathbb{N}$$

Solutions $\hat{u}_\kappa(x) = C \operatorname{Re}(x_1 + i|x_2|)^\kappa$ (polynomials when $\kappa = 2m$)

- For $n \geq 3$?

$$u_0 \in C^{1,\alpha} \Rightarrow \kappa \geq 1 + \alpha$$

1.5. Proposition $\kappa \geq 3/2$. Moreover

$$\kappa = \frac{3}{2} \quad \text{or} \quad \kappa \geq 2$$

1.6 Lemma If u_0 homogen of degree κ and $1 < \kappa < 2$

Then $\kappa = 3/2$ and $u_0(x) = C_n \operatorname{Re}(x_{n-1} + i|x_n|)^{3/2}$ (after rot in $\mathbb{R}^{n-1}$)

Proof of Lemma 1.6

Take $e \in \mathbb{R}^{n-1}$, $|e|=1$. Then $w^{\pm} = (\partial_e u_0)^{\pm}$ satisfy:

(*) $w^{\pm} \geqslant 0$, $\Delta w^{\pm} \geqslant 0$, $w^{+} \cdot w^{-} = 0$ in B_1

1.7 Lemma (Alt-Caffarelli-Friedman monotonicity formula)

Let $w^{\pm}$ satisfy (*). Then

$$\Phi(r) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla w^+|^2}{|x|^{n-2}} \int_{B_r} \frac{|\nabla w^-|^2}{|x|^{n-2}} \quad \nearrow \text{ in } r \in (0,1)$$

Proof is difficult

Now for $w^{\pm} = (\partial_e u_0)^{\pm}$; $\partial_e u_0$ is homogeneous of degree $\kappa-1$.

$\Rightarrow$

$$\Phi(r) = C r^{2(2\kappa-4)} \nearrow$$

This is possible only if $C=0$, since $\kappa < 2$ $\quad \Phi(r) \equiv 0$

$\Rightarrow$ either $(\partial_e u_0)^+ \equiv 0$ or $(\partial_e u)^- \equiv 0$ in $\mathbb{R}^n$

$\Rightarrow$ u_0 depends only on one tangential variable

$\Rightarrow$ Problem reduced to 2D $\Rightarrow$ $\kappa = \frac{3}{2}$

Thus, we get a decomposition

$$\Gamma = \Gamma_{3/2} \cup \bigcup_{\kappa \geqslant 2} \Gamma_\kappa$$

We call $\Gamma_{3/2}$ regular set

1.8 $\Gamma_{3/2}$ is relatively open in Γ

Proof $x_0 \mapsto \kappa(x_0)$ is upper semicontinuous

$$\Gamma_{3/2} = \kappa^{-1}\{\tfrac{3}{2}\} = \kappa^{-1}((-\infty, 2))$$

| κ missing values between $\frac{3}{2}$ and 2

1.9. Weiss-type monotonicity formula

Lemma If $x_0 \in \Gamma_\kappa$ then

$$W_\kappa(r) = \frac{1}{r^{n+2\kappa-2}} \int_{B_r(x_0)} |\nabla u|^2 - \frac{\kappa}{r^{n+2\kappa-1}} \int_{\partial B_r(x_0)} u^2 \quad \nearrow \text{ in } r \in (0,1)$$

$$= \frac{1}{r^{n+2\kappa-2}} D(r) - \frac{\kappa}{r^{n+2\kappa-1}} H(r) = \frac{H(r)}{r^{n+2\kappa-1}} (N(r) - \kappa)$$

Proof Using the same diff identities as for $N(r)$ $|_{x_0=0}$

$H'(r) = \frac{n-1}{r} H(r) + 2 D(r)$

$D'(r) = \frac{n-2}{r} D(r) + 2 \int_{\partial B_r} u_\nu^2$

$$\Rightarrow \quad W_\kappa'(r) = \frac{2}{r^{n+2\kappa}} \int_{\partial B_r} (x \nabla u - \kappa u)^2 .$$

1.10 Homogeneous rescalings

$$u_r^{(\kappa)}(x) = \frac{u(rx)}{r^\kappa}$$

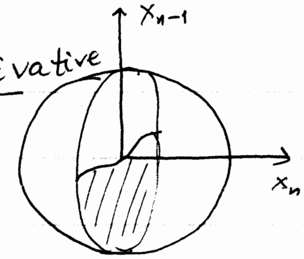
$$W_\kappa(\rho, u_r^{(\kappa)}) = W_\kappa(r\rho, u) \geq 0$$

$u_{r_j}^{(\kappa)}(x) \longrightarrow u_0^{(\kappa)}(x)$ solution of Signorini problem
homogeneous blowup

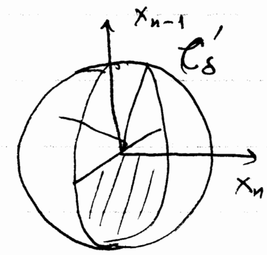
$x_0 \in \Gamma_\kappa \Rightarrow H(r) \leq C r^{n-1+2\kappa} \Rightarrow u_0^{(\kappa)}$ exist

$\int_{\partial B_1} (u_{r_j}^{(\kappa)})^2 \leq C$ However, it is not obvious that $u_0^{(\kappa)} \not\equiv 0$

2. Regularity of $\Gamma_{3/2}$: Directional derivative approach



$$\begin{cases} \Delta u = 0 & \text{in } B_1^{\pm} \\ u \geq 0, \quad -u_{x_n} \geq 0, \quad u u_{x_n} = 0 & \text{on } B_1' \end{cases}$$

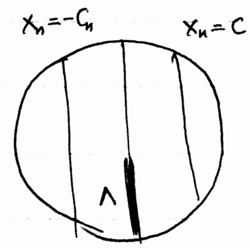


Suppose that $0 \in \Gamma_{3/2}$:

$$u_{r_j}(x) = \frac{u(r_j x)}{\left(\frac{1}{r_j^{n-1}} \int_{\partial B_{r_j}} u^2\right)^{1/2}} \longrightarrow u_0 = C_n \operatorname{Re}(x_{n-1} + i x_n)^{3/2}$$

$$\mathcal{C}_\delta' = \{x : x_n = 0, \ x_{n-1} > \delta |x''|\} \quad \text{thin cone}$$

$$\partial_e u_0(x) = \frac{3}{2} C_n (e \cdot e_{n-1}) \operatorname{Re}(x_{n-1} + i x_n)^{1/2}$$
$$= \frac{3}{2} C_n (e \cdot e_{n-1}) \sqrt{\sqrt{x_{n-1}^2 + x_n^2} + x_{n-1}}$$



$\Rightarrow$ $\quad \partial_e u_0 \geq 0 \quad$ in B_1

$\quad\quad \partial_e u_0 \geq 2\eta_0 \quad$ in $B_1 \cap \{|x_n| \geq C_n\}$

Now, if $u_{r_j} \to u_0$ in $C^{1,\alpha}(B_1^{\pm} \cup B_1') \Rightarrow$ for large j

$\quad \partial_e u_{r_j} \geq -\varepsilon \quad$ in B_1

$\quad \partial_e u_{r_j} \geq \eta_0 \quad$ in $B_1 \cap \{|x_n| \geq C_n\}$

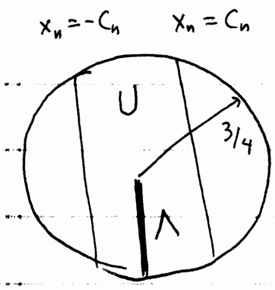
2.1. Lemma $\exists\, \varepsilon_0 > 0$ s.t. if ~~$\varepsilon \leq \varepsilon_0$ then $\partial_e u \geq 0$ in~~

$$\left.\begin{array}{ll} \partial_e u \geq -\varepsilon_0 & \text{in } B_1 \\ \partial_e u \geq \eta_0 & \text{in } B_1 \cap \{|x_n| \geq C_n\} \end{array}\right| \Rightarrow \partial_e u \geq 0 \text{ in } B_{1/2}$$

<u>Proof</u> For $C_n = \frac{1}{16\sqrt{n}}$. Suppose $\exists\, x_0 \in \{|x_n| < C_n\} \cap B_{1/2}$: $\partial_e u(x_0) < 0$

Let $\quad w(x) = \partial_e u(x) + \frac{\alpha_0}{n-1} |x' - x_0'|^2 - \alpha_0 x_n^2 \quad$ in $\quad U = (B_{3/4} \setminus \Lambda) \cap \{|x_n| < C_n\}$

$w(x_0) < 0, \quad \Delta w = 0$ in $U \quad \Rightarrow \quad \inf_{\partial U} w < 0$



On ∂U

1) on Λ : $w \geq 0$

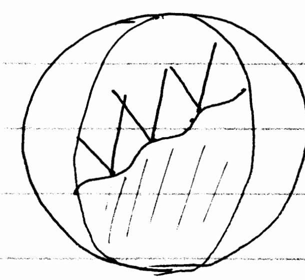
2) on $\{|x_n| = C_n\} \cap \partial U$

$$w \geq \eta_0 - \alpha_0 C_n^2 > 0 \quad \text{if } \alpha_0 \text{ small}$$

3) on $\{|x_{\bullet}| = \frac{3}{4}\} \cap \partial U$

$$w \geq -\varepsilon_0 + \alpha_0 \left(\tfrac{1}{n-1} \left[\tfrac{1}{4} - C_n \right]^2 - C_n^2 \right) > 0 \quad \text{if } \varepsilon_0 \text{ small}$$

2.2 <u>Theorem</u> (Lip regularity of $\Gamma_{3/2}$)



If $0 \in \Gamma_{3/2} \Rightarrow \forall \delta > 0 \; \exists r_\delta > 0$ s.t

$$\partial_e u \geq 0 \text{ in } B_{r_\delta} \text{ for any } e \in \mathcal{C}_\delta'$$

$$\Rightarrow \exists f : B''_{r_\delta} \to \mathbb{R} \quad \text{s.t} \quad |\nabla_{x''} f| \leq \delta$$

$$\{u = 0\} \cap B'_{r_\delta} = \{x_{n-1} \leq f(x'')\} \cap B'_{r_\delta}$$

$$\Gamma \cap B'_{r_\delta} = \{x_{n-1} = f(x'')\} \cap B'_{r_\delta}$$

<u>2.2.1</u> Actually, it is easy to see that $\Gamma \cap B'_{r_0}$ is C^1

2.3 <u>Theorem</u> In Theorem 2.2 $f \in C^{1,\alpha}(B''_{r_\delta}) \Rightarrow$

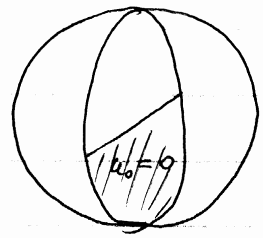
$\Gamma \cap B'_{r_\delta}$ is $C^{1,\alpha}$ graph in $\mathbb{R}^{n-1}$

<u>Proof</u> Boundary Harnack principle in $B_1 \setminus \Lambda$

See Lecture 3, Section 4

3. Regularity of $\Gamma_{3/2}$: Epiperimetric inequality approach

Recall $\quad W_\kappa(r,u) = \dfrac{1}{r^{n+2\kappa-2}} \displaystyle\int_{B_r} |\nabla u|^2 - \dfrac{\kappa}{r^{n+2\kappa-1}} \displaystyle\int_{\partial B_r} u^2$



$u_0 = \mathrm{Re}\,(x_{n-1} + i x_n)^{3/2}$

Note $\quad W_{3/2}(r, u_0) \equiv 0$

If $\quad u_{r_j} \to u_0 \quad$ then $\quad \| u_{r_j} - u_0 \|_{W^{1,2}(B_1)} < \delta$

3.1. <u>Theorem</u> (<u>Epiperimetric inequality</u>) There exist $\delta > 0$ and $\theta \in (0,1)$

s.t. if $\cdot\ v$ is $3/2$-homogeneous $\quad v(\lambda x) = \lambda^{3/2} v(x)$

$\quad v \geq 0$ on B_1'

$\quad \| v - u_0 \|_{W^{1,2}(B_1)} < \delta$

Then $\exists\, v^* \in W^{1,2}(B_1)$ s.t. $\quad v = v^*$ on ∂B_1, $\; v^* \geq 0$ on B_1' and

$$W_{3/2}(1, v^*) \leq \theta\, W_{3/2}(1, v)$$

<u>Proof</u> Indirect, by contradiction

3.2 Different rescalings

Recall $\left(\displaystyle\fint_{\partial B_r} u^2 \right)^{1/2} \leq C r^{3/2}$

Almgren:

$$u_{r_j}(x) = \frac{u(r_j x)}{\left(\frac{1}{r_j^{n-1}} \int_{\partial B_{r_j}} u^2 \right)^{1/2}}$$

$$\downarrow$$

$$C_n u_0(x) \qquad C_n \neq 0,\ C_n > 0$$

$3/2$-homogeneous

$$u_{r_j}^{(3/2)} = \frac{u(r_j x)}{r_j^{3/2}}$$

$$\downarrow$$

$$a u_0(x) \qquad a \geq 0, \text{ possibly } a = 0$$

7 EI applicable to u_{r_j}, j large $\quad \Rightarrow$ applicable to any positive multiple $u_{r_j}^{(3/2)}$, j large

Using Epiper-Ineq.

3.3. <u>Preposition</u> There exists $\gamma > 0$ s.t

$$0 \leq W_{3/2}(r,u) \leq C r^{\gamma} \quad \text{for} \quad 0 < r < r_0$$

<u>Proof</u> • $W_{3/2}(r,u) = \dfrac{H(r)}{r^{n+2\kappa-1}}\left(N(r)-\kappa\right) \geq 0 \qquad (\kappa = 3/2)$

• Upper bound from EI

$$\frac{d}{dr} W_{3/2}(r,u) = \frac{n+1}{r}\left[W_{3/2}(1, w_r) - W_{3/2}(1, u_r^{(3/2)})\right] + \frac{1}{r}\int_{\partial B_1}\left(\partial_\nu u_r^{(3/2)} - \frac{3}{2}u_r^{(3/2)}\right)^2$$

Here $u_r^{(3/2)}(x) = \dfrac{u(rx)}{r^{3/2}}$ $\qquad w_r(x) = |x|^{3/2}\, u^{(3/2)}(x/|x|)$ $\quad$ 3/2-homogeneous

By Epiperim Ineq

$$W_{3/2}(1, u_r^{(3/2)}) \leq \theta\, W_{3/2}(1, w_r) \qquad \text{for } r < r_0$$

$$\Downarrow$$

$$\frac{d}{dr} W_{3/2}(r,u) \geq \frac{n+1}{r}\left(\frac{1}{\theta}-1\right) W_{3/2}(r,u), \qquad r < r_0$$

$$\Rightarrow \quad W_{3/2}(r,u) \leq C r^{\gamma} \qquad \gamma = (n+1)\frac{1-\theta}{\theta}$$

3.4 <u>Proposition</u> (Control of rotation) r_0, γ as above. Then

$$\int_{\partial B_1} |u_t^{(3/2)} - u_s^{(3/2)}| \leq C t^{\gamma/2} \qquad 0 < s < t < r_0$$

Proof

$$\int_{\partial B_1} |u_t^{(3/2)} - u_s^{(3/2)}| \leq \int_{\partial B_1}\int_s^t \left|\frac{d}{dr} u_r^{(3/2)}(x)\right| = \int_{\partial B_1} \frac{|\partial_\nu u_r^{(3/2)} - 3/2\, u_r^{(3/2)}|}{r}$$

$$\leq \left(\int_s^t \frac{1}{r}\,dr\right)^{1/2}\left(\int_s^t \frac{d}{dr} W_{3/2}(r,u)\right)$$

$$\leq C\left(\log\frac{t}{s}\right)^{1/2} t^{\gamma/2}$$

$$\Rightarrow \int_{\partial B_1} |u_t^{(3/2)} - u_s^{(3/2)}| \leq C t^{\gamma/2} \quad \text{by dyadic argument}$$

3.5 Corollary If $u_{r_j}^{(3/2)} \longrightarrow u_0^{(3/2)}$ over some $r_j \to 0$

$$\int_{\partial B_1} |u_r^{(3/2)} - u_0^{(3/2)}| \leq C r^{\gamma/2} \quad , r < r_0$$

$\Rightarrow$ $u_r^{(3/2)} \longrightarrow u_0^{(3/2)}$ uniqueness of blowup

$$u_0^{(3/2)}(x) = a \operatorname{Re}(x_{n-1} + i|x_n|)^{3/2}$$

3.6. Nondegeneracy $u_0^{(3/2)} \not\equiv 0 \Leftrightarrow a \neq 0$ (Here we assume $0 \in \Gamma_{3/2}$)

Proof Assume not. Then

$$\int_{\partial B_1} |u_r^{(3/2)}| \leq C r^{\gamma/2}$$

Let $u_r = \dfrac{u(rx)}{\left(\frac{1}{r^{n-1}} H(r)\right)^{1/2}}$, $H(r) = \int_{\partial B_r} u^2$ Almgren rescaling

$$\Rightarrow \quad \int_{\partial B_1} |u_r| \leq C \left(\frac{r^{n+2}}{H(r)}\right) r^{\gamma/2}$$

$0 \in \Gamma_{3/2} \Rightarrow \frac{d}{dr} H(r) = \frac{n-1+2N(r)}{r} < \frac{n+2+\varepsilon}{r}$ if $r < r_\varepsilon$

$$\Rightarrow \quad H(r) \geq C r^{n+2+\varepsilon} \quad , r < r_\varepsilon$$

$$\Rightarrow \quad \int_{\partial B_1} |u_r| \leq C r^{(\gamma-\varepsilon)/2} \to 0 \quad \text{if } \varepsilon < \gamma$$

$$\Rightarrow \quad \int_{\partial B_1} |u_0| = 0 \qquad u_0 = C_n \operatorname{Re}(x_{n-1} + i|x_n|)^{3/2} \quad \text{contradiction}$$

3.7. Varying centers

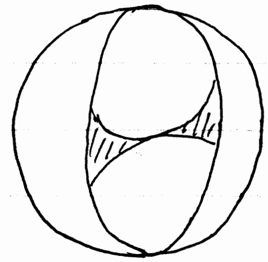
$$u_{r,x_0}^{(3/2)} = \frac{u(x_0 + rx)}{r^{3/2}} \longrightarrow a_{x_0} \operatorname{Re}(x' \cdot e_{x_0} + i|x_n|)^{3/2} = u_{0,x_0}^{(3/2)} \quad , |e_{x_0}| = 1$$

Proposition
$$\int_{\partial B_1} |u_{0,x_0}^{(3/2)} - u_{0,y_0}^{(3/2)}| \leq C|x_0 - y_0|^\beta \quad , \quad x_0, y_0 \in \Gamma_{3/2} \cap B'_{r_0}$$

$$\Rightarrow a_{x_0} \in C^\beta, \quad e_{x_0} \in C^\beta \quad \to \quad \Gamma_{3/2} \cap B'_{r_0} \in C^{1,\beta}$$

Lecture 3

1. Singular free boundary points



$$\begin{cases} \Delta u = 0 & \text{in } B_1^{\pm} \\ u \geq 0, \ -u_{x_n} \geq 0, \ u u_{x_n} = 0 & \text{on } B_1' \end{cases}$$

Wy say that $x_0 \in \Gamma(u)$ is singular if
$$\liminf_{r \to 0} \frac{|\Lambda(u) \cap B_r'(x_0)|}{|B_r'|} = 0$$

$$\Sigma = \{x_0 \in \Gamma : x_0 \text{ singular}\}$$

1.1. Proposition $x_0 \in \Sigma \Leftrightarrow \kappa(x_0) = N^{x_0}(0+, u) = 2m, \ m \in \mathbb{N}$

$$\Sigma = \bigcup_{m \in \mathbb{N}} \Gamma_{2m}$$

Proof $(\Rightarrow)$ $\quad u_r(x) = \dfrac{u(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{1/2}}$

$\Delta u_r = 2 (u_r)_{x_n} \mathcal{H}^{n-1} \lfloor_{\Lambda(u_r)}$ $\qquad$ 0 singular $\Leftrightarrow |\Lambda(u_{r_j})| \to 0$ for $r = r_j \to 0$

Let $u_{r_j} \to u_0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$ $\Rightarrow$ $\Delta u_0 = 0$ in $\mathbb{R}^n$.

We have

$\Delta u_0 = 0$ in $\mathbb{R}^n$, $u_0 \geq 0$ on $\mathbb{R}^{n-1}$, $u(x', -x_n) = u(x', x_n)$ $\Big|$ Liouville thm $\Rightarrow$ u_0 polynomial

u_0 homogeneous of degree κ $\qquad \kappa = 2m$

$(\Leftarrow)$ Let $u_{r_j} \to u_0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$

Then $\Delta u_0 \leq 0$ in $\mathbb{R}^n$, $\Delta u_0 = 0$ in $\mathbb{R}^n_{\pm}$ $\Big|$ $\cancel{u_0 \geq 0 \text{ on } \mathbb{R}^{n-1}}$

u_0 homogeneous of degree $2m$, $m \in \mathbb{N}$

1.2 Lemma (Monneau) If u_0 is as above $\Rightarrow \Delta u_0 = 0$ in $\mathbb{R}^n$

2 Monneau's monotonicity formula

2.1 Lemma Let $0 \in \Gamma_{2m} \subset \Sigma$. Then

$$M_{2m}(r) = \frac{1}{r^{n+4m-1}} \int_{\partial B_r} (u - p_{2m})^2 \nearrow \quad \text{for } r \in (0,1)$$

for any polynomial p_{2m}

- p_{2m} homogeneous of degree $2m$
- $\Delta p_{2m} = 0$, $p_{2m}(x', -x_n) = p_{2m}(x', x_n)$, $p_{2m}(x', 0) \geq 0$

Proof Let $w = u - p_{2m}$

- $$\frac{d}{dr} M_{2m}(r) = \frac{d}{dr}\left(\frac{1}{r^{n+4m-1}} \int_{\partial B_r} w^2 \right) = \frac{d}{dr}\left(\int_{\partial B_1} \frac{w^2(rx)}{r^{4m}} \right) = \int_{\partial B_1} \frac{2w(rx)\left(rx\nabla w(rx) - 2m\, w(rx)\right)}{r^{4m+1}}$$

$$= \frac{2}{r^{n+4m}} \int_{\partial B_r} w\left(x \nabla w(x) - 2m w\right)$$

- $$W_{2m}(r,u) = W_{2m}(r,u) - W_{2m}(r, p_{2m})$$

$$= \frac{1}{r^{n+4m-2}} \int_{B_r} |\nabla w|^2 + 2\nabla w \nabla p_{2m} - \frac{2m}{r^{n+4m-1}} \int_{\partial B_r} w^2 + 2 w p_{2m} = \cdots$$

$$= \frac{1}{r^{n+4m-2}} \int_{B_r} (-w\Delta w) + 2(-w \overset{0}{\overset{\parallel}{\Delta p_{2m}}}) + \frac{1}{r^{n+4m-1}} \int_{\partial B_r} w\left(x\nabla w - 2m w\right)$$

$$+ \frac{2}{r^{n+4m-1}} \int_{\partial B_r} w \overset{0}{\overset{\parallel}{\left(x \nabla p_{2m} - 2m\, p_{2m}\right)}}$$

$\boxed{\text{Key fact} \quad w \Delta w \geq 0}$

$$\leq \frac{1}{r^{n+4m-1}} \int_{\partial B_r} w\left(x\nabla w - 2m w\right)$$

$$\Rightarrow \quad \frac{d}{dr} M_{2m} \geq \frac{2}{r} W_{2m}(r,u) \geq 0$$

2.2 Corollary If $0 \in \Gamma_{2m} \subset \Sigma \Rightarrow$ 2m-homogeneous blowup is unique

$$u_r^{(2m)}(x) = \frac{u(rx)}{r^{2m}} \longrightarrow u_0^{(2m)} \quad \text{over } r = r_j \to 0$$

2m-homogeneous blowup

We know that $u_0^{(2m)} = p_{2m}$

Then $\quad M(r, u, p_{2m}) = \int_{\partial B_1} (u_r^{(2m)} - p_{2m})^2 \longrightarrow 0$ over $r = r_j \to 0+$

But $M(r) \nearrow \quad \Rightarrow \quad M(r) \to 0$ over all $r \to 0+$

$\Rightarrow \quad u_r^{(2m)} \to p_{2m}$ over all $r \to 0+$.

2.3 Preposition (Nondegeneracy) $\quad u_0^{(2m)} \not\equiv 0$

Proof Suppose $u_0^{(2m)} \equiv 0$. Then

$$\frac{1}{r^{n-1+4m}} \int_{\partial B_r} u^2 \to 0 \quad \Leftrightarrow \quad \left(\frac{H(r)}{r^{n-1}}\right)^{1/2} = h_r = o(r^{2m})$$

Consider Almgren rescalings and blowups

$$u_r(x) = \frac{u(rx)}{h_r} \longrightarrow u_0 = p_{2m} \not\equiv 0 \quad \text{as } \int_{\partial B_1} p_{2m}^2 = 1$$

$$M_{2m}(0+, u, p_{2m}) = \int_{\partial B_1} p_{2m}^2 = \frac{1}{r^{n+4m-1}} \int_{\partial B_r} p_{2m}^2$$

We must have

$$M_{2m}(r, u, p_{2m}) \geq M_{2m}(0+, u, p_{2m})$$

$$\frac{1}{r^{n+4m-1}} \int_{\partial B_r} (u - p_{2m})^2 \geq \frac{1}{r^{n+4m-1}} \int_{\partial B_r} p_{2m}^2$$

$$\Rightarrow \int_{\partial B_r} u^2 - 2u p_{2m} \geq 0 \quad \Leftrightarrow \quad \int_{\partial B_1} h_r^2 u_r^2 - 2 h_r r^{2m} p_{2m} \geq 0$$

$$\Rightarrow \int_{\partial B_1} \frac{h_r}{r^{2m}} u_r^2 - 2 u_r p_{2m} \geq 0 \quad \Rightarrow \quad \int_{\partial B_1} -2 p_{2m}^2 \geq 0 \quad \Rightarrow \quad p_{2m} \equiv 0 \text{ contradiction}$$

2.4 Varying centers

For $x_0 \in \Gamma_{2m} \subset \Sigma$ consider rescalings and blowups (2m-homogen.)

$$u^{(2m)}_{r,x_0} = \frac{u(x_0+rx)}{r^{2m}} \longrightarrow p^{x_0}_{2m}(x) \quad \text{as } r \to 0$$

unique blowup, $p^{x_0}_{2m} \not\equiv 0$

Proposition $x_0 \mapsto p^{x_0}_{2m}$ continuous on Γ_{2m}

Proof Similar to uniqueness

$$M^{x_0}(r, u, p^{x_0}_{2m}) = \frac{1}{r^{n+4m-1}} \int_{\partial B_r} (u(x_0+x) - p^{x_0}_{2m})^2$$

$\exists r_\varepsilon > 0$ s.t. $M^{x_0}(r_\varepsilon, u, p^{x_0}_{2m}) < \varepsilon$

Then vary x_0

$$M^{x_0'}(r_\varepsilon, u, p^{x_0}_{2m}) < 2\varepsilon \quad \text{if } |x_0' - x_0| < \delta_\varepsilon$$

From Monneau's monot. formula

$$M(1, p^{x_0'}_{2m}, p^{x_0}_{2m}) \leq 2\varepsilon \qquad \int_{\partial B_1} (p^{x_0'}_{2m} - p^{x_0}_{2m})^2 \leq 2\varepsilon$$

2.5 Proposition Asymptotic expansion

For $x_0 \in \Gamma_{2m}$ we have

$$u(x) = p^{x_0}_{2m}(x - x_0) + o(|x-x_0|^{2m})$$ uniformly for compact subsets of Γ_{2m}

$$|u(x) - p^{x_0}_{2m}(x-x_0)| \leq \sigma_K(|x-x_0|)\,|x-x_0|^{2m} \quad \text{for } K \subset\subset \Gamma_{2m}.$$

3. Structure of singular set

Suppose $x_0 \in \Gamma_{2m} \subset \Sigma$ and $p_{2m}^{x_0}$ is the $2m$-homog. blowup at x_0

$$\Pi_{x_0}^{(2m)} = \{ \xi \in \mathbb{R}^{n-1} : \partial_\xi p_{2m}^{x_0}(x',0) \equiv 0 \}$$

$$d_{x_0}^{(2m)} = \dim \Pi_{x_0}^{(2m)} \in \{0, \dots, n-2\}$$

$$\Gamma_{2m}^{d} = \{ x_0 \in \Gamma_{2m} : d_{x_0}^{(2m)} = d \}$$

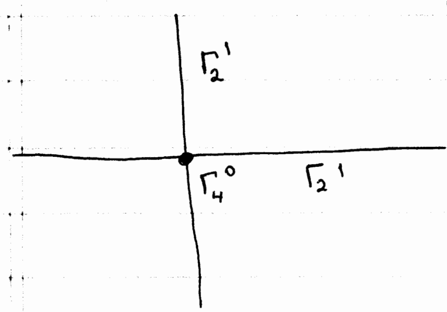
3.1 <u>Proposition</u> Γ_{2m}^{d} is contained in a countable union of d-dimensional C^1 manifolds

<u>Proof</u> Whitney's extension theorem.

3.2 <u>Example</u> $u = x_1^2 x_2^2 - (x_1^2 + x_2^2) x_3^2 + \frac{1}{3} x_3^4$ in $\mathbb{R}^3$

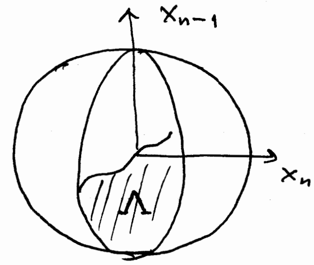
harmonic extension to $\mathbb{R}^3$ of

$u(x_1, x_2, 0) = x_1^2 x_2^2$



Cover after 2.3 in Lecture 2

4. Higher regularity of $\Gamma_{3/2}$



Slit domain $\quad B_1 \setminus \Lambda$

$\Lambda = \{x_n = 0, \; x_{n-1} \leq g(x'')\}$

$\Gamma = \{x_n = 0, \; x_{n-1} = g(x'')\}$

Suppose u and U harmonic in $B_1 \setminus \Lambda$, vanishing continuously on Λ

4.0 "Standard" Boundary Harnack $\qquad U > 0$ in $B_1 \setminus \Lambda$

If Γ is Lip $\Rightarrow \dfrac{u}{U} \in C^{\alpha}(B_{1/2})$ $\qquad$ even in x_n

4.1. Theorem (Higher order boundary Harnack of De Silva–Savin)

$\Gamma \in C^{k,\alpha} \Rightarrow \dfrac{u}{U} \in C^{k,\alpha}(x_1, \ldots, x_{n-1}, r) \qquad r = \sqrt{\operatorname{dist}(x', \Gamma)^2 + x_n^2}$

$k \geq 1$

4.2 Application to Signorini problem

Recall, if $0 \in \Gamma_{3/2}$ $\quad \partial_e u \geq 0$ for a cone of directions $e \in \mathcal{C}_\delta'$

In particular $\Lambda(u)$ is given with Lip g

Then "standard" boundary Harnack implies

$$\frac{\partial_{e_i} u}{\partial_{e_n} u} \in C^{\alpha}(B_{1/2}) \Rightarrow g \in C^{1,\alpha} \quad (\Gamma \in C^{1,\alpha})$$

Next, higher order boundary Harnack implies

$$\Gamma \in C^{k,\alpha} \Rightarrow \frac{\partial_{e_i} u}{\partial_{e_n} u} \in C^{k,\alpha} \Rightarrow \Gamma \in C^{k+1,\alpha}$$

$$\Rightarrow \Gamma \in C^{\infty}$$