A PARABOLIC ALMOST MONOTONICITY FORMULA

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Abstract. We prove the parabolic counterpart of the almost monotonicity formula of Caffarelli, Jerison and Kening for pairs of functions $u_\pm(x,s)$ in the strip $S_1=\mathbb{R}^n\times(-1,0]$ satisfying

$$u_\pm\geq 0,\quad (\Delta-\partial_s)u_\pm\geq -1,\quad u_+\cdot u_-=0\quad\text{in } S_1.$$

We also establish a localized version of the formula as well as prove one of its variants. At the end of the paper we give an application to a free boundary problem related to the caloric continuation of heat potentials.

Introduction

In [Caf93] Caffarelli established the following monotonicity formula for caloric functions in disjoint domains: If $u_\pm(x,s)$ are two continuous functions in the unit strip $S_1=\mathbb{R}^n\times(-1,0]$ satisfying

$$u_\pm\geq 0,\quad (\Delta-\partial_s)u_\pm\geq 0,\quad u_+\cdot u_-=0\quad\text{in } S_1,$$

then the functional

$$\Phi(r)=\frac{1}{r^4}\int_{-r^2}^0\int_{\mathbb{R}^n}|\nabla u_+|^2G(x,-s)\,dx\,ds\int_{-r^2}^0\int_{\mathbb{R}^n}|\nabla u_-|^2G(x,-s)\,dx\,ds$$

is monotone increasing in r, $0<r<1$, provided $u_\pm$ have moderate growth at infinity. Here $G(x,t)$ is the heat kernel. This is a direct parabolic analogue of the celebrated monotonicity formula of Alt, Caffarelli, and Friedman [ACF84], which says that for continuous functions $u_\pm(x)$ in the unit ball B_1, satisfying

$$u_\pm\geq 0,\quad \Delta u_\pm\geq 0,\quad u_+\cdot u_-=0\quad\text{in } B_1,$$

the functional

$$\varphi(r)=\frac{1}{r^4}\int_{B_r}\frac{|\nabla u_+|^2}{|x|^{n-2}}dx\int_{B_r}\frac{|\nabla u_-|^2}{|x|^{n-2}}dx$$

is monotone increasing in r, $0<r<1$.

These monotonicity formulas (and their variations) have been extremely important in the regularity theory of elliptic and parabolic free boundary problems, see e.g. [ACF84, Caf87, Caf89, Caf88, Caf95, CK98, CKS00, CPS04, Ura01]. One significance of the monotonicity formulas is the ability to produce the following kind of

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estimates (say, in the elliptic case):

$$c_n|\nabla u_+(0)|^2|\nabla u_-(0)|^2 \leq \varphi(0+) \leq \varphi(1/2) \leq C_n\|u_+\|^2_{L^2(B_1)}\|u_-\|^2_{L^2(B_1)},$$

which makes them a central tool in establishing the optimal regularity of the solutions in two-phase and other free boundary problems.

Recently, Caffarelli, Jerison, and Kenig [CJK02] generalized the elliptic monotonicity formula to the functions $u_\pm$ satisfying

$$u_\pm \geq 0, \quad \Delta u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad \text{in } B_1.$$

The function φ, however, is no longer monotone but still has an estimate

$$\varphi(r) \leq C_n \left(1 + \|u_+\|^2_{L^2(B_1)} + \|u_-\|^2_{L^2(B_1)}\right)^2, \quad 0 < r \leq 1/2.$$

This is known in the literature as the *almost monotonicity formula*. In particular, one still has a control of $|\nabla u_+(0)||\nabla u_-(0)|$. The main objective of this paper is to establish a parabolic version of this almost monotonicity formula.

One of the differences between the elliptic and parabolic monotonicity formulas is that the elliptic ones are "local" in the sense that $u_\pm$ must be defined only in a ball, say B_1. For the parabolic formula, however, $u_\pm$ must be defined in an infinite strip such as $S_1 = \mathbb{R}^n \times (-1, 0]$. One of the ways to obtain a localized version of the formula for subsolutions $u_\pm$ defined only in a parabolic cylinder $Q_1^- = B_1 \times (-1, 0]$ is to multiply them with a cutoff function $\psi(x)$ thus extending them to S_1. This introduces a small error in the computations, which, however, can be easily controlled.

Finally, let us make a remark on the proof of the parabolic almost monotonicity formula. Caffarelli-Jerison-Kenig's proof of the elliptic formula (Theorem 1.3 in [CJK02]) consists of two independent parts: the first ("technical") part establishes recursive inequalities based on the properties of subsolutions $u_\pm$; the second ("arithmetic") part is purely arithmetic and uses the recursive inequalities proved in the first part to obtain the required inequality. This means that we only need to establish the parabolic counterpart of the "technical" part and can reuse the "arithmetic" part.

Structure of the paper. In Section 1 we prove the monotonicity formula for solutions in the infinite strip (Theorem I). In Section 2 we prove the localized form of the monotonicity formula (Theorem II). Section 3 is devoted to a variation of the almost monotonicity formula (Theorem III) with more features of the monotonicity under additional assumptions on the growth of $u_\pm$ near the origin. Finally, in Section 4 we give an applications in a parabolic free boundary problem.

Notation. Throughout the paper we will use the following notations:

$$\begin{aligned}
&B_r(x) = \{y \in \mathbb{R}^n : |x-y| < r\} && \text{(spatial ball)}\\
&B_r = B_r(0) &&\\
&Q_r^- = B_r \times (-r^2, 0] && \text{(lower parabolic cylinder)}\\
&S_r = \mathbb{R}^n \times (-r^2, 0] && \text{(infinite strip)}
\end{aligned}$$

$$
\begin{aligned}
&G(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, && x \in \mathbb{R}^n, t > 0 \quad \text{(the heat kernel)}\\
&d\gamma(x,s) = G(x,-s)\,dx\,ds && \\
&d\gamma^s(x) = G(x,-s)\,dx && \\
&d\nu = d\gamma^{-1/2} && \text{(the standard Gaussian measure)}\\
&\Delta u = \sum_{i=1}^n \partial_{x_i x_i} u && \text{(Laplacian)}\\
&\nabla u = \nabla_x u = (\partial_{x_1} u, \dots, \partial_{x_n} u) && \text{(spatial gradient)}
\end{aligned}
$$

For integrals in space and time we use the double-integral sign $\iint$, regardless of the space dimension, while for the integrals in space only we use the single-integral sign $\int$.

1. The global case

Theorem I (Almost Monotonicity Formula)**.** *Suppose we have two continuous functions $u_\pm(x,s)$ is the unit strip S_1, which satisfy*

$$
u_\pm \geq 0, \quad (\Delta - \partial_s) u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad \textit{in} \quad S_1.
$$

Assume also that $u_\pm$ have moderate growth at infinity, for instance

$$
|u_\pm(x,s)| \leq C e^{|x|^2/(8+\epsilon)}, \quad (x,s) \in S_1
$$

for some $\epsilon > 0$. Then the functional

$$
\Phi(r) := r^{-4} A_+(r) A_-(r), \quad \textit{where} \quad A_\pm(r) := \iint_{S_r} |\nabla u_\pm|^2\, d\gamma,
$$

satisfies

$$
\Phi(r) \leq C(1 + A_+(1) + A_-(1))^2, \quad 0 < r \leq 1,
$$

for an absolute constant C.

Remark 1.1. Everywhere in this paper we assume that the inequality $(\Delta - \partial_s)u_\pm \geq -1$ is satisfied in the sense of distributions. The standard energy inequality implies that $\nabla u_\pm$ are in fact in $L^2_{\text{loc}}(S_1)$. Moreover, as we will see later in Proposition 1.1, $A_\pm(r)$ are finite.

As we mentioned in Introduction, to prove this theorem we only need to establish the parabolic counterpart of the "technical" part of the proof of Theorem 1.3 in [CJK02]. This will consist of six propositions below. For reader's convenience we give direct references to the corresponding parts in [CJK02].

Before we proceed, we also remark that some of the constants that appear in the elliptic case are dimension-dependent, while the corresponding constants in the parabolic case are actually absolute. In the local case (Section 2), the constants again depend on the dimension.

Proposition 1.1 (cf. Remark 1.5 in [CJK02])**.** *Let $u \geq 0$ and $(\Delta - \partial_s)u \geq -1$ in S_2. Assume also that $|u(x,s)| \leq C e^{|x|^2/(32+\epsilon)}$, $(x,s) \in S_2$. Then there exists an absolute constant C such that we have the following estimates*

$$
\iint_{S_1} |\nabla u|^2 d\gamma \leq C + \left[\int_{\mathbb{R}^n} u(\cdot,-1)^2 d\gamma^{-1} \right]^{1/2} + \frac{1}{2} \int_{\mathbb{R}^n} u(\cdot,-1)^2 d\gamma^{-1} \tag{1.1}
$$

$$\iint_{S_1} |\nabla u|^2 d\gamma \leq C + C \inf_{s\in[-4,-1]} \int_{\mathbb{R}^n} u(\cdot,s)^2 d\gamma^s, \tag{1.2}$$

and

$$\iint_{S_1} |\nabla u|^2\, d\gamma \leq C + C \iint_{S_2\setminus S_1} u^2\, d\gamma. \tag{1.3}$$

Proof. We start with an observation that if u^ϵ are mollifications of u (convolutions with a mollifier) then u^ϵ still satisfy the same assumptions in the proposition as u. Moreover, once we prove the inequalities (1.1)–(1.3) for u^ϵ, the corresponding inequalities for u will follow simply by passing to the limit. Thus, without loss of generality, we may assume that u is a C^∞ function both in x and in t.

The assumptions $(\Delta - \partial_s)u \geq -1$ and $u \geq 0$ imply that

$$(\Delta - \partial_s)(u^2) \geq -2u + 2|\nabla u|^2$$

and therefore

$$2\iint_{S_r} |\nabla u|^2 d\gamma \leq \iint_{S_r} (\Delta - \partial_s)(u^2) d\gamma + 2\iint_{S_r} u\, d\gamma =: I_1(r) + I_2(r)$$

for any $0 < r < 2$. We next estimate each of the integrals I_1 and I_2.

1) Using that $(\Delta + \partial_s)G(x,-s) = 0$ for $s < 0$ and integrating by part in x-variables, we obtain

$$\begin{aligned} I_1(r) &= \iint_{S_r} (\Delta - \partial_s)(u^2) G(x,-s) dx ds = -\int_{-r^2}^0 \int_{\mathbb{R}^n} \partial_s(u^2 G(x,-s)) dx ds \\ &\leq \int_{\mathbb{R}^n} u(\cdot,-r^2)^2 d\gamma^{-r^2}. \end{aligned}$$

Note that there are no spatial boundary terms after integration by parts in x-variables, because of the growth assumption on u.

2) To estimate $I_2(r)$ notice that $(\Delta - \partial_s)(u(x,s) - s) \geq 0$, which implies that

$$\int_{\mathbb{R}^n} (u(\cdot,s_1) - s_1)\, d\gamma^{s_1} \leq \int_{\mathbb{R}^n} (u(\cdot,s_2) - s_2)\, d\gamma^{s_2}$$

for any $-r^2 < s_2 \leq s_1 \leq 0$, $0 < r < 2$. Consequently, for such s_1, s_2

$$\int_{\mathbb{R}^n} u(\cdot,s_1)\, d\gamma^{s_1} \leq r^2 + \int_{\mathbb{R}^n} u(\cdot,s_2)\, d\gamma^{s_2} \leq r^2 + \left[\int_{\mathbb{R}^n} u(\cdot,s_2)^2\, d\gamma^{s_2}\right]^{1/2}$$

and therefore

$$\begin{aligned} I_2(r) &\leq 2r^4 + 2r^2 \left[\int_{\mathbb{R}^n} u(\cdot,-r^2)^2 d\gamma^{-r^2}\right]^{1/2} \\ &\leq 3r^4 + \int_{\mathbb{R}^n} u(\cdot,-r^2)^2 d\gamma^{-r^2}, \quad 0 < r < 2. \end{aligned}$$

Now, collecting the estimates for I_1 and I_2 for $r = 1$, we immediately obtain (1.1). Further,

$$2\iint_{S_1} |\nabla u|^2 d\gamma \leq 2\iint_{S_r} |\nabla u|^2 d\gamma \leq 3r^4 + 2\int_{\mathbb{R}^n} u(x,-r^2)^2 d\gamma^{-r^2}$$

for any $1 \le r < 2$. Taking the infimum of the latter quantity for all such r, we obtain

$$\iint_{S_1} |\nabla u|^2 d\gamma \le 24 + \inf_{s\in[-4,-1]} \int_{\mathbb{R}^n} u(\cdot,s)^2 d\gamma^s,$$

which gives (1.2). Finally, (1.3) is a simple consequence from (1.2). □

Proposition 1.2 (cf. Lemma 2.1 in [CJK02])**.** *Let $u \ge 0$, $(\Delta - \partial_s)u \ge -1$ in S_1 and $\Omega := \{u > 0\}$. Suppose*

$$\iint_{\Omega\cap S_1} |\nabla u|^2 d\gamma = \alpha < \infty$$

and

$$\iint_{\Omega\cap S_{1/4}} |\nabla u|^2 d\gamma \ge \frac{\alpha}{256}.$$

Then

$$|\Omega \cap (S_{1/2} \setminus S_{1/4})| \ge c_0 > 0,$$

provided $\alpha > \alpha_0$ for sufficiently large α_0. Here

$$|E| = \gamma(E) = \iint_E d\gamma, \qquad \text{for} \quad E \subset \mathbb{R}^n \times (-\infty, 0).$$

To prove this proposition, we will need the log-Sobolev inequality of Gross [Gro75].

Lemma 1.1 (Log-Sobolev inequality)**.** *For any $f \in L^2(\mathbb{R}^n, d\nu)$ with $\nabla f \in L^2(\mathbb{R}^n, d\nu)$ one has*

$$\int_{\mathbb{R}^n} f^2 \log f^2 \, d\nu \le \int_{\mathbb{R}^n} f^2 \, d\nu \log \int_{\mathbb{R}^n} f^2 \, d\nu + 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\nu. \tag{1.4}$$

Here $d\nu = (2\pi)^{-n/2} e^{-|x|^2/2} dx = d\gamma^{-1/2}$ is the standard Gaussian measure.

We will also need the following corollary from the log-Sobolev inequality.

Lemma 1.2. *For $f \in L^2(\mathbb{R}^n, d\nu)$ with $\nabla f \in L^2(\mathbb{R}^n, d\nu)$ let $\omega = \{|f| > 0\}$. Then*

$$\log \frac{1}{|\omega|} \int_{\mathbb{R}^n} f^2 \, d\nu \le 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\nu, \tag{1.5}$$

where $|\omega| = \nu(\omega) = \int_\omega d\nu$.

Proof. Let us define $\psi(y) = y \log y$ for $y > 0$ and $\psi(0) = 0$. Then the log-Sobolev inequality can be rewritten as

$$\int_{\mathbb{R}^n} \psi(f^2) \, d\nu \le \psi\left(\int_{\mathbb{R}^n} f^2 \, d\nu\right) + 2 \int_{\mathbb{R}^n} |\nabla f|^2 \, d\nu.$$

On the other hand, since ψ is convex on $[0, \infty)$, by Jensen's inequality we have

$$\frac{1}{|\omega|} \int_{\mathbb{R}^n} \psi(f^2) \, d\nu \ge \psi\left(\frac{1}{|\omega|} \int_{\mathbb{R}^n} f^2 \, d\nu\right).$$

Combining these inequalities and using the identity $\lambda\,\psi\left(\frac{a}{\lambda}\right) - \psi(a) = a \log \frac{1}{\lambda}$, we arrive at (1.5). □

Proof of Proposition 1.2. We want to apply (1.5) to function $u(\cdot,s)$ with respect to the Gaussian measure $d\gamma^s$, for $s\in[-(\frac12)^2,-(\frac14)^2]$. So, let
$$\omega(s):=\{u(\cdot,s)>0\}.$$
Then
$$\log\frac{1}{|\omega(s)|}\int_{\mathbb{R}^n}u(\cdot,s)^2\,d\gamma^s\leq C\int_{\mathbb{R}^n}|\nabla u(\cdot,s)|^2\,d\gamma^s.$$
We now use Proposition 1.1, more precisely the estimate (1.2). Then
$$\frac{\alpha}{256}\leq\int_{S_{1/4}}|\nabla u|^2\,d\gamma\leq C+C\inf_{s\in[-(1/2)^2,-(1/4)^2]}\int_{\mathbb{R}^n}u(\cdot,s)^2d\gamma^s$$
Now, if $\alpha>512\,C$ then
$$\int_{\mathbb{R}^n}u(\cdot,s)^2d\gamma^s\geq\frac{\alpha}{512C}\quad\text{for all } s\in[-(\tfrac12)^2,-(\tfrac14)^2]$$
On the other hand, since
$$\iint_{S_1}|\nabla u|^2\,d\gamma=\alpha,$$
we have
$$\int_{\mathbb{R}^n}|\nabla u(\cdot,s)|^2\,d\gamma^s\leq16\alpha,$$
for any $s\in(-1,0)$ except a set of linear measure at most $1/16$. Since the length of the interval $[-(\frac12)^2,-(\frac14)^2]$ is $3/16$ we obtain that for a set of $s\in[-(\frac12)^2,-(\frac14)^2]$ of linear measure at least $1/8$ we must have
$$\log\frac{1}{|\omega(s)|}\frac{\alpha}{512C}\leq16\,\alpha\,C$$
which implies that
$$|\omega(s)|\geq e^{-2^{13}C^2}$$
and consequently
$$|\Omega\cap(S_{1/2}\setminus S_{1/4})|\geq\frac{e^{-2^{13}C^2}}{8}.$$
□

Proposition 1.3 (cf. Lemma 2.3 in [CJK02]). *Let $u\geq0$, $(\Delta-\partial_s)u\geq-1$ in S_1, and $\Omega=\{u>0\}$. Suppose*
$$\iint_{\Omega\cap S_1}|\nabla u|^2\,d\gamma=\alpha<\infty.$$
Suppose also there exists $\lambda>0$ such that
$$|\Omega\cap(S_{1/2}\setminus S_{1/4})|<(1-\lambda)|S_{1/2}\setminus S_{1/4}|.$$
Then there exists $\mu<1$, depending only λ, such that
$$\iint_{\Omega\cap S_{1/4}}|\nabla u|^2\,d\gamma\leq\mu\iint_{\Omega\cap S_1}|\nabla u|^2\,d\gamma,$$
provided $\alpha>\alpha_0$ for sufficiently large α_0.

Proof. Since $|\Omega\cap(S_{1/2}\setminus S_{1/4})| = \int_{-(1/2)^2}^{-(1/4)^2}|\omega(s)|ds$ it follows that $|\omega(s)|\leq(1-\lambda/2)$ for s in a set $E_\lambda\subset[-(\frac{1}{2})^2,-(\frac{1}{4})^2]$ of the linear measure $|E_\lambda|\geq(\lambda/2)|S_{1/2}\setminus S_{1/4}|$. Thus by Lemma 1.1

$$\int_{\mathbb{R}^n}u(\cdot,s)^2\,d\gamma^s\leq C_\lambda\int_{\mathbb{R}^n}|\nabla u(\cdot,s)|^2\,d\gamma^s,\qquad s\in E_\lambda, \tag{1.6}$$

where $C_\lambda=-C/\log(1-\lambda/2)$. Now, if

$$\iint_{\Omega\cap S_{1/4}}|\nabla u|^2\,d\gamma\leq\alpha/2$$

then there is nothing to prove. Otherwise we apply Proposition 1.1, which gives

$$\frac{\alpha}{2}\leq C+C\inf_{s\in[-(1/2)^2,-(1/4)^2]}\int_{\mathbb{R}^n}u(\cdot,s)^2\,d\gamma^s$$

and if $\alpha>\alpha_0$ for sufficiently large α_0, we will have

$$\frac{\alpha}{4C}\leq\int_{\mathbb{R}^n}u(\cdot,s)^2\,d\gamma^s\leq C_\lambda\int_{\mathbb{R}^n}|\nabla u(\cdot,s)|^2\,d\gamma^s$$

for all $s\in E_\lambda$. This implies that

$$\iint_{\Omega\cap(S_{1/2}\setminus S_{1/4})}|\nabla u|^2\,d\gamma\geq\frac{\alpha|E_\lambda|}{4CC_\lambda}$$

and the proposition follows with $\mu=(1-3\lambda/32CC_\lambda)$. □

Proposition 1.4 (cf. Lemma 2.4 in [CJK02]). *Let $u_\pm$ and $A_\pm$ be as in Theorem I. There exists an absolute constant C_0 such that if $A_\pm(r)\geq C_0$ for all $r\in[\frac{1}{4},1]$, then*

$$\Phi'(r)\geq-C\left(\frac{1}{\sqrt{A_+(r)}}+\frac{1}{\sqrt{A_-(r)}}\right)\Phi(r).$$

for all $r\in[\frac{1}{4},1]$. (Recall that $\Phi(r)=r^{-4}A_+(r)A_-(r)$.)

Proof. We start with the same remark as in the proof of Lemma 2.4 in [CJK02]. The functions $A_\pm$ are continuous nondecreasing functions, hence Φ' is the sum of a nonnegative singular measure and an absolutely continuous part and we need to obtain the bound on Φ' at the points r that are Lebesgue point both for both $A_\pm$. Thus, we assume that r is such that

$$B_\pm(r)=\int_{\mathbb{R}^n}|\nabla u_\pm(\cdot,-r^2)|^2\,d\gamma^{-r^2}<\infty.$$

For the sake of notational convenience assume that $r=1$ and abbreviate $A^\pm=A_\pm(1)$, $B^\pm=B_\pm(1)$. Then

$$\frac{\Phi'(1)}{\Phi(1)}=-4+2\,\frac{B^+}{A^+}+2\,\frac{B^-}{A^-}$$

Next, by inequality (1.1) in Proposition 1.1, we have

$$2A^+\leq C_1+C_1\left[\int_{\mathbb{R}^n}u_+(\cdot,-1)^2\,d\gamma^{-1}\right]^{1/2}+\int_{\mathbb{R}^n}u_+(\cdot,-1)^2d\gamma^{-1}$$

for some absolute constant C_1. Before we proceed, observe that $u_+(\cdot,-1)$ cannot vanish identically on $\mathbb{R}^n$ if the constant C_0 in the statement of Proposition 1.4 is sufficiently large. Indeed, otherwise we would have $A^+ \le C_1/2$, which would be a contradiction. Similarly, $u_-(\cdot,-1)$ cannot vanish identically on $\mathbb{R}^n$. Then

$$\omega^\pm = \{u_\pm(\cdot,-1) > 0\}$$

are nonempty. The eigenvalue inequality for ω_+ now says

$$\lambda_+ \int_{\omega_+} f^2\, d\gamma^{-1} \le \int_{\omega_+} |\nabla f|^2\, d\gamma^{-1},$$

which implies

$$2A^+ \le C_1 + C_1\sqrt{B^+/\lambda_+} + B^+/\lambda_+. \tag{1.7}$$

Clearly we have a similar inequality for u_-. The proof will follow now by simple arithmetic from the eigenvalue inequality of Beckner, Kenig and Pipher [BKP98], which says

$$\lambda_+ + \lambda_- \ge 2\lambda_0 = 1, \tag{1.8}$$

where λ_0 is the eigenvalue corresponding to the halfspace[1]. We have the following several cases:

1) $B^+ \ge 2A^+$ (or $B^- \ge 2A^-$). Then

$$\frac{\Phi'(1)}{\Phi(1)} = -4 + 2\,\frac{B^+}{A^+} + 2\,\frac{B^-}{A^-} \ge 0.$$

2) $B^+ \le 2A^+$ and $\lambda_+ \ge 1$ (or $B^- \le 2A^-$ and $\lambda_- \ge 1$). Then by (1.7), if $A^+ \ge C_0$ is sufficiently large

$$2A^+ \le C_2\sqrt{A^+} + B^+.$$

It follows then

$$\frac{\Phi'(1)}{\Phi(1)} \ge -4 + 2\,\frac{B^+}{A^+} \ge -\frac{2C_2}{\sqrt{A^+}}.$$

3) $B^\pm \le 2A^\pm$ and $\lambda_\pm \le 1$. Then by (1.7), if $A^\pm \ge C_0$ are sufficiently large

$$2\lambda_\pm A^\pm \le C_3\sqrt{A^\pm} + B^\pm.$$

It follows then

$$\begin{aligned}\frac{\Phi'(1)}{\Phi(1)} &= -4 + 2\,\frac{B^+}{A^+} + 2\,\frac{B^-}{A^-} \ge -\frac{2C_2}{\sqrt{A^+}}\\ &\ge -4 + 4(\lambda_+ + \lambda_-) - C_3\left(\frac{1}{\sqrt{A^+}} + \frac{1}{\sqrt{A^-}}\right).\end{aligned}$$

Then the estimate for $\Phi'(1)$ follows now from (1.8). In the exact same way we prove the estimate for $\Phi'(r)$ for any $r \in [\frac{1}{4}, 1]$ and the proof is complete. □

As we already mentioned, the propositions above constitute the technical core of the proof of Theorem I. The rest of the proof is of purely arithmetic nature and is exactly the same as in [CJK02]. Let

$$A_k^\pm = A_\pm(4^{-k}), \quad b_k^\pm = 4^{4k} A_k^\pm.$$

[1] Since [BKP98] is not published, we refer to Section 2.4 in [CK98] for the proof of (1.8).

One should treat $b_k^\pm$ as the correctly rescaled versions of $A_k^\pm$, which is explained as follows. If $u \geq 0$ satisfies $(\Delta - \partial_s)u \geq -1$ then the rescaling $u_r(x,s) := r^{-2}u(rx, r^2 s)$ will satisfy exactly the same inequalities and one will have

$$\iint_{S_1} |\nabla u_r|^2 \, d\gamma = r^{-4} \iint_{S_r} |\nabla u|^2 d\gamma. \tag{1.9}$$

Note also that

$$\Phi(4^{-k}) = 4^{4k} A_k^+ A_k^-.$$

Proposition 1.5 (cf. Lemma 2.8 in [CJK02]). *There exists an absolute constant C such that if $b_k^\pm \geq C$, then*

$$4^4 A_{k+1}^+ A_{k+1}^- \leq A_k^+ A_k^- (1 + \delta_k) \quad \textit{with} \quad \delta_k = \frac{C}{\sqrt{b_k^+}} + \frac{C}{\sqrt{b_k^-}}$$

Proof. The proof in [CJK02] is obtained from Lemma 2.4 by arithmetic arguments. Using Proposition 1.4 instead, we will obtain the proof of Proposition 1.5. □

Proposition 1.6 (cf. Lemma 2.9 in [CJK02]). *There exist an absolute constant $\epsilon > 0$ such that if $b_k^\pm \geq C_0$ and $4^4 A_{k+1}^+ \geq A_k^+$, then $A_{k+1}^- \leq (1 - \epsilon) A_k^-$.*

Proof. See the proof in [CJK02]. We just use Propositions 1.2 and 1.3 instead of their counterparts, Lemmas 2.1 and 2.3, respectively. □

Proof of Theorem I. As already noted in [CJK02], Theorem 1.3 there is obtained from Lemmas 2.8 and 2.9 by pure arithmetic. Using Propositions 1.5 and 1.6 instead, we obtain the proof of Theorem I. □

2. The local case

Theorem II (Localized Almost Monotonicity Formula). *Suppose now we have two continuous L^2 functions $u_\pm(x,s)$ in the lower parabolic half-cylinder Q_3^-, which satisfy*

$$u_\pm \geq 0, \quad (\Delta - \partial_s) u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad \textit{in} \quad Q_3^-.$$

Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function in x such that

$$0 \leq \psi \leq 1, \quad \operatorname{supp} \psi \subset B_2, \quad \psi\big|_{B_1} = 1.$$

Then if $w_\pm(x,s) = u_\pm(x,s)\psi(x)$ for $(x,s) \in S_3$, the functional

$$\Phi(r) := r^{-4} A_+(r) A_-(r), \quad \textit{where} \quad A_\pm(r) := \iint_{S_r} |\nabla w_\pm|^2 \, d\gamma,$$

satisfies

$$\Phi(r) \leq C_n \left(1 + \|u_+\|^2_{L^2(Q_3^-)} + \|u_-\|^2_{L^2(Q_3^-)}\right)^2, \quad 0 < r \leq 1,$$

where C_n depends only on the dimension n and the cutoff function ψ.

The proof follows the lines of the proof of Theorem I. However, we need to state the appropriate scaled versions of the estimates, since we want to have the dependence on constants M and n only.

Proposition 2.1 (cf. Proposition 1.1)**.** *Let $u \geq 0$ and $(\Delta - \partial_s)u \geq -1$ in Q_3^-. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function as in Theorem II and $w(x,s) = u(x,s)\psi(x)$, $(x,s) \in S_3$. Then for $0 < r \leq 1$,*

$$\iint_{S_r} |\nabla w|^2 d\gamma \leq C_M r^4 + r^2 \left[\int_{\mathbb{R}^n} w(\cdot,-r^2)^2 d\gamma^{-r^2}\right]^{1/2} + \frac{1}{2}\int_{\mathbb{R}^n} w(\cdot,-r^2)^2 d\gamma^{-r^2}, \tag{2.1}$$

$$\iint_{S_r} |\nabla w|^2 d\gamma \leq C_M r^4 + C_M \inf_{s\in[-4r^2,-r^2]} \int_{\mathbb{R}^n} w(\cdot,s)^2 d\gamma^s, \tag{2.2}$$

and

$$\iint_{S_r} |\nabla w|^2 d\gamma \leq C_M r^4 + \frac{C_M}{r^2}\iint_{S_{2r}\setminus S_r} w^2 d\gamma, \tag{2.3}$$

with C_M, depending only on $M = \|u\|_{L^2(Q_2^-)}$, the dimension n only and the cutoff function ψ.

Proof. We revisit the proof of Proposition 1.1. First note that for the function $w(x,s)$ we have the following inequality

$$\begin{aligned}(\Delta - \partial_s)w =& \psi(\Delta - \partial_s)u + u\Delta\psi + 2\nabla\psi\nabla u\\ &\geq -1 + u\Delta\psi + 2\nabla\psi\nabla u,\end{aligned}$$

where the last two terms are supported outside the ball B_1, where $\psi \equiv 1$. Next, we obtain

$$\begin{aligned}(\Delta - \partial_s)(w^2) &= 2|\nabla w|^2 + 2w(\Delta - \partial_s)w\\ &\geq 2|\nabla w|^2 + 2w(-1 + u\Delta\psi + 2\nabla\psi\nabla u)\\ &\geq 2|\nabla w|^2 - 2w + 2u^2\psi\Delta\psi + \nabla(u^2)\nabla(\psi^2),\end{aligned}$$

where again the last two terms are supported outside B_1. We also notice that the standard energy estimates in this case implies

$$\iint_{Q_2^-} |\nabla u|^2 \leq C_n + C_n \iint_{Q_3^-} u^2. \tag{2.4}$$

This, together with the estimate of the Gaussian function

$$0 \leq G(x,t) \leq \frac{C_n}{t^{n/2}} e^{-1/4t} \leq C_{n,m} t^m, \quad |x| \geq 1,\ t > 0, \tag{2.5}$$

which holds with any $m > 0$, will help us to control those additional extra terms that come from the cutoff function.

We then estimate

$$\begin{aligned}2\iint_{S_r} |\nabla w|^2 d\gamma \leq& \iint_{S_r} (\Delta - \partial_s)(w^2) d\gamma + 2\iint_{S_r} w\, d\gamma\\ &- \iint_{S_r} [2u^2\psi\Delta\psi + \nabla(u^2)\nabla(\psi^2)] d\gamma =: I_1(r) + I_2(r) + I_3(r)\end{aligned}$$

for any $0 < r \leq 2$.

1) Arguing precisely as in the proof of Proposition 1.1, we obtain

$$I_1(r) \leq \int_{\mathbb{R}^n} w^2(\cdot,-r^2) d\gamma^{-r^2}, \quad 0 < r \leq 2.$$

2) To estimate $I_2(r)$, consider the auxiliary function $\hat{w}(x,s) = w(x,s) - s$, which satisfies

$$(\Delta - \partial_s)\hat{w} \geq u\Delta\psi + 2\nabla\psi\nabla u.$$

Integrating by parts, for any $-4 \leq s_2 \leq s_1 < 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \hat{w}(x,s_1)d\gamma^{s_1} - \int_{\mathbb{R}^n} \hat{w}(x,s_2)d\gamma^{s_2} &= -\int_{s_2}^{s_1}\int_{\mathbb{R}^n}(\Delta - \partial_s)\hat{w}\, d\gamma\\ &\leq -\int_{s_2}^{s_1}\int_{B_2\setminus B_1}[u\Delta\psi + 2\nabla\psi\nabla u]G(x,-s)dx\, ds\\ &\leq C_M|s_2|,\end{aligned}$$

where we have used the inequalities (2.4) and (2.5) with $m = 1$ in the last step. Consequently, for $-r^2 \leq s_2 \leq s_1 < 0$ with $0 < r \leq 2$, we have

$$\int_{\mathbb{R}^n} w(\cdot, s_1)\, d\gamma^{s_1} \leq C_M r^2 + \int_{\mathbb{R}^n} w(\cdot, s_2)\, d\gamma^{s_2} \leq C_M r^2 + \left[\int_{\mathbb{R}^n} w(\cdot, s_2)^2\, d\gamma^{s_2}\right]^{1/2}$$

and therefore

$$\begin{aligned}I_2(r) &\leq C_M r^4 + 2r^2\left[\int_{\mathbb{R}^n} w(\cdot, -r^2)^2 d\gamma^{-r^2}\right]^{1/2}\\ &\leq (C_M+1)r^4 + \int_{\mathbb{R}^n} w(\cdot,-r^2)^2 d\gamma^{-r^2}, \quad 0 < r \leq 2.\end{aligned}$$

3) Using the energy estimate (2.4) and estimate (2.5) on G outside B_1 with $m = 1$, we easily obtain

$$I_3(r) \leq C_M r^4, \quad 0 < r \leq 2$$

Thus, collecting the estimates for I_1, I_2, I_3 for $r = 1$, we will immediately obtain (2.1). Further, we have

$$\begin{aligned}\iint_{S_r} |\nabla w|^2 d\gamma \leq \iint_{S_\rho} |\nabla w|^2 d\gamma &= \frac{1}{2}[I_1(\rho) + I_2(\rho) + I_3(\rho)]\\ &\leq C_M\rho^4 + \int_{\mathbb{R}^n} w(\cdot, -\rho^2)^2 d\gamma^{-\rho^2},\end{aligned}$$

for any $r \leq \rho \leq 2r$ and $0 < r \leq 1$. Taking infimum by all such ρ, we obtain (2.2) and consequently (2.3). □

The next five propositions are obtained one after another from Proposition 2.1 exactly as Propositions 1.2–1.6 are obtained from Proposition 1.1. The only difference is that we have to work with the scaled versions all the time and the function $w = u\psi$ instead of u. Also, all involved constants are now not absolute but depend on M, n, and ψ (which we indicate by subscript M). Because of this, the proofs are omitted.

Proposition 2.2 (cf. Proposition 1.2)**.** *Let $u(x,s)$ and $w(x,s) = u(x,s)\psi(x)$ be as in Proposition 2.1 and $\Omega := \{w > 0\}$. Suppose*

$$\iint_{\Omega\cap S_r} |\nabla w|^2 d\gamma = \alpha r^4 < \infty$$

and
$$\iint_{\Omega\cap S_{r/4}} |\nabla w|^2 d\gamma \geq \frac{\alpha r^4}{256},$$
for some $0 < r \leq 1$. *Then*
$$|\Omega \cap (S_{r/2} \setminus S_{r/4})| \geq c_M r^2 > 0,$$
provided $\alpha > \alpha_M$ *for sufficiently large* α_M. *(Here* $|E| = \gamma(E) = \iint_E d\gamma$, *for* $E \subset \mathbb{R}^n \times (-\infty, 0)$.*)* □

Proposition 2.3 (cf. Proposition 1.3)**.** *Let* $u(x,s)$ *and* $w(x,s) = u(x,s)\psi(x)$ *be as in Proposition 2.1 and* $\Omega := \{w > 0\}$. *Suppose*
$$\iint_{\Omega\cap S_r} |\nabla w|^2 \, d\gamma = \alpha r^4 < \infty,$$
for some $0 < r \leq 1$. *Suppose also there exists* $\lambda > 0$ *such that*
$$|\Omega \cap (S_{r/2} \setminus S_{r/4})| < (1-\lambda)|S_{r/2} \setminus S_{r/4}|.$$
Then there exists $\mu < 1$, *depending on* λ, M, n, *and* ψ *such that*
$$\iint_{\Omega\cap S_{r/4}} |\nabla w|^2 \, d\gamma \leq \mu \iint_{\Omega\cap S_r} |\nabla w|^2 \, d\gamma,$$
provided $\alpha > \alpha_M$ *for sufficiently large* α_M. □

Proposition 2.4 (cf. Proposition 1.4)**.** *Let* $w_\pm$ *and* $A_\pm$ *be as in Theorem II. There exists a constant* C_M *such that if* $A_\pm(\rho) \geq C_M$ *for all* $\rho \in [\frac{1}{4}r^2, r^2]$, $0 < r \leq 1$ *then*
$$\Phi'(\rho) \geq -C_M r^2 \left(\frac{1}{\sqrt{A_+(\rho)}} + \frac{1}{\sqrt{A_-(\rho)}} \right) \Phi(\rho).$$
for all $\rho \in [\frac{1}{4}r^2, r^2]$. □

Hereafter, $M = \max\left\{ \|u_\pm\|_{L^2(Q_3^-)} \right\}$. In the next propositions
$$A_k^\pm = A_\pm(4^{-k}), \quad b_k^\pm = 4^{4k} A_k^\pm,$$
where $A_\pm$ are evaluated for $w_\pm$ as in Theorem II. Recall also that
$$\Phi(4^{-k}) = 4^{4k} A_k^+ A_k^-.$$

Proposition 2.5 (cf. Proposition 1.5)**.** *There exists a constant* C_M *such that if* $b_k^\pm \geq C_M$, *then*
$$4^4 A_{k+1}^+ A_{k+1}^- \leq A_k^+ A_k^- (1 + \delta_k) \quad \textit{with} \quad \delta_k = \frac{C_M}{\sqrt{b_k^+}} + \frac{C_M}{\sqrt{b_k^-}}. \quad \square$$

Proposition 2.6 (cf. Proposition 1.6)**.** *There exist a constant* $\epsilon_M > 0$ *such that if* $b_k^\pm \geq C_M$ *and* $4^4 A_{k+1}^+ \geq A_k^+$, *then* $A_{k+1}^- \leq (1 - \epsilon_M) A_k^-$. □

Proof of Theorem II. Similarly to the local case, Propositions 2.2–2.6 imply the following inequality:
$$\Phi(r) \leq C_M(1 + A_+(1) + A_-(1))^2, \quad 0 < r \leq 1. \tag{2.6}$$
To obtain more explicit dependence on the constant $M = \max\{\|u_\pm\|_{L^2(Q_3^-)}\}$, we can argue as follows. Define
$$\tilde{u}_\pm = \frac{u_\pm}{1 + M}.$$

Then $(\Delta - \partial_s)\tilde{u} \geq -1$ in Q_3^-, $\|\tilde{u}\|_{L^2 Q_3^-} \leq 1$ and we can apply the inequality (2.6) to $\tilde{u}_\pm$ with $M = 1$. This will give

$$\Phi(r, \tilde{u}_\pm) \leq C_n(1 + A_+(1, \tilde{u}_+) + A_-(1, \tilde{u}_-))^2,$$

where C_n depends only on the dimension n and the cutoff function ψ. Hence

$$\frac{1}{(1+M)^4}\Phi(r) \leq C_n \left(1 + \frac{1}{(1+M)^2}A_+(1) + \frac{1}{(1+M)^2}A_-(1)\right)^2$$

and consequently

$$\Phi(r) \leq C_n \left((1+M)^2 + A_+(1) + A_-(1)\right)^2.$$

Finally, revisiting the proof of Proposition 2.1 and tracing the constants, we can establish the following more precise version of (2.3), namely

$$\iint_{S_r} |\nabla w|^2 d\gamma \leq C_n(1+M)r^4 + \frac{1}{3r^2}\iint_{S_{2r}\setminus S_r} w^2 d\gamma.$$

For $r = 1$ this gives

$$A_\pm(1) \leq C_n(1 + M^2)$$

and therefore

$$\Phi(r) \leq C_n(1+M)^4.$$

This completes the proof of the theorem. □

3. Another variant of the formula

Under assumptions on the growth of functions $u_\pm$ near the origin, it is possible to obtain more precise versions of Theorems I and II.

Theorem III. *Let $u_\pm$, ψ, $w_\pm$, $A_\pm$ and Φ be as in Theorem II. Assume additionally that $|u_\pm(x,s)| \leq C_\epsilon(|x|^2 + |s|)^{\epsilon/2}$ for $(x,s) \in Q_3^-$ for some $0 < \epsilon \leq 1$. Then*

$$\Phi(r) \leq (1 + \rho^\epsilon)\Phi(\rho) + C_M\rho^\epsilon, \quad 0 < r \leq \rho \leq 1,$$

where C_M depends only on $M = \|u_+\|_{L^2(Q_3^-)} + \|u_-\|_{L^2(Q_3^-)}$, n, ψ, ϵ and the constant C_ϵ.

Proof. We will use the notation

$$A_\pm(r) = \iint_{S_r} |\nabla w_\pm|^2 d\gamma, \quad B_\pm(r) = \int_{\mathbb{R}^n} |\nabla w_\pm(\cdot, -r^2)|^2 d\gamma^{-r^2}.$$

Then $\Phi(r) = r^{-4}A_+(r)A_-(r)$ and therefore

$$\begin{aligned}\Phi'(r) &= -4r^{-5}A_+(r)A_-(r) + r^{-4}A_+'(r)A_-(r) + r^{-4}A_+(r)A_-'(r)\\ &= 2r^{-5}[r^2B_+(r)A_-(r) + r^2B_-(r)A_+(r) - 2A_+(r)A_-(r)],\end{aligned}$$

where we used that

$$A_\pm'(r) = 2rB_\pm(r).$$

1) We now claim that the additional growth assumption implies that

$$A_\pm(r) \leq C_M r^{2\epsilon} \tag{3.1}$$

Indeed, let w be either w_+ or w_-. Then by estimate (2.3) in Proposition 2.1, we have

$$\begin{aligned}\iint_{S_r} |\nabla w|^2 d\gamma &\leq C_M r^4 + \frac{C_M}{r^2} \iint_{S_{2r}\setminus S_r} w^2 d\gamma \\ &\leq C_M r^4 + C_M r^{2\epsilon} + \iint_{S_{2r}\setminus (Q^-_{2r}\cup S_r)} w^2 G(x,-s) dx\, ds \\ &\leq C_M r^{2\epsilon} + C_M r^2 \int_r^\infty G(\xi, r^2) d\xi \leq C_M r^{2\epsilon}\end{aligned}$$

This implies (3.1).

2) Let now $\omega_\pm(s) = \{w_\pm(\cdot, s) > 0\}$ and let $\lambda_\pm(r)$ be the largest number such that

$$\lambda_\pm(r) \int_{\omega_\pm(-r^2)} f^2 d\gamma^{-1} \leq \int_{\omega_\pm(-r^2)} |\nabla f|^2 d\gamma^{-1}$$

for any f. Recall also that by the eigenvalue inequality of Beckner, Kenig and Pipher [BKP98]

$$\lambda_+(r) + \lambda_-(r) \geq 1.$$

Scaling, we also have

$$\lambda_\pm(r) \int_{\omega_\pm(-r^2)} f^2 d\gamma^{-r^2} \leq r^2 \int_{\omega_\pm(-r^2)} |\nabla f|^2 d\gamma^{-r^2}.$$

Then the inequality (2.1) in Proposition 2.1 implies

$$2\lambda_\pm(r) A_\pm(r) \leq C_M \lambda_\pm(r) r^4 + 2r^3 \sqrt{B_\pm(r)\lambda_\pm(r)} + r^2 B_\pm(r). \tag{3.2}$$

Based now on (3.2), we obtain estimates on $\Phi'(r)$ by considering three possibilities.

Case 1: $r^2 B_+(r) \geq 2A_+(r)$ or $r^2 B_-(r) \geq 2A_-(r)$. Then from the formula above we easily have $\Phi'(r) \geq 0$.

Case 2: $r^2 B_\pm(r) \leq 2A_+(r)$ and $\lambda_\pm(r) \leq 1$. Then by (3.2) we have

$$2\lambda_\pm(r) A_\pm(r) \leq C_M r^4 + C r^2 \sqrt{A_\pm(r)} + r^2 B_\pm(r).$$

Using now that $\lambda_+(r) + \lambda_-(r) \geq 1$ one then finds

$$\begin{aligned}\Phi'(r) &= 2r^{-5}\left\{[r^2 B_+(r) - 2\lambda_+(r) A_+(r)] A_-(r) + [r^2 B_-(r) - 2\lambda_-(r) A_-(r)] A_+(r)\right\} \\ &\geq -C_M r^{-1} [A+(r) + A_-(r)] - C r^{-3} \left[\sqrt{A_+(r)} A_-(r) + \sqrt{A_-(r)} A_+(r)\right] \\ &\geq -C_M r^{-1+2\epsilon} - C r^{-1} \sqrt{\Phi(r)} \left[\sqrt{A_+(r)} + \sqrt{A_-(r)}\right] \\ &\geq -C_M r^{-1+2\epsilon} - C r^{-1+\epsilon} \sqrt{\Phi(r)}\end{aligned}$$

Case 3: $r^2 B_\pm(r) \leq 2A_+(r)$ and $\lambda_+(r) \geq 1$ (or $\lambda_-(r) \geq 1$). Then by (3.2) we have

$$2A_+(r) \leq C_M r^4 + C r^2 \sqrt{A_+(r)} + r^2 B_+(r).$$

One then finds

$$\begin{aligned}\Phi'(r) &= 2r^{-5}\left\{[r^2 B_+(r) - 2A_+(r)]A_-(r) + r^2 B_-(r)A_+(r)\right\}\\ &\geq 2r^{-5}[r^2 B_+(r) - 2A_+(r)]A_-(r)\\ &\geq -C_M r^{-1} A_-(r) - Cr^{-3}\sqrt{A_+(r)}A_-(r)\\ &\geq -C_M r^{-1+2\epsilon} - Cr^{-1}\sqrt{\Phi(r)}\sqrt{A_-(r)}\\ &\geq -C_M r^{-1+2\epsilon} - Cr^{-1+\epsilon}\sqrt{\Phi(r)}\end{aligned}$$

So, we see that in all cases we obtain the inequality

$$\Phi'(r) \geq -C_M r^{-1+2\epsilon} - Cr^{-1+\epsilon}\sqrt{\Phi(r)}$$

It is now easy to show that

$$\frac{d}{dr}\left[\left(\Phi(r) + C_M r^{2\epsilon}\right)^{1/2} + C_M r^{\epsilon}\right] \geq 0$$

and that

$$\Phi(\rho) \leq (1 + r^{\epsilon})\Phi(r) + C_M r^{\epsilon}, \quad 0 < r \leq \rho \leq 1,$$

arguing precisely as at the end of the proof of Theorem 3.8 in [CJK02]. □

4. An Application

In a typical application, the functions $u_\pm$ are the positive and negative parts of a solution of a two-phase free boundary problem, see e.g. [ACF84, Caf88, Caf95, CJK02]. In yet another class of problems (obstacle-type problems), the monotonicity formulas can be applied to the positive and negative parts $(\partial_e u)^\pm$ of the directional derivatives of solutions, see e.g. [CKS00, CPS04, Sha03, Ura01]. In this section we give an application of the latter kind in a parabolic free boundary problem related to the caloric continuation of the heat potentials, see [CPS04].

Let $u(x,s)$ be a solution of the equation

$$\Delta u - \partial_s u = f(x,s)\chi_\Omega \quad \text{in } Q_1^-, \tag{4.1}$$

where

$$\Omega = Q_1^- \setminus \{u = |\nabla u| = 0\} \tag{4.2}$$

and f satisfies

$$\sup_{Q_1^-} |f(x,s)| \leq K < \infty. \tag{4.3}$$

We assume that u and ∇u are continuous functions and that (4.1) is satisfied in the sense of distributions. In the formulation of Duvaut [Duv73], the famous Stefan problem of the melting of the ice can be written as (4.1)–(4.3) with $f \equiv 1$. In that model, however, both u and $\partial_s u$ are nonnegative ($\partial_s u$ meaning the temperature) which significantly simplifies the problem. Without sign assumptions on u and $\partial_s u$ the problem has been studied recently by Caffarelli, Petrosyan, and Shahgholian [CPS04] for $f \equiv 1$. This paper makes an extensive use of the monotonicity formula of Caffarelli [Caf93] for caloric functions in disjoint domains. We would like to show here that some of the results in [CPS04] (if not all) can be extended to the case of $f(x,s)$ satisfying the following additional assumption:

$$|f(x,t) - f(y,s)| \leq L(|x-y|^2 + |t-s|)^{1/2}, \tag{4.4}$$

i.e. that f is Lipschitz continuous with respect to the parabolic distance. In fact, this condition can be relaxed to

$$|\nabla_x f| \le L, \quad \text{and} \quad f \text{ is Dini-continuous w.r.t. the parabolic distance.}$$

The Dini-continuity is needed for interior $C^2_x \cap C^1_t$-estimates for solutions of the equation $(\Delta - \partial_s)w = f(x,s)$, see for instance Notes to Chapter IV in Lieberman's book [Lie96].

Particularly, we prove the following result.

Theorem 4.1. *Let u be a solution of (4.1)–(4.4) with $\|u\|_{L^\infty(Q_1^-)} \le M$. Then there exists a constant $C = C(K,L,M,n)$ such that*

$$\sup_{Q^-_{1/4}\cap\Omega} |\partial_{x_i x_j} u| \le C, \qquad \sup_{Q^-_{1/4}\cap\Omega} |\partial_s u| \le C, \quad i,j = 1,\dots,n$$

Observe that the $C^{1,1}_x \cap C^{0,1}_s$-regularity is optimal for solutions of (4.1), as $f\chi_\Omega$ may be discontinuous. Also, the boundedness of both u and $(\Delta - \partial_s)u$ alone implies that $D^2u, \partial_s u \in L^p_{\mathrm{loc}}(Q_1^-)$ for any $1 < p < \infty$. Consequently u is $C^{1,\alpha}_x \cap C^{0,(1+\alpha)/2}_s$-regular for any $0 < \alpha < 1$. However, to push the regularity to $C^{1,1}_x \cap C^{0,1}_s$, one needs to use the structure of the right-hand side.

The proof of Theorem 4.1 is based on the following growth lemma.

Lemma 4.1. *Let u be a solution of (4.1)–(4.4) with $\|u\|_{L^\infty(Q_1^-)} \le M$. Suppose additionally that $u = |\nabla u| = 0$ at the origin. Then there exists $C = C(K,L,M,n)$ such that*

$$\sup_{Q_r^-} |u| \le Cr^2, \quad 0 < r < 1.$$

But first, we need to established the following fact, which will allow us to apply the almost monotonicity formula.

Lemma 4.2. *Let u be as in Theorem 4.1. Then for any spatial direction e, the functions $v^\pm = (\partial_e u)^\pm = \max\{\pm\partial_e u, 0\}$ satisfy*

$$(\Delta - \partial_s)v^\pm \ge -L \quad \text{in } Q_1^-.$$

Proof. In the sense of distributions, we have that

$$(\Delta - \partial_s)(\partial_e u) = \partial_e(\Delta - \partial_s)u = \partial_e f \ge -L \quad \text{in } \Omega.$$

This implies that

$$(\Delta - \partial_s)v^+ \ge -L \quad \text{in } \Omega_+ = \Omega \cap \{\partial_e u > 0\}.$$

Moreover,

$$v^+ = 0 \quad \text{on } Q_1^- \setminus \Omega_+.$$

To conclude from here that v^+ satisfies

$$(\Delta - \partial_s)v^+ \ge -L \quad \text{in } Q_1^-$$

is just one step, since v^+ is a limit of truncations $v_\delta = \max\{v, \delta\}$ for $\delta > 0$ which obviously satisfy the inequality $(\Delta - \partial_s)v_\delta \ge -L$ as the maximums of two subsolutions.

The proof for v^- is similar. □

We are now ready to prove Lemma 4.1.

Proof of Lemma 4.1. The proof is based on a compactness argument and follows the lines of the proof of Theorem in [CPS04].

Let $\mathcal{P} = \mathcal{P}(K, L, M, n)$ denote the family of all functions u satisfying the assumptions in Lemma 4.1. Further, for $u \in \mathcal{P}$ denote.

$$S_k = S_k(u) = \sup_{Q^-_{2^{-k}}} |u|, \quad k = 0, 1, \dots. \tag{4.5}$$

The statement of the lemma is equivalent to saying that

$$S_k \le C\, 2^{-2k}, \quad k = 0, 1, \dots, \tag{4.6}$$

for some constant C depending on the class $\mathcal{P}$ only. We claim that

$$S_{k+1} \le \max\left\{2^{-2} S_k, 2^{-4} S_{k-1}, \dots, 2^{-2(k+1)} S_0, C\, 2^{-2(4k+1)}\right\}, \tag{4.7}$$

for $C = C(\mathcal{P})$. A simple inductive argument then shows that (4.7) implies (4.6). So the lemma will follow once we establish (4.7).

Now, suppose that (4.7) fails. Then there exists a sequence of solutions $u_j \in \mathcal{P}$ and integers k_j such that

$$S_{k_j+1}(u_j) \ge j 2^{-2(k_j+1)}, \quad S_{k_j+1}(u_j) \ge 2^{-2} S_{k_j}(u_j). \tag{4.8}$$

Define $\widetilde{u}_j$ as

$$\widetilde{u}_j(x,t) = \frac{u_j(2^{-k_j} x, 2^{-2k_j} t)}{S_{k_j+1}(u_j)} \qquad \text{in } Q^-_1.$$

Then

$$\sup_{Q^-_1} |(\Delta - \partial_s)(\widetilde{u}_j)| \le \frac{2^{-2k_j} K}{S_{k_j+1}(u_j)} \le \frac{4K}{j} \to 0, \tag{4.9}$$

$$\sup_{Q^-_{1/2}} |\widetilde{u}_j| = 1, \qquad \text{(by (4.5))}, \tag{4.10}$$

$$\sup_{Q^-_1} |\widetilde{u}_j| \le \frac{S_{k_j}(u_j)}{S_{k_j+1}(u_j)} \le 4 \qquad \text{(by (4.8))}, \tag{4.11}$$

$$\widetilde{u}_j(0,0) = |\nabla \widetilde{u}_j(0,0)| = 0. \tag{4.12}$$

Now by (4.9)–(4.12) we will have a subsequence of $\widetilde{u}_j$ converging in $C^{1,\alpha}_x \cap C^{0,\alpha}_t(Q^-_1)$ to a nonzero caloric function u_0 in Q^-_1, satisfying $u_0(0,0) = |\nabla u_0(0,0)| = 0$. Moreover, from (4.10), we will have

$$\sup_{Q^-_{1/2}} |u_0| = 1. \tag{4.13}$$

For any spatial unit vector e define

$$v = \partial_e u_0, \qquad v_j = \partial_e u_j, \qquad \widetilde{v}_j = \partial_e \widetilde{u}_j.$$

Then, over a subsequence, $\widetilde{v}_j$ converges in $C^{0,\alpha}_x \cap C^{0,\alpha}_t(Q^-_1)$ to a function v satisfying $(\Delta - \partial_s) v = 0$.

To proceed, for a fixed cut-off function $\psi(x)$ with $\psi|_{B_{1/2}} = 1$ and $\operatorname{supp}\psi \subset B_{3/4}$ and $u \in \mathcal{P}$ denote

$$\Phi(r, (\partial_e u)^{\pm}\psi) = \frac{1}{r^4} A(r; (\partial_e u)^+ \psi)\, A(r; (\partial_e u)^- \psi),$$

where A are defined as in Theorem II. By Lemma 4.2, we may apply Theorem II to $v_j^\pm = (\partial_e u_j)^\pm$, which will give the estimate

$$\Phi(r, v_j^\pm \psi) \le C_0 \tag{4.14}$$

for a constant C_0 depending on the class $\mathcal{P}$ and ψ only. Let now $\psi_j(x) = \psi(2^{-k_j}x)$. Then rescaling the estimate (4.14), we obtain

$$\Phi(1; \widetilde{v}_j\psi_j) \le \left(\frac{2^{-2k_j}}{S_{k_j+1}}\right)^4 \Phi(2^{-k_j}; v_j\psi) \le C_0 \left(\frac{2^{-2k_j}}{S_{k_j+1}}\right)^4 \tag{4.15}$$

for k_j large enough. Since $\psi_j = 1$ in $B_{2^{k_j-1}}$, we will have

$$|\nabla(\widetilde{v}_j\psi_j)|^2 \ge |\nabla\widetilde{v}_j|^2\chi_{B_1}.$$

Hence for $\epsilon > 0$ (small and fixed) we have

$$c_{n,\epsilon}\int_{-1}^{-\epsilon}\int_{B_1} |\nabla\widetilde{v}_j^\pm|^2\,dx\,ds \le \int_{-1}^{0}\int_{B_1} |\nabla\widetilde{v}_j^\pm\psi_j|^2 G(x,-s)\,dx\,ds = A(1, \widetilde{v}_j^\pm\psi_j).$$

This estimate, in combination with Poincare's inequality, gives

$$\int_{-1}^{-\epsilon}\int_{B_1} |\widetilde{v}_j^\pm - M^\pm(s)|^2\,dx\,ds \le C_n\int_{-1}^{-\epsilon}\int_{B_1} |\nabla\widetilde{v}_j^\pm|^2\,dx\,ds \le C_{n,\epsilon}A(1, \widetilde{v}_j^\pm\psi_j),$$

where $M_j^\pm(s)$ denotes the corresponding mean value of $\widetilde{v}_j^\pm$ on the s-section.

Using this and (4.15), we will have

$$\begin{aligned}\left(\int_{-1}^{-\epsilon}\int_{B_1} |\widetilde{v}_j^+ - M_j^+(s)|^2\,dx\,ds\right)\left(\int_{-1}^{-\epsilon}\int_{B_1} |\widetilde{v}_j^- - M_j^-(s)|^2\,dx\,ds\right)&\\ \le C_{n,\epsilon}\,\Phi(1, v_j\psi) \le C_{n,\epsilon}\left(\frac{2^{-2k_j}}{S_{k_j+1}}\right)^4&.\end{aligned}$$

Using (4.8) and letting $j \to \infty$ (and then $\epsilon \to 0$), we obtain

$$\int_{-1}^{0}\int_{B_1} |v^+ - M^+(s)|^2 \int_{-1}^{0}\int_{B_1} |v^- - M^-(s)|^2 = 0, \tag{4.16}$$

where $M^\pm(s)$ denotes the corresponding mean value of $v^\pm$ on s-sections over B_1. Obviously, (4.16) implies that either of $v^\pm$ is equivalent to $M^\pm(s)$ in Q_1^- and is thus independent of the spatial variables. Let us assume $v^- = M^-(s)$. Then $-\partial_s v^- = (\Delta - \partial_s)v^- = 0$, i.e. M^- is constant in Q_1^-. Since $v(0,0) = 0$, we must have $M^- = 0$, i.e. $v \ge 0$ in Q_1^-. Hence by the minimum principle $v \equiv 0$ in Q_1^-. Since $v = \partial_e u_0$, and e is an arbitrary direction, we conclude that u_0 is constant in Q_1^-. Also $u_0(0,0) = 0$ implies that the constant must be zero, i.e $u_0 \equiv 0$ in Q_1^-. This contradicts (4.13) and the lemma is proved. □

Finally, we prove Theorem 4.1.

Proof of Theorem 4.1. For $(x_0, s_0) \in \Omega \cap Q_{1/4}^-$ let

$$d = d^-(x_0, s_0) = \sup\{r : Q_r^-(x_0, s_0) \subset \Omega \cap Q_1^-\}.$$

Consider then two possibilities:

1) If $d \ge 1/2$ then by the parabolic interior Schauder estimates (see Theorem 4.9 in Chapter IV of [Lie96]) $\partial_{x_ix_j}u(x_0, s_0)$ and $\partial_s u(x_0, s_0)$ are bounded by a constant depending only on K, L, M and n.

2) If $d < 1/2$ then the parabolic boundary of $Q_d^-(x_0, s_0)$ must contain a point from $\partial\Omega$ and therefore by Lemma 4.1

$$|u(x,s)| \le Cd^2 \quad \text{in } Q_d^-(x_0,s_0),$$

where $C = C(K, L, M, n)$. Now consider the rescaling

$$u_d(x,s) = \frac{u(x_0 + d\,x, s_0 + d^2 s)}{d^2} \quad \text{in } Q_1^-.$$

It satisfies $(\Delta - \partial_s)u_d = f_d(x,s)$ in Q_1^-, where

$$f_d(x,s) = f(x_0 + dx, s_0 + d^2 s).$$

Moreover, $|u_d| \le C$ in Q_1^-. Hence, applying by the parabolic interior Schauder estimates to u_d we obtain that $\partial_{x_i x_j} u_d(0,0) = \partial_{x_i x_j} u(x_0, s_0)$, and $\partial_s u_d(0,0) = \partial_s u(x_0, s_0)$ are bounded by a constant depending only on K, L, M, n, which is the desired result. The theorem is proved. $\square$

Remark 4.1. In the elliptic case there is a more direct proof of the corresponding analogue of Theorem 4.1 by using the elliptic almost monotonicity formula, see Shahgholian [Sha03]. Moreover, the result in [Sha03] is proved for a certain class of right hand sides (Lipschitz in x and semimonotone in u) , which includes also the so-called two-phase obstacle problem, see Ural'tseva [Ura01].

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