Some Extremal Problems for Analytic Functions

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Abstract

The paper mainly concerns with functions f, analytic in $S : |Imz| < 1$ and bounded by a constant $M > 1$. We state sharp estimates for $\sup_{\mathbf{R}} |f'|$ under the additional condition $\sup_{\mathbf{R}} |f| \leq 1$. Using these estimates we deduce well-known Bernstein's inequality and some its generalizations for entire functions of a finite type with respect to an arbitrary proximate order.

Parallely we investigate also the next extremal problem, related to the mentioned class of functions: if $f(\zeta) = f(\overline{\zeta}) = 1$, for some $\zeta \in S$, what is the minimal value of $\sup_{\mathbf{R}} |f|$?

Also we present the description of extremal functions for these problems.

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1 Introduction

For a given $M > 1$ consider the class $\mathcal{B} = \mathcal{B}_M(S)$ of functions f analytic in the strip $S : |Imz| < 1$ and satisfying the conditions

$$\sup_S |f| \leq M, \quad \sup_{\mathbf{R}} |f| \leq 1.$$

By the compactness of the class $\mathcal{B}$, there exists $a < \infty$ such that $|f'(0)| \leq a$ for each $f \in \mathcal{B}$. Moreover, since the class $\mathcal{B}$ is closed under translations along $\mathbf{R}$, we may conclude that $\sup_{\mathbf{R}} |f'| \leq a$ for each $f \in \mathcal{B}$. Obviously, the sharpest a is

$$a = a(M) = \sup_{f \in \mathcal{B}_M(S)} |f'(0)|$$

One of the main purposes of this paper is to investigate the value of $a(M)$ and to describe extremal functions, attaining these values.

The problem was suggested by Norair Arakelian and to some extent was considered in [1].

Parallelly we investigate the next extremal problem.

For given $A > 1$ and $\zeta \in S$ consider the class $\mathcal{H}_{A,\zeta}$ of functions f, analytic in S and satisfying the conditions

$$\sup_S |f| \le A, \quad f(\zeta) = f(\overline{\zeta}) = 1.$$

Let us denote

$$\varepsilon = \varepsilon(A,\zeta) = \inf\{\sup_{\mathbf{R}} |f| : f \in \mathcal{H}_{A,\zeta}\}.$$

The problem is to investigate the smallness of ε, its asymptotic behavior as A tends to ∞, and to describe the extremal functions, attaining ε.

This problem was posed in [2]. The estimates for ε and extremal functions may be applied to construct an approximation of the Cauchy kernel.

The contest of paper is as follows. In section 2 we state the main theorems, estimating a and ε. Besides we deduce Bernstein's inequality [4] from the estimates of $a(M)$. Sections 3, 4, 5 are basic for the proofs of the main theorems, located in section 6. In section 7 we describe extremal functions of the investigated problems. In concluding section 8 we give an application of the estimates of the quantity $a(M)$ to prove an asymptotic analogue of Bernstein's inequality for entire functions of a finite type with respect to an arbitrary proximate order.

The author would like to express thanks to Norair Arakelian for suggestion of the problems investigated here and for guidance during the work on this paper.

2 Main theorems

Theorem 2.1 *The quantity* $a(M) = \sup_{f\in\mathcal{B}_M(S)} |f'(0)|$ *satisfies the inequalities*

$$\log(M + \sqrt{M^2-1}) \le a(M) \le \log(M + \sqrt{M^2+1}), \ M > 1. \tag{1}$$

Particularly

$$\lim_{M\to\infty} \frac{a(M)}{\log M} = 1. \tag{2}$$

Besides, the exact value of $a(M)$ *may be expressed in terms of an elliptic integral:*

$$a(M) = \frac{M^2}{M^2+1} \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k_M'^2 t^2)}}, \quad k_M' = \frac{M^2-1}{M^2+1}. \tag{3}$$

Remark. It is easy to establish for $a(M)$ the estimate $a(M) \le c \log M$, where c is an absolute constant, by using the 2-constant theorem. See the proof of lemma 6 in [1]. But our purpose is to describe $a(M)$ exactly and therefore we have to use more delicate methods. However, the lower estimate in (1) does not need such methods. Simply consider $\sin a_i(M)z$, where $a_i(M) = \log(M + \sqrt{M^2-1})$, for which $|\sin a_i(M)z| \le \cosh a_i(M) = M,\ z \in S,\ \sin|a_i(M)x| \le 1,\ x \in \mathbf{R}$ and $(\sin a_i(M)z)'\big|_{z=0} = a_i(M)$.

Corollary 2.2 (Bernstein's inequality) *If g is an entire function of exponential type $\sigma_g \le \sigma$, such that*

$$\sup_{\mathbf{R}} |g| \le 1,$$

then its derivative g' satisfies the inequality

$$\sup_{\mathbf{R}} |g'| \le \sigma.$$

Proof. First we recall that for such g the following estimate is fulfilled: $|g(z)| \le \exp(\sigma|y|)$ for each $z = x+iy$. Therefore functions $g_h(z) = g(hz)$ belong to the classes $\mathcal{B}_{\exp(\sigma h)}(S)$, for each $h > 0$ and hence

$$\sup_{\mathbf{R}} |g'| = \frac{1}{h} \sup_{\mathbf{R}} |g_h'| \le \frac{a(\exp(\sigma h))}{h}.$$

Tending $h \to +\infty$ and using asymptotic equality (2), we receive the result.

Theorem 2.3 *The quantity $\varepsilon(A,\zeta) = \inf\{\sup_{\mathbf{R}} |f| : f \in \mathcal{H}_{A,\zeta}\}$ satisfies the following inequalities*

$$\frac{1}{\cosh \eta a_e} \le \varepsilon(A,\zeta) \le \frac{1}{\cosh \eta a_i} \tag{4}$$

where $\eta = Im\zeta$ and numbers a_e and a_i are chosen such that $\sinh a_e / \cosh \eta a_e = A = \cosh a_i / \cosh \eta a_i$. Particularly

$$\lim_{A\to\infty} \frac{\varepsilon(A,\zeta)}{2A^{-\frac{\eta}{1-\eta}}} = 1 \tag{5}$$

Remark. As it is noted in [2], the upper estimate in (4) may be easily obtained by consideration of the function $f(z) = \cos a_i(z-\xi)/\cosh \eta a_i$, where $\xi = Re\zeta$.

3 Symmetrization

We will prove Theorems 2.1 and 2.3 in the following four sections.

Let us begin with some evident notes. First we note that $\varepsilon(A,\zeta) = \varepsilon(A, i\eta)$, where $\eta = Im\zeta$. Therefore, with no loss of generality we assume in further that $\zeta = i\eta$ and moreover $0 \le \eta < 1$.

Next we note that to attain $\varepsilon(A,\zeta)$ it is sufficient to take inf by $f \in \mathcal{H}_{A,\zeta}$ that are symmetric with respect to $\mathbf{R}$, i.e. $f(\overline{z}) = \overline{f(z)}$ for each $z \in S$. Indeed, if $f(z)$ is replaced by $f_1(z) = \frac{1}{2}[f(z) + \overline{f(\overline{z})}]$, $z \in S$ then $\sup_{\mathbf{R}} |f_1| \le \sup_{\mathbf{R}} |f|$.

Further we claim that

$$\varepsilon(A,\zeta) = c_0(M,\zeta)^{-1} \quad \text{for } M = A/\varepsilon(A,\zeta),$$

where

$$c_0(M,\zeta) = \sup_{f\in\mathcal{B}_0} f(\zeta),$$

$$\mathcal{B}_0 = \left\{ f \in \mathcal{B}_M(S) : f(\overline{z}) = \overline{f(z)}, \quad z \in S \text{ and } f(\zeta) > 0 \right\}.$$

It follows from the possibility to transfer any symmetric $f \neq 0$ from $\mathcal{H}_{A,\zeta}$ to $\mathcal{B}_0$ by dividing it on $\sup_{\mathbf{R}} |f|$, and on the other hand, to transfer any g, $g(\zeta) \neq 0$, from $\mathcal{B}_0$ to $\mathcal{H}_{A,\zeta}$ by dividing it on $g(\zeta)$, if $g(\zeta) > \varepsilon(A,\zeta)^{-1}$.

Symmetrization is applicable also to the quantity $a(M)$. Namely,

$$a(M) = \sup_{f \in \mathcal{B}_1} |f'(0)|,$$

where

$$\mathcal{B}_1 = \left\{ f \in \mathcal{B}_M(S) : f(\overline{z}) = \overline{f(z)},\ f(-\overline{z}) = -\overline{f(z)}, \quad z \in S \text{ and } f'(0) > 0 \right\}$$

so to attain $a(M)$ it is enough to take sup by such f that are symmetric with respect to the real and imaginary axes. Indeed, if $f \in \mathcal{B}_M(S)$ and $\arg f'(0) = \theta$ then, setting successively

$$f_0(z) = e^{-i\theta} f(z),\ f_1(z) = \frac{1}{2}[f_0(z) + \overline{f_0(\overline{z})}],\ f_2(z) = \frac{1}{2}[f_1(z) - f_1(-z)], \quad z \in S$$

we come to such a function $f_2 \in \mathcal{B}_1$ that

$$f_2'(0) = f_1'(0) = Re f_0'(0) = |f'(0)|.$$

This implies the statement.

4 The covering mapping p

To solve the extremal problems from introduction we use the harmonic measure principle, combined with the symmetrization from the previous section.

Let $\mathcal{E}_M = D_M \backslash I$, where D_M is the open disk $|z| < M$ and $I = [-1, 1]$. The boundary of the doubly-connected region $\mathcal{E}_M$ consists of the segment I and the circle $\Gamma_M : |z| = M$. Denote by ω_M the harmonic measure of Γ_M with respect to the region $\mathcal{E}_M$ (see, for example [7]). We define ω_M also on I as identically equal to 0 and thus obtain continuous subharmonic function in D_M.

Consider now the class of all functions f, analytic in S and symmetric with respect to $\mathbf{R}$, i.e. $f(\overline{z}) = \overline{f(z)}$ for all $z \in S$, which is equivalent to the inclusion $f(\mathbf{R}) \subset \mathbf{R}$. By the Schwarz reflection principle, this class may be identified with the class of all functions, analytic in $S^+ = \{z \in S : Imz > 0\}$ and having on $\mathbf{R}$ real boundary values. Functions of this type belong to the class $\mathcal{B}_M(S)$ iff

$$f(\mathbf{R}) \subset I, \quad f(S^+) \subset D_M,$$

where $f(\mathbf{R})$ is the set of boundary values on $\mathbf{R}$ of the function f. Under these conditions, by the harmonic measure principle it follows that

$$\omega_M(f(z)) \leq Imz, \quad z \in S^+ \tag{6}$$

since Imz is the harmonic measure of the line $Imz = 1$ with respect to the strip S^+. This inequality follows also from observation that $\omega_M(f(z))$ is a bounded subharmonic function in S^+, with boundary values not exceeding 1 on $Imz = 1$ and equal to 0 on $Imz = 0$, so that Imz is its harmonic majorant. Hence, by the maximum principle, the equality sign in (6) for some $z \in S^+$ implies the equality sign for all such z.

Let us describe now all functions f that possess this property. For this purpose consider the universal covering surface $\mathcal{E}_M^\infty$ of the annual region $\mathcal{E}_M$. Since the boundary of $\mathcal{E}_M$ consists of two continuums, $\mathcal{E}_M^\infty$ may be identified with the strip S^+. Then the covering mapping

$$p : S^+ \longrightarrow \mathcal{E}_M$$

may be constructed as follows. There exist a number $\tau = \tau(M)$ and a conformal mapping of the rectangle $Q_\tau^+ : |Rez| < \tau,\ 0 < Imz < 1$ to the region $\mathcal{E}_M^+ = \{w \in \mathcal{E}_M : Imw > 0\}$, that carries the upper side of the rectangle to the semicircle $\Gamma_M^+ = \{w \in \Gamma_M : Imw \geq 0\}$ and the lower side to the segment I. Since this conformal mapping takes real values on vertical sides of Q_τ^+, it may be analytically continued by the reflection principle to the whole strip S^+. The resulting mapping is the covering mapping $p : S^+ \to \mathcal{E}_M$. It has a primitive period 4τ and is symmetric with respect to the imaginary axis, i.e. $p(-\overline{z}) = -\overline{p(z)},\ z \in S^+$. Besides,

$$p(\mathbf{R}) = I, \quad p(\{Imz = 1\}) = \Gamma_M,$$

where $p(\mathbf{R})$ and $p(\{Imz = 1\})$ are the boundary values of p on $\mathbf{R}$ and $Imz = 1$ respectively. It implies immediately the equality

$$\omega_M(p(z)) = Imz, \quad z \in S^+, \tag{7}$$

since the both sides are bounded harmonic functions, having the same boundary values.

Let now a function f be such that there is equality sign in (6) for some and consequently all $z \in S^+$. We claim that there exists $\xi \in \mathbf{R}$ such that $f(z) = p(z + \xi),\ z \in S^+$. Indeed, equality in (6) implies particularly the inequality $0 < \omega_M(f(z)) < 1,\ z \in S^+$, which is equivalent to inclusion $f(S^+) \subset \mathcal{E}_M$. From universality of the covering p there exists a lifting $\widetilde{f} : S^+ \to S^+$ of the mapping f along p, which satisfies the equality $f = p \circ \widetilde{f}$. But then

$$Im\widetilde{f}(z) = \omega_M(p \circ \widetilde{f}(z)) = \omega_M(f(z)) = Imz$$

for all $z \in S^+$, which is possible only if $\widetilde{f}(z) = z + \xi$ for some real constant ξ. Therefore $f(z) = p(z + \xi)$, as claimed.

5 Main Lemmas

The covering mapping p described in previous section and its shifting on $-\tau$ along $\mathbf{R}$ are natural extremal functions for a and ε, as it is seen from the following lemma. Besides the quantities $a(M)$ and $c_0(M, \zeta)$, we introduce the quantity

$$c_1(M, \zeta) = \sup \{|f(\zeta)| : f \in \mathcal{B}_1\}$$

for $\zeta = i\eta,\ 0 \le \eta < 1$.

Lemma 5.1 *Let $\zeta = i\eta,\ 0 \le \eta < 1$. Then*
(i) $c_0(M, \zeta) = p(\zeta + \tau)$;
(ii) $c_1(M, \zeta) = \frac{1}{i} p(\zeta)$;
(iii) $a(M) = p'(0) = \frac{\partial}{\partial \eta} c_1(M, i\eta)\big|_{\eta=0+} = (\frac{\partial}{\partial v} \omega_M(iv)\big|_{v=0+})^{-1}$.

For the proof we use the additional lemma.

Lemma 5.2 *The harmonic measure ω_M is symmetric with respect to the real and imaginary axes. It strongly increases along all radiuses $\arg w = \theta$ as $|w|$ increases, $w \in \mathcal{E}_M$.*

Proof. The statement that ω_M is symmetric is an immediate consequence of symmetry of both $\mathcal{E}_M$ and Γ_M. Therefore the only last statement of the lemma needs to be proved.

For $0 < t < 1$ consider the function $\omega_M(tw)$. It is harmonic in $D_M \backslash [-\frac{1}{t}, \frac{1}{t}]$ and so is ω_M. Comparing the boundary values on Γ_M and $[-\frac{1}{t}, \frac{1}{t}]$ we conclude that $\omega_M(tw) \le \omega_M(w)$.

Proof of lemma 5.1. We suppose that p is continued by the Schwarz reflection principle to the whole strip S. Then $p \in \mathcal{B}_1$ and $p(\cdot + \tau) \in \mathcal{B}_0$, as it follows from the construction of p in the previous section.

(i) If $f \in \mathcal{B}_0$, then by (6) and (7) we have that

$$\omega_M(f(\zeta)) \le \omega_M(p(\zeta + \tau))$$

and the conclusion of (i) follows from lemma 5.2, since $\arg f(\zeta) = \arg p(\zeta + \tau) = 0$ and moreover $1 \le p(\zeta + \tau) < M$.

(ii) If $f \in \mathcal{B}_1$, then again by (6) and (7) we will have that

$$\omega_M(f(\zeta)) \le \omega_M(p(\zeta))$$

and again the conclusion of (ii) follows from lemma 5.2.

(iii) First we note that all numbers $c_1(M, \zeta),\ \zeta = i\eta$, are attained on the same extremal function p. Therefore, if $f \in \mathcal{B}_1$, then

$$\begin{aligned} f'(0) &= \lim_{\eta \to 0+} \frac{f(i\eta) - f(0)}{i\eta} = \lim_{\eta \to +0} \frac{|f(i\eta)|}{\eta} \le \lim_{\eta \to 0+} \frac{c_1(M, i\eta)}{\eta} \\ &= \lim_{\eta \to 0+} \frac{c_1(M, i\eta) - c_1(M, 0)}{\eta} = \frac{\partial}{\partial \eta} c_1(M, i\eta)\big|_{\eta=0+} \end{aligned}$$

and

$$p'(0) = \frac{\partial}{\partial \eta} c_1(M, i\eta)\big|_{\eta=0+}.$$

We used that $f(0) = c_1(M, 0) = 0$; $\frac{1}{i} f(i\eta) = |f(i\eta)|$, for sufficiently small η, because $f'(0) > 0$ and $\frac{1}{i} f(i\eta) \in \mathbf{R}$; $\frac{1}{i} p(i\eta) = c_1(M, i\eta)$ by (ii). Therefore

$$a(M) = p'(0).$$

The last equality sign in (iii) follows from (7). Indeed,

$$1 = \lim_{\eta \to 0+} \frac{\omega_M(p(i\eta))}{\eta} = (\frac{\partial}{\partial v} \omega_M(iv)\big|_{v=0+})^{-1} p'(0).$$

It remains to note that $\frac{\partial}{\partial v} \omega_M(iv)\big|_{v=0+}$ truly exists: the part of ω_M in the upper semi-disk D_M^+ may be continued harmonically trough the interval $(-1, 1)$ by the reflection principle. The derivative needed simply coincides with the partial derivative of the obtained harmonic function along the imaginary axis in 0.

The next lemma relates to estimates of $c_0(M, \zeta)$, $c_1(M, \zeta)$, where $\zeta = i\eta$, by elementary functions. But first we give estimates for the harmonic measure ω_M.

Let $\mathcal{E}_M^i$ and $\mathcal{E}_M^e$ (i-interior, e-exterior) be regions, obtained by removing the segment I from the interiors of the ellipses Γ_M^i and Γ_M^e with focuses in ± 1 touching the circle Γ_M inside and outside of it respectively, so that $\mathcal{E}_M^i \subset \mathcal{E}_M \subset \mathcal{E}_M^e$. By the function $z(w) = w + \sqrt{w^2 - 1}$, the inverse of the Zhukovski function $w(z) = \frac{1}{2}(z + z^{-1})$, the regions $\mathcal{E}_M^i$ and $\mathcal{E}_M^e$ are mapped onto rings $1 < |z| < a_i(M)$ and $1 < |z| < a_e(M)$, where

$$a_i(M) = \log(M + \sqrt{M^2 - 1}) \text{ and } a_e(M) = \log(M + \sqrt{M^2 + 1}).$$

Then, if ω_M^i and ω_M^e denote the harmonic measures of Γ_M^i and Γ_M^e with respect to $\mathcal{E}_M^i$ and $\mathcal{E}_M^e$ respectively, we obtain that

$$\omega_M^i(w) = \frac{\log |z(w)|}{a_i(M)}, \ w \in \mathcal{E}_M^i, \text{ and } \omega_M^e(w) = \frac{\log |z(w)|}{a_e(M)}, \ w \in \mathcal{E}_M^e.$$

By virtue of the Carleman extension principle (see [7]) we have that

$$\omega_M(w) \le \omega_M^i(w), \ w \in \mathcal{E}_M^i, \text{ and } \omega_M^e(w) \le \omega_M(w), \ w \in \mathcal{E}_M. \tag{8}$$

Now we can easily prove the following lemma.

Lemma 5.3 *If* $\zeta = i\eta$, $0 \le \eta < 1$, *then the following inequalities hold*
(i) $\cosh \eta a_i(M) \le c_0(M, \zeta) \le \cosh \eta a_e(M)$;
(ii) $\sinh \eta a_i(M) \le c_1(M, \zeta) \le \sinh \eta a_e(M)$.

Proof. The points $c_0(M, \zeta)$ and $ic_1(M, \zeta)$ belong to the level line $\omega_M(w) = \eta$, as it follows from lemma 5.1. and equality (7). On the other hand, (8) and lemma 5.2 imply that the level line $\omega_M(w) = \eta$ is placed inside of the level line $\omega_M^e(w) = \eta$ and outside of the level line $\omega_M^i(w) = \eta$, which , according to elementary properties of the Zhukovski function are ellipses with focuses in ± 1, and semi-axes $\cosh \eta a_e(M)$, $\sinh \eta a_e(M)$, and $\cosh \eta a_i(M)$, $\sinh \eta a_i(M)$. This implies inequalities (i) and (ii)

6 Proof of main theorems

Proof of inequality (1). From lemma 5.1 (iii) and lemma 5.3 (ii) we obtain that

$$\begin{aligned} a(M) &= \lim_{\eta\to 0+} \frac{c_1(M,i\eta)}{\eta}, \\ \frac{\sinh \eta a_i(M)}{\eta} &\le \frac{c_1(M,i\eta)}{\eta} \le \frac{\sinh \eta a_e(M)}{\eta} \end{aligned}$$

and tending $\eta \to 0+$ we conclude that

$$a_i(M) \le a(M) \le a_e(M),$$

which proves (1).

Proof of inequality (4). Let us apply the equality

$$\varepsilon(A,\zeta) = c_0(M,\zeta)^{-1}, \ M = A/\varepsilon(A,\zeta)$$

and the lemma 5.3 (i). Then

$$\frac{1}{\cosh \eta a_e(M)} \le \varepsilon(A,\zeta) \le \frac{1}{\cosh \eta a_i(M)}$$

and moreover

$$\frac{\sinh a_e(M)}{\cosh \eta a_e(M)} \le A \le \frac{\cosh a_i(M)}{\cosh \eta a_i(M)},$$

which imply that $a_e(M) \le a_e$ and $a_i \le a_i(M)$ and hence (4) is also proved.

Asymptotic equalities (2) and (5) are immediate corollaries of (1) and (4) respectively.

It remains to prove the equality (3). To do this we go back to section 4 and review the construction of the universal covering mapping $p : S^+ \to \mathcal{E}_M$, which is based on the conformal mapping $Q_\tau^+ \to \mathcal{E}_M^+$ that carries vertices $z = -\tau+i, \ -\tau, \ \tau, \ \tau+i$ to $w = -M, \ -1, \ 1, \ M$ in the mentioned order. Moreover, the value $\tau = \tau(M)$ is defined by this condition uniquely. To construct this conformal mapping, recall (see [3]) that the elliptic integral

$$z(\zeta) = \frac{1}{K'} \int_0^\zeta \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad 0 < k < 1$$

maps conformally the upper half-plane $Im\zeta > 0$ to the rectangle Q_τ^+ for $\tau = \frac{K}{K'}$, where K and K' are complete elliptic integrals of parameters k and k', i.e.

$$\begin{aligned} K &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \\ K' &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}} = \int_1^{1/k} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}} \end{aligned}$$

and k' is conjugate to k by $k^2 + k'^2 = 1$. This mapping carries points $\zeta = -\frac{1}{k},\ -1,\ 1,\ \frac{1}{k}$ to the vertices $z = -\tau + i,\ -\tau,\ \tau,\ \tau + i$ of Q_τ^+. Further, the function

$$\zeta(w) = \frac{M + 1/M}{w/M + M/w}$$

maps conformally the region $\mathcal{E}_M^+$ to the half-plane $Im\zeta > 0$, carrying the points $w = -M,\ -1,\ 1,\ M$ to $\zeta = -\frac{1}{k},\ -1,\ 1,\ \frac{1}{k}$ respectively, where $k = k_M = \frac{2M}{M^2+1}$. Indeed, it is elementary consequence of the properties of the Zhukovski function. The composition $z(\zeta(w))$ of the mentioned functions is the inverse to p conformal mapping from $\mathcal{E}_M^+$ to Q_τ^+. Therefore for ω_M we receive the expression

$$\omega_M(w) = Imz(\zeta(w)) = \frac{1}{K'} Im \int\limits_0^{\zeta(w)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad k = k_M.$$

Now we are able to prove (2.3). By lemma 5.1 (iii)

$$\begin{aligned} a(M)^{-1} &= \frac{\partial}{\partial v}\omega_M(iv)\big|_{v=0+} = \lim_{v\to 0+} \frac{\omega_M(iv)}{v} = \\ &= \frac{1}{K'} Im \lim_{v\to 0+} \frac{1}{v} \int\limits_0^{\zeta(iv)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{\zeta'(0)}{K'} = \frac{M^2+1}{M^2} \cdot \frac{1}{K'}, \end{aligned}$$

and hence (3) follows.

7 Extremal functions

The following two theorems describe the extremal functions for a and ε.

Theorem 7.1 *The unique extremal functions $f \in \mathcal{B}_M(S)$ for which $|f'(0)| = a(M)$ are functions of the view*

$$f(z) = \lambda p(z), \quad z \in S,$$

where λ is a constant with $|\lambda| = 1$

Theorem 7.2 *The unique extremal function $g \in \mathcal{H}_{A,\zeta},\ \zeta = i\eta$, for which $\sup_{\mathbf{R}} |g| = \varepsilon(A, \zeta)$ is the function*

$$g(z) = \varepsilon(A, \zeta) p(z + \tau), \quad z \in S.$$

To prove these theorems we need the following lemma.

Lemma 7.3 *For any $w,\ |w| < M$, the following inequality holds*

$$\omega_M(w) \geq \omega_M(|w|).$$

Proof. By the Riesz decomposition theorem (see [6], theorem 3.14) the measure ω_M, which is subharmonic in D_M and harmonic in $\mathcal{E}_M = D_M \backslash I$ admits the representation

$$\omega_M(w) = 1 - \int\limits_I g_M(w,t)d\mu(t), \quad |w| < M,$$

where $g_M(w,t)$ is the Green function of the region D_M with a pole in t, i.e.

$$g_M(w,t) = \log\left|\frac{M^2 - w\bar{t}}{M(w-t)}\right|;$$

μ is a positive Borel measure on the segment I, in this case symmetric with respect to 0, since ω_M is itself symmetric. Altogether we obtain the representation

$$\omega_M(w) = 1 - \int\limits_0^1 \left[g_M(w,t) + g_M(w,-t)\right] d\mu(t), \quad |w| < M.$$

Observe now that

$$g_M(w,t) + g_M(w,-t) = g_{M^2}(w^2, t^2)$$

and always holds inequalities

$$g_{M_1}(w_1, t_1) \le g_{M_1}(|w_1|, |t_1|), \quad |w_1|, |t_1| < M_1.$$

As a result we have

$$\omega_M(w) = 1 - \int\limits_0^1 g_{M^2}(w^2, t^2)d\mu(t) \ge 1 - \int\limits_0^1 g_{M^2}(|w|^2, t^2)d\mu(t) = \omega_M(|w|),$$

as stated.

Proof of theorem 7.1. First we assume that a function $f \in \mathcal{B}_M(S)$ for which $|f'(0)| = a(M)$ belongs also to the class $\mathcal{B}_1$ and particularly $f(0) = 0$ and $f'(0) = a(M)$. Then we consider the quotient

$$q(z) = \frac{f(z)}{p(z)}$$

in the rectangle $Q_\tau : |Rez| < \tau, |Imz| < 1$. It is analytic since p has the unique and simple zero in $z = 0$ and $f(0) = 0$. Moreover, $q(0) = f'(0)/p'(0) = a(M)/a(M) = 1$. On the other hand we claim that $\limsup_{z\to z^*} |q(z)| \le 1$, where $z \in Q_\tau$ and $z^* \in \partial Q_\tau$. By the maximum modulus principle this would imply that $q(z) = 1$ for all $z \in Q_\tau$, which is equivalent to the equality $f = p$.

So, let $z^* \in \partial Q_\tau$. By the symmetry of $|q|$ with respect to the real and imaginary axes, with no loss of generality we may assume that either $Imz^* = 1$ or $z^* = \tau + i\eta$ for some $0 \le \eta < 1$. In the first case $\lim_{z\to z^*} |p(z)| = M$ and

$\limsup_{z\to z^*}|f(z)|\le M$ and therefore $\limsup_{z\to z^*}|q(z)|\le 1$. In the second case, by (6) and (7) and also lemma 7.3 we have

$$\omega_M(|f(z^*)|)\le\omega_M(f(z^*))\le\omega_M(p(z^*))=\omega_M(|p(z^*)|),$$

since $1\le p(z^*)<M$. By lemma 5.2 we conclude that $|f(z^*)|\le p(z^*)$ and hence $\lim_{z\to z^*}|q(z)|=|q(z^*)|\le 1$ and the consideration of the case $f\in\mathcal{B}_1$ is completed.

Consider now the general case. If $|f'(0)|=a(M)$ and $\arg f'(0)=\theta$ then symmetrizing f as in section 3 we obtain

$$f_0(z)=e^{-i\theta}f(z),\ f_1(z)=\frac{1}{2}[f_0(z)+\overline{f_0(\overline{z})}],\ f_2(z)=\frac{1}{2}[f_1(z)-f_1(-z)],\quad z\in S$$

and $f_2\in\mathcal{B}_1,\ f_2'(0)=a(M)$. By the considerations above $f_2=p$ and therefore

$$\frac{1}{2}[f_1(z)-f_1(-z)]=p(z),\quad z\in S.$$

Tend $z\to z^*$, $Imz^*=\pm1$. Then $|p(z)|\to M$. But $\frac{1}{2}[f_1(z)-f_1(-z)]$ may tend to M only if $[p(z)-f_1(z)]\to 0$, and $[p(z)+f_1(-z)]\to 0$, since $|f_1(z)|\le M$ and $|f_1(-z)|\le M$. Hence $p-f_1$ tends to 0 as $z\to z^*$, $Imz^*=\pm1$. But $p-f_1$ is bounded and we obtain that $f_1=p$. Similarly, from the equality

$$\frac{1}{2}[f_0(z)+\overline{f_0(\overline{z})}]=p(z),\quad z\in S$$

we obtain that $f_0=p$ and therefore $f(z)=e^{i\theta}p(z),\ z\in S$.

Proof of theorem 7.2. A function $g\in\mathcal{H}_{A,\zeta}$, symmetric with respect to $\mathbf{R}$ and such that $\sup_{\mathbf{R}}|g|=\varepsilon(A,\zeta)$, admits evidently a representation

$$g(z)=\varepsilon(A,\zeta)f(z),\quad z\in S,$$

where $f\in\mathcal{B}_0$ and $f(\zeta)=c_0(M,\zeta)$. Particularly f satisfies the condition

$$\omega_M(f(\zeta))=\omega_M(p(\zeta))=Im\zeta.$$

By section 4 it is possible only if $f(z)=p(z+\xi)$ for some real constant ξ, which can be chosen from interval $[-2\tau,2\tau)$ since p is 4τ-periodical. But $f\in\mathcal{B}_0$ and hence $f(i\eta)=p(\xi+i\eta)>0$ which implies that $\xi=\tau$. We obtain that

$$g(z)=\varepsilon(A,\zeta)p(z+\tau),\quad z\in S.$$

The general case may be derived from the considered case as it was done in the proof of the previous theorem.

8 Application: an asymptotic analogue of Bernstein's inequality

To make quantities $a(M)$ more flexible and applicable we continue as follows.

Let D be a simply connected region, symmetric with respect to the real axis $\mathbf{R}$. We define corresponding classes $\mathcal{B}_M(D)$ for $M > 1$ to be the classes of all functions f, analytic in D and satisfying the conditions

$$\sup_D |f| \leq M, \quad \sup_{D\cap\mathbf{R}} |f| \leq 1$$

By the Riemann mapping theorem there exists such a conformal mapping $w : D \longrightarrow S$ that $w(\mathbf{R}) \subset \mathbf{R}$. Then to each $f \in \mathcal{B}_M(D)$ we have the unique $g \in \mathcal{B}_M(S)$ corresponded such that $f = g \circ w$. Since $f' = (g' \circ w)w'$ we obtain that

$$|f'(x)| \leq \frac{a(M)}{h_D(x)}, \quad x \in D \cap \mathbf{R}, \tag{9}$$

where

$$h_D(x) = 1/w'(x)$$

for $x \in D\cap\mathbf{R}$. This quantity is well-defined, since w is unique up to translations along $\mathbf{R}$. By an analogy with the conformal radius, we call $h_D(x)$ a *conformal semi-width of D in x.* For example, the conformal semi-width of the strip $S_h : |Imz| <$ is equal to h in each real x.

Under some restrictions on D we are able to replace the conformal semi-width in (9) by a metric characteristic of D. Namely, symmetric D is said to be a L-strip in the sense of Warschawski if it is given by inequalities $x > x_0,\ |y| < \theta_D(x)$, where θ_D is a positive continuous function with the following regularity condition at infinity:

$$\lim_{x_1,x_2\to+\infty} \frac{\theta_D(x_1) - \theta_D(x_2)}{x_1 - x_2} = 0.$$

Warschawski's theorem (see [8], theorem X) states that

$$\lim_{x\to+\infty} \frac{\theta_D(x)}{h_D(x)} = 1.$$

In further we will use the notation $\alpha \sim \beta$ to indicate that $\alpha/\beta \to 1$. Thus $\theta_D(x) \sim h_D(x)$ as $x \to +\infty$. By (9) we conclude that

$$\limsup_{x\to+\infty} |f'(x)|\theta_D(x) \leq a(M) \tag{10}$$

for each $f \in \mathcal{B}_M(D)$. Note that the quantity in right-hand side of (10) is the sharpest possible for the class $\mathcal{B}_M(D)$, since there are periodic extremal functions $f \in \mathcal{B}_M(S)$ with $f'(0) = a(M)$ and therefore there is a function $g \in \mathcal{B}_M(D)$ and a sequence $x_n \to +\infty$ such that $g'(x_n) = a(M)/h_D(x_n)$.

Further note, that (10) remains valid, if f is not from the class $\mathcal{B}_M(D)$, but satisfies the conditions

$$\limsup_{z\to\infty, z\in D} |f(z)| \leq M, \quad \limsup_{x\to+\infty} |f(z)| \leq 1.$$

Below we apply (10) to prove an asymptotic analogue of Bernstein's inequality for entire functions of a finite type with respect to an arbitrary proximate order $\rho(r)$. But first recall a proximate order.

A function $\rho(r) \geq 0,\ r \geq 0$ is said to be a proximate order (see [5], chapter II, §2) iff it is continuously differentiable and there exists limit $\rho = \lim_{r\to+\infty} \rho(r)$ and $0 = \lim_{r\to+\infty} \rho'(r) r \log r$. If we denote $\nu(r) = r^{\rho(r)}$, this conditions are equivalent to the condition

$$\rho = \lim_{r\to+\infty} \frac{r\nu'(r)}{\nu(r)}.$$

Therefore we would operate rather with ν and call it a proximate growth. For our purposes we may assume that $\rho > 0$ and, additionally, that ν is monotony increasing and $\nu(0) = 0$.

Further, it is said an entire function f to have ν-type σ_f iff

$$\sigma_f = \limsup_{r\to+\infty} \frac{\log M_f(r)}{\nu(r)},$$

where $M_f(r) = \sup_{|z|=r} |f(z)|$.

Theorem 8.1 *If g is an entire function of ν-type $\sigma_f \leq \sigma$ and*

$$\limsup_{x\to+\infty} |g(x)| \leq 1$$

then its derivative g' satisfies the asymptotic inequality

$$\limsup_{x\to+\infty} \frac{|g'(x)|}{\nu'(x)} \leq \sigma. \tag{11}$$

Remark. It is unknown to author if there is any growth ν, for which the σ in the right-hand side of (11) is not sharp. However, if $\nu(r) = r^n$ for a positive integer n, the inequality is sharp. Simply consider the function $g(z) = \exp(iz^n)$. If we replace σ by $c_\nu\sigma$, where c_ν is a constant depending only ν, the proof will be easier, the Cauchy inequality is enough for this purpose. See [2], the proof of theorem 3. From this inequality follows the necessary condition on a function f, bounded on $\mathbf{R}$, to be uniformly approximable on $\mathbf{R}$ by bounded on $\mathbf{R}$ entire functions of finite ν-types. Namely, the composition $f \circ \mu$ must be uniformly continuous on $\mathbf{R}$, where μ is the inverse of ν, continued on $\mathbf{R}$ as an odd function. This is also a sufficient condition (see [2], theorem 3).

Proof. With no loss of generality we may assume that $|g(x)| \leq 1$ for $x \geq 0$.

We intend to apply an asymptotic inequality for appropriately chosen L-strips D. More exactly, for any $\lambda > 1$ and $t > 0$ we construct such L-strips $D = D_{\lambda,t} : x > 0, |y| < \theta_{\lambda,t}(x)$ that

$$\theta_{\lambda,t}(x) \sim \frac{tx}{\lambda\rho\nu(x)} \sim \frac{t}{\lambda\nu'(x)} \quad \text{as } x \to \infty$$

and $|g(z)| \le \exp(\sigma t)$ if $z = x + iy \in D$ and $x > x_0$ for some fixed x_0. Then by (10) we would have that

$$\limsup_{x\to+\infty} \frac{|g'(x)|}{\nu'(x)} \le \lambda \frac{a(\exp(\sigma t))}{t}$$

and tending $\lambda \to 1+$, $t \to +\infty$ would obtain in right-hand side the required σ. Therefore we need only to construct L-strips $D_{\lambda,t}$ with described properties.

For this purpose consider that the quantity

$$h_g(\varphi) = \limsup_{r\to+\infty} \frac{\log|g(re^{i\varphi})|}{\nu(r)},$$

called the generalized indicator function (with respect to ν). Obviously, $h_g(\varphi) \le \sigma_g$. Moreover, the indicator function is ρ-trigonometric convex. For the given g we have also $h_g(0) = 0$. Altogheter it gives for $|\varphi| \le \frac{\pi}{2\rho}$,

$$h_g(\varphi) \le \sigma|\sin\rho\varphi|.$$

But we are able to state something stronger:

Lemma 8.2 *Under conditions of theorem 8.1, for each $\lambda > 1$ there exists r_λ such that*

$$\log|g(re^{i\varphi})| \le \lambda\sigma\nu(r)|\sin\rho\varphi|$$

if $|\varphi| \le \alpha_0 = \min\{\frac{\pi}{2\rho}, \frac{\pi}{2}\}$ and $r > r_\lambda$.

Suppose at moment that this lemma is already proved.

Define L-strips $D_{\lambda,t}$ in the polar coordinates as follows

$$D_{\lambda,t} = \{z = re^{i\varphi} : |\varphi| < \alpha_0, r < r(\varphi)\}$$

where $r(\varphi)$ is determined uniquely from the equality

$$\nu(r(\varphi)) = \frac{t}{\lambda|\sin\rho\varphi|} \quad \text{if } \varphi \ne 0 \quad \text{and } r(0) = +\infty. \tag{12}$$

Since $r(\varphi)$ increases monotonely as $|\varphi|$ decreases, in the Cartesian coordinates $D_{\lambda,t}$ is given by

$$D_{\lambda,t} : x > 0, |y| < \theta_{\lambda,t}(x),$$

where the function $\theta_{\lambda,t}$ satisfies the equality

$$\theta_{\lambda,t}(x(\varphi)) = |y(\varphi)|$$

for $x(\varphi) = r(\varphi)\cos\varphi$ and $y(\varphi) = r(\varphi)\sin\varphi$. Therefore,

$$\theta_{\lambda,t}(x(\varphi)) = \frac{t|y(\varphi)|}{t} \sim \frac{tr(\varphi)}{\lambda\rho\nu(r(\varphi))} \quad \text{as } |\varphi| \to 0+$$

Further, recall that for proximate growths ν there exist limits (see [5], chapter II, theorem 2.2)

$$\lim_{r\to\infty}\frac{\nu(\kappa r)}{\nu(r)}=\kappa^{\rho} \tag{13}$$

uniformly on every compact set of κ. Particularly, if $r'/r\to 1$, then $\nu(r')/\nu(r)\to 1$. Therefore

$$\theta_{\lambda,t}(x(\varphi))\sim\frac{tx(\varphi)}{\lambda\rho\nu(x(\varphi))}.$$

But if $|\varphi|$ runs α_0 downward to 0, $x(\varphi)$ runs $x(\alpha_0)$ upward to $+\infty$, without any skips. So, we conclude that

$$\theta_{\lambda,t}(x)\sim\frac{tx}{\lambda\rho\nu(x)}\sim\frac{t}{\lambda\nu'(x)}\quad\text{as } x\to+\infty.$$

Further, by construction and lemma 8.2

$$\log|g(re^{i\varphi})|\le\lambda\sigma\nu(r)|\sin\rho\varphi|\le\sigma t\frac{\nu(r)}{\nu(r(\varphi))}\le\sigma t$$

if $r_\lambda<r<r(\varphi)$, where r_λ is that of lemma 8.2. Therefore

$$|g(z)|\le\exp(\sigma t)\quad\text{for } z\in D_{\lambda,t}\text{ such that } x>r_\lambda.$$

Thus, $D_{\lambda,t}$ possesses prescribed properties. But it remains to check the regularity condition for $\theta_{\lambda,t}$ at infinity, to be assured that $D_{\lambda,t}$ is true L-strip. It is easy to see that $\theta_{\lambda,t}$ is differentiable and consequently the regularity condition is equivalent to

$$\lim_{x\to+\infty}\theta'_{\lambda,t}(x)=0,$$

which is in its turn equivalent to

$$\lim_{\varphi\to+0}\frac{y'(\varphi)}{x'(\varphi)}=\lim_{\varphi\to+0}\frac{r'(\varphi)\sin\varphi+r(\varphi)\cos\varphi}{r'(\varphi)\cos\varphi-r(\varphi)\sin\varphi}=0.$$

The terms $r'(\varphi)\cos\varphi$ and $-r(\varphi)\sin\varphi$ in the denominator of the second fraction have the same signs. Removing the second term, we only strengthen the condition. Then, the fraction becomes equal to $\tan\varphi+r(\varphi)/r'(\varphi)$. Obviously, $\tan\varphi\to 0$ and we have to prove that $r'(\varphi)/r(\varphi)\to\infty$. We derive it from (12). Indeed, denote by μ the inverse function to ν. Then

$$\lim_{r\to\infty}\frac{r\mu'(r)}{\mu(r)}=\frac{1}{\rho}.$$

By (12) $r(\varphi)=\mu(t/(\lambda\sin\rho\varphi))$. Differentiating this equality, we obtain that $r'(\varphi)/r(\varphi)\sim C/\varphi$ for some constant C, as $\varphi\to 0+$. The regularity of $\theta_{\lambda,t}$ at infinity is proved and hence $D_{\lambda,t}$ is true L-strip.

To complete the proof of theorem 8.1 it remains to prove lemma 8.2.

Proof of lemma 8.2. By [5], chapter II, theorem 5.1 there exists a function f, analytic in $|\varphi| \le \alpha_0$ such that

$$f(z) = r^{\rho(r)-\rho}(1 + o(1)), \tag{14}$$

where $o(1)$ tends to 0 uniformly as $r \to \infty$. Additionally we may assume that $f(\overline{z}) = \overline{f(z)}$, otherwise we consider the symmetrization $f_1(z) = \frac{1}{2}[f(z) + \overline{f(\overline{z})}]$. Denote

$$U(z) = Im(f(z)z^\rho)$$

in the angle $|\varphi| \le \frac{\pi}{2\rho}$. We claim that

$$U(z) = r^{\rho(r)} \sin \rho\varphi(1 + o(1)) \tag{15}$$

Indeed if $f(z) = u(z) + iv(z)$ then

$$U(z) = r^\rho u(z) \sin \rho\varphi + r^\rho v(z) \cos \rho\varphi.$$

Besides, (14) exactly means that

$$u(z) = r^{\rho(r)-\rho}(1 + o(1)), \quad v(z) = o(r^{\rho(r)-\rho}) \tag{16}$$

and we note that (15) will be proved if we prove for $v(z)$ the stronger representation

$$v(z) = |\sin \rho\varphi| o(r^{\rho(r)-\rho}). \tag{17}$$

For this purpose consider v in disks Δ_r with center at r and radius κr, where $\kappa = \sin \alpha_0$. By (8.8), for an arbitrarily chosen $\delta > 0$ there exist r_δ such that $|v(z)| \le \delta r^{\rho(r)-\rho}$ if $r > r_\delta$. Moreover, in the view of (13) we may assume that $|v| \le \delta r^{\rho(r)-\rho}$ in the whole disk Δ_r. Further, it is easy to see that $v = 0$ on $\mathbf{R}$. Therefore, applying the 2-constant theorem, we receive that

$$|v(r + iy)| \le \frac{4}{\pi} \delta r^{\rho(r)-\rho} \arctan \frac{|y|}{r}$$

for each $r > r_\delta$ and $|y| \le \kappa r$. This is equivalent to (17) and hence (15) is proved.

To finish the proof, let us fix λ' such that $1 < \lambda' < \lambda$ and consider the difference $q(z) = \log |g(z)| - \lambda' \sigma U(z)$ in the closed angle $0 \le \varphi \le \alpha_0$. On the sides of this angle $q(z)$ is bounded above since $\limsup_{r\to\infty} |q(z)| < 0$ on these sides. Further, $q(z)$ is subharmonic function of finite ν-type. Since the measure of the considered angle is less than π/ρ, we may apply the Phragmén-Lindelöf principle and conclude that $q(z) \le C$ for some constant C. Additionally we have that $q(x) \le 0$ for each real x. Once again applying the 2-constant theorem this time for the angle we obtain that $q(z) \le C_1\varphi$. Thus we conclude that

$$\log |g(re^{i\varphi})| \le \lambda' \sigma(1 + o(1))\nu(r) \sin \rho\varphi + O(\varphi) \le \lambda\sigma\nu \sin \rho\varphi$$

for $0 \le \varphi \le \alpha_0$ and sufficiently large r. Symmetrically this inequality is also valid for the angle $-\alpha_0 \le \varphi \le 0$.

This concludes the proof of the theorem.

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