An Epiperimetric Inequality Approach to the Thin and Fractional Obstacle Problems

Geometric Analysis
Free Boundary Problems
& Measure Theory
MPI Leipzig, June 15–17, 2015

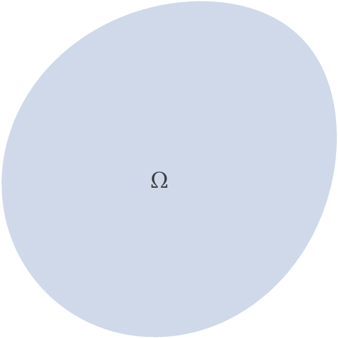
Arshak Petrosyan
(joint with Nicola Garofalo, Camelia Pop, and Mariana Smit Vega Garcia)

Purdue University · West Lafayette · IN · USA

The thin obstacle problem

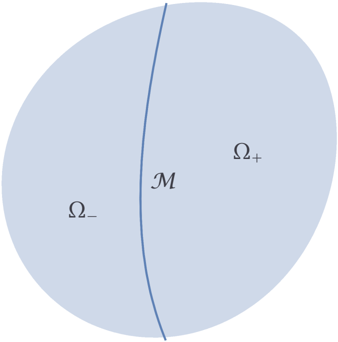
The thin obstacle problem

- Given
  - Ω domain in $\mathbb{R}^n$



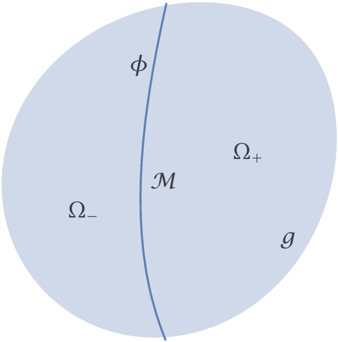
The thin obstacle problem

- Given
  - Ω domain in $\mathbb{R}^n$
  - $\mathcal{M}$ smooth hypersurface, $\Omega \setminus \mathcal{M} = \Omega_+ \cup \Omega_-$



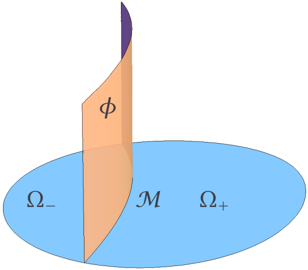
The thin obstacle problem

- Given
  - Ω domain in $\mathbb{R}^n$
  - $\mathcal{M}$ smooth hypersurface,
    $\Omega \setminus \mathcal{M} = \Omega_+ \cup \Omega_-$
  - $\phi : \mathcal{M} \to \mathbb{R}$ (*thin obstacle*),
    $g : \partial\Omega \to \mathbb{R}$ (*boundary values*),
    $g > \phi$ on $\mathcal{M} \cap \partial\Omega$.



The thin obstacle problem

- Given
  - Ω domain in $\mathbb{R}^n$
  - $\mathcal{M}$ smooth hypersurface,
    $\Omega \setminus \mathcal{M} = \Omega_+ \cup \Omega_-$
  - $\phi : \mathcal{M} \to \mathbb{R}$ (*thin obstacle*),
    $g : \partial\Omega \to \mathbb{R}$ (*boundary values*),
    $g > \phi$ on $\mathcal{M} \cap \partial\Omega$.



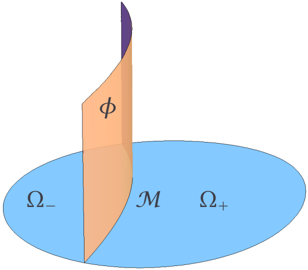
The thin obstacle problem

- Given
  - Ω domain in $\mathbb{R}^n$
  - $\mathcal{M}$ smooth hypersurface,
    $\Omega \setminus \mathcal{M} = \Omega_+ \cup \Omega_-$
  - $\phi : \mathcal{M} \to \mathbb{R}$ (*thin obstacle*),
    $g : \partial\Omega \to \mathbb{R}$ (*boundary values*),
    $g > \phi$ on $\mathcal{M} \cap \partial\Omega$.
- Minimize the Dirichlet integral

$$D_\Omega(u) = \int_\Omega |\nabla u|^2 dx$$

on the closed convex set

$$\mathfrak{K} = \{u \in W^{1,2}(\Omega) \mid u = g \text{ on } \partial\Omega,\, u \geq \phi \text{ on } \mathcal{M} \cap \Omega\}.$$



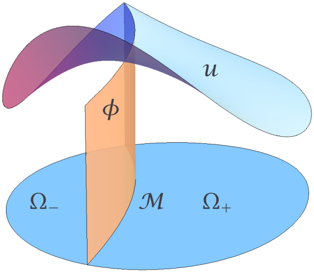
The thin obstacle problem

- Given
  - Ω domain in $\mathbb{R}^n$
  - $\mathcal{M}$ smooth hypersurface,
    $\Omega \setminus \mathcal{M} = \Omega_+ \cup \Omega_-$
  - $\phi : \mathcal{M} \to \mathbb{R}$ (*thin obstacle*),
    $g : \partial\Omega \to \mathbb{R}$ (*boundary values*),
    $g > \phi$ on $\mathcal{M} \cap \partial\Omega$.
- Minimize the Dirichlet integral

$$D_\Omega(u) = \int_\Omega |\nabla u|^2 dx$$

on the closed convex set

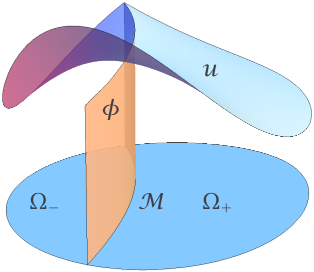
$$\mathfrak{K} = \{u \in W^{1,2}(\Omega) \mid u = g \text{ on } \partial\Omega,\, u \geq \phi \text{ on } \mathcal{M} \cap \Omega\}.$$



The thin obstacle problem

The minimizer u satisfies

- $\Delta u = 0 \quad \text{in } \Omega \setminus \mathcal{M} = \Omega_+ \cup \Omega_-$

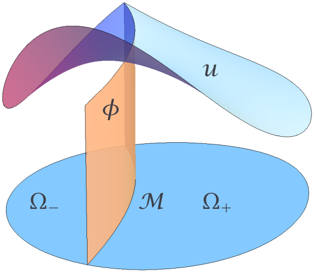


The thin obstacle problem

The minimizer u satisfies

- $\Delta u = 0 \quad \text{in } \Omega \setminus \mathcal{M} = \Omega_+ \cup \Omega_-$
- *Signorini (complementarity) conditions* on $\mathcal{M}$

$$\begin{aligned} u - \phi &\geq 0 \\ \partial_{\nu^+} u + \partial_{\nu^-} u &\geq 0 \\ (u - \phi)(\partial_{\nu^+} u + \partial_{\nu^-} u) &= 0 \end{aligned}$$

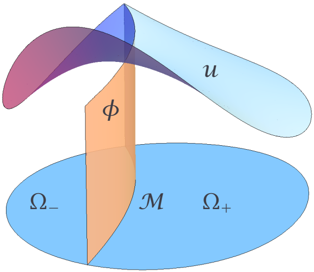


The thin obstacle problem

The minimizer u satisfies

- $\Delta u = 0 \quad \text{in } \Omega \setminus \mathcal{M} = \Omega_+ \cup \Omega_-$
- *Signorini (complementarity) conditions* on $\mathcal{M}$

$$\begin{aligned} u - \phi &\geq 0 \\ \partial_{\nu^+} u + \partial_{\nu^-} u &\geq 0 \\ (u - \phi)(\partial_{\nu^+} u + \partial_{\nu^-} u) &= 0 \end{aligned}$$



- Main objectives of study
  - Regularity of u
  - Structure and regularity of the **free boundary**

$$\Gamma(u) := \partial_{\mathcal{M}}\{x \in \mathcal{M} \mid u = \phi\}$$

The thin obstacle problem

The thin obstacle problem arises in a variety of situations of interest for the applied sciences:

The thin obstacle problem

The thin obstacle problem arises in a variety of situations of interest for the applied sciences:

- It presents itself in elasticity, when an elastic body is at rest, partially laying on a surface $\mathcal{M}$.

The thin obstacle problem

The thin obstacle problem arises in a variety of situations of interest for the applied sciences:

- It presents itself in elasticity, when an elastic body is at rest, partially laying on a surface $\mathcal{M}$.
- It models the flow of a saline concentration through a semipermeable membrane when the flow occurs in a preferred direction.

The thin obstacle problem

The thin obstacle problem arises in a variety of situations of interest for the applied sciences:

- It presents itself in elasticity, when an elastic body is at rest, partially laying on a surface $\mathcal{M}$.
- It models the flow of a saline concentration through a semipermeable membrane when the flow occurs in a preferred direction.
- It also arises in financial mathematics in situations in which the random variation of an underlying asset changes discontinuously.

The thin obstacle problem

The thin obstacle problem arises in a variety of situations of interest for the applied sciences:

- It presents itself in elasticity, when an elastic body is at rest, partially laying on a surface $\mathcal{M}$.
- It models the flow of a saline concentration through a semipermeable membrane when the flow occurs in a preferred direction.
- It also arises in financial mathematics in situations in which the random variation of an underlying asset changes discontinuously.
- Obstacle problem for the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$

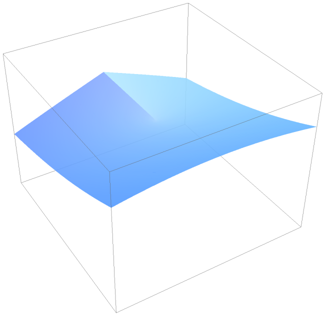
$$u - \phi \geq 0, \quad (-\Delta)^s u \geq 0, \quad (u - \phi)(-\Delta)^s u = 0 \quad \text{in } \mathbb{R}^n.$$

The thin obstacle problem corresponds to $s = \frac{1}{2}$.

Regularity of the minimizer u: smooth $\mathcal{M}$

- Generally, it is easy to realize that u is not smooth in Ω, as it may develop a Lipschitz corner across $\mathcal{M}$.
  Explicit example:

$$u(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$$

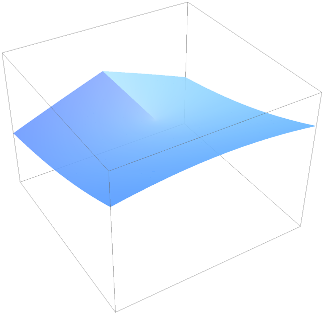


Regularity of the minimizer u: smooth $\mathcal{M}$

- Generally, it is easy to realize that u is not smooth in Ω, as it may develop a Lipschitz corner across $\mathcal{M}$.
  Explicit example:

$$u(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$$

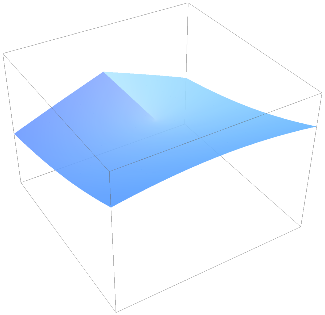
- However, on $\mathcal{M}$ and consequently on $\Omega_\pm \cup \mathcal{M}$, u is better:
  $u \in C^{1,\alpha}(\Omega_\pm \cup \mathcal{M})$ for some $\alpha > 0$.



Regularity of the minimizer u: smooth $\mathcal{M}$

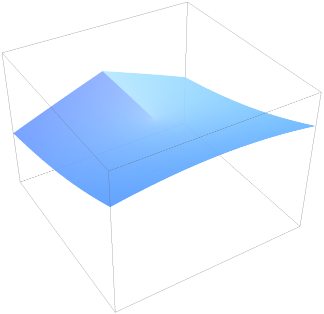
- Generally, it is easy to realize that u is not smooth in Ω, as it may develop a Lipschitz corner across $\mathcal{M}$.
  Explicit example:

$$u(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$$



- However, on $\mathcal{M}$ and consequently on $\Omega_\pm \cup \mathcal{M}$, u is better:
  $u \in C^{1,\alpha}(\Omega_\pm \cup \mathcal{M})$ for some $\alpha > 0$.
  - [Caffarelli'79]: for flat $\mathcal{M}$, using *semiconvexity* in tangential directions to $\mathcal{M}$.

Regularity of the minimizer u: smooth $\mathcal{M}$



- Generally, it is easy to realize that u is not smooth in Ω, as it may develop a Lipschitz corner across $\mathcal{M}$.
  Explicit example:

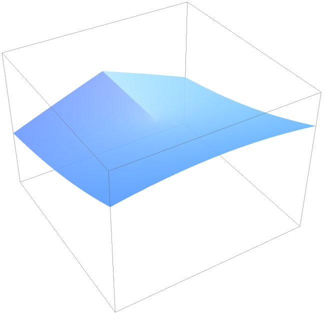
$$u(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$$

- However, on $\mathcal{M}$ and consequently on $\Omega_\pm \cup \mathcal{M}$, u is better:
  $u \in C^{1,\alpha}(\Omega_\pm \cup \mathcal{M})$ for some $\alpha > 0$.
  - [Caffarelli'79]: for flat $\mathcal{M}$, using *semiconvexity* in tangential directions to $\mathcal{M}$.
  - [Kinderlehrer'81]: For smooth $\mathcal{M}$ (or variable coefficient equations) by using a *filling holes* method.

Regularity of the minimizer u: smooth $\mathcal{M}$

- Generally, it is easy to realize that u is not smooth in Ω, as it may develop a Lipschitz corner across $\mathcal{M}$.
  Explicit example:

$$u(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$$

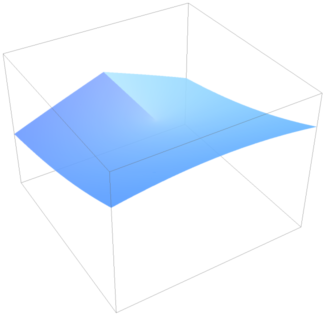


- However, on $\mathcal{M}$ and consequently on $\Omega_\pm \cup \mathcal{M}$, u is better: $u \in C^{1,\alpha}(\Omega_\pm \cup \mathcal{M})$ for some $\alpha > 0$.
  - [Caffarelli'79]: for flat $\mathcal{M}$, using *semiconvexity* in tangential directions to $\mathcal{M}$.
  - [Kinderlehrer'81]: For smooth $\mathcal{M}$ (or variable coefficient equations) by using a *filling holes* method.
  - [Uraltseva'85]: Less regular $\mathcal{M}$, *De Giorgi type* method.

Regularity of the minimizer u: smooth $\mathcal{M}$

- Generally, it is easy to realize that u is not smooth in Ω, as it may develop a Lipschitz corner across $\mathcal{M}$.
  Explicit example:

$$u(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$$



- However, on $\mathcal{M}$ and consequently on $\Omega_\pm \cup \mathcal{M}$, u is better: $u \in C^{1,\alpha}(\Omega_\pm \cup \mathcal{M})$ for some $\alpha > 0$.
  - [Caffarelli'79]: for flat $\mathcal{M}$, using *semiconvexity* in tangential directions to $\mathcal{M}$.
  - [Kinderlehrer'81]: For smooth $\mathcal{M}$ (or variable coefficient equations) by using a *filling holes* method.
  - [Uraltseva'85]: Less regular $\mathcal{M}$, *De Giorgi type* method.
  - In dimension $n = 2$ this was known at least by [Lewy'70] (C^1) and [Richardson'78] ($C^{1,1/2}$).

Optimal regularity of the minimizer u: flat $\mathcal{M}$, $\phi = 0$

- As the explicit example shows, u cannot be better than $C^{1,1/2}$ on $\Omega_\pm \cup \mathcal{M}$.

Optimal regularity of the minimizer u: flat $\mathcal{M}$, $\phi = 0$

- As the explicit example shows, u cannot be better than $C^{1,1/2}$ on $\Omega_{\pm} \cup \mathcal{M}$.
- Optimal regularity: $u \in C^{1,1/2}(\Omega_{\pm} \cup \mathcal{M})$

Optimal regularity of the minimizer u: flat $\mathcal{M}$, $\phi = 0$

- As the explicit example shows, u cannot be better than $C^{1,1/2}$ on $\Omega_\pm \cup \mathcal{M}$.
- Optimal regularity: $u \in C^{1,1/2}(\Omega_\pm \cup \mathcal{M})$
- For flat $\mathcal{M}$ and $\phi = 0$: [Athanasopoulos-Caffarelli'04]

Optimal regularity of the minimizer u: flat $\mathcal{M}$, $\phi = 0$

- As the explicit example shows, u cannot be better than $C^{1,1/2}$ on $\Omega_\pm \cup \mathcal{M}$.
- Optimal regularity: $u \in C^{1,1/2}(\Omega_\pm \cup \mathcal{M})$
- For flat $\mathcal{M}$ and $\phi = 0$: [Athanasopoulos-Caffarelli'04]
  - Using *semiconvexity* (convexity on $\mathcal{M}$) and the *monotonicity* of

$$\psi(r) = \frac{1}{r} \int_{B_r^+} \frac{|\nabla w|^2}{|x|^{n-2}}, \quad w = \partial_\nu u$$

akin to *Alt-Caffarelli-Friedman monotonicity formula.*

Optimal regularity of the minimizer u: flat $\mathcal{M}$, $\phi = 0$

- As the explicit example shows, u cannot be better than $C^{1,1/2}$ on $\Omega_\pm \cup \mathcal{M}$.
- Optimal regularity: $u \in C^{1,1/2}(\Omega_\pm \cup \mathcal{M})$
- For flat $\mathcal{M}$ and $\phi = 0$: [Athanasopoulos-Caffarelli'04]
  - Using *semiconvexity* (convexity on $\mathcal{M}$) and the *monotonicity* of

    $$\psi(r) = \frac{1}{r}\int_{B_r^+} \frac{|\nabla w|^2}{|x|^{n-2}}, \quad w = \partial_\nu u$$

    akin to *Alt-Caffarelli-Friedman monotonicity formula.*
- A different proof is given by [Athanasopoulos-Caffarelli-Salsa'08] using monotonicity of *Almgren's frequency function:*

$$N(r) = \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2}$$

Optimal regularity of the minimizer u: flat $\mathcal{M}$, $\phi \neq 0$

- Nonzero ϕ (still flat $\mathcal{M}$): [Caffarelli-Salsa-Silvestre'08]

Optimal regularity of the minimizer u: flat $\mathcal{M}$, $\phi \neq 0$

- Nonzero ϕ (still flat $\mathcal{M}$): [Caffarelli-Salsa-Silvestre'08]
  - Using *truncated Almgren's frequency function*

$$\Phi(r) = r e^{r^\sigma} \frac{d}{dr} \log \max\left\{\int_{\partial B_r} v^2, r^{n+3}\right\}, \quad v = u - \phi$$

Optimal regularity of the minimizer u: flat $\mathcal{M}$, $\phi \neq 0$

- Nonzero ϕ (still flat $\mathcal{M}$): [Caffarelli-Salsa-Silvestre'08]
  - Using *truncated Almgren's frequency function*

$$\Phi(r) = re^{r^\sigma} \frac{d}{dr} \log \max \left\{ \int_{\partial B_r} v^2, r^{n+3} \right\}, \quad v = u - \phi$$

  - Finer properties can be studied by looking at finer truncations [Garofalo-P.'09]

$$\Phi_k(r) = re^{r^\sigma} \frac{d}{dr} \log \max \left\{ \int_{\partial B_r} v_k^2, r^{n-1+2k+\delta} \right\},$$
$$v_k = u - P_k(x) - [\phi(x') - P_k(x', 0)],$$

  where $\Delta P_k = 0$ kills the k-th Taylor polynomial of ϕ at 0, $k \geq 2$, $\delta > 0$

Optimal regularity of the minimizer u: non-flat $\mathcal{M}$

- Straighten $\mathcal{M} \rightsquigarrow B_1' \coloneqq \{x_n = 0\} \cap B_1$ and consider the minimizer of
$$\int_{B_1} \nabla u \cdot \mathcal{A}(x) \nabla u, \quad u(\cdot, 0) \geq \phi \quad \text{on } B_1'$$

Optimal regularity of the minimizer u: non-flat $\mathcal{M}$

- Straighten $\mathcal{M} \rightsquigarrow B_1' \coloneqq \{x_n = 0\} \cap B_1$ and consider the minimizer of
$$\int_{B_1} \nabla u \cdot \mathcal{A}(x) \nabla u, \quad u(\cdot, 0) \geq \phi \quad \text{on } B_1'$$
- Then $v(x) = u(x) - \phi(x', 0)$ satisfies the complementarity conditions

$$L_{\mathcal{A}} v = \operatorname{div}(\mathcal{A} \nabla v) = f \coloneqq -L_{\mathcal{A}} \phi \quad \text{in } B_1^{\pm}$$
$$v \geq 0, \;\; \nu^+ \cdot \mathcal{A} \nabla v + \nu^- \cdot \mathcal{A} \nabla v \geq 0, \;\; v(\nu^+ \cdot \mathcal{A} \nabla v + \nu^- \cdot \mathcal{A} \nabla v) = 0 \quad \text{on } B_1'.$$

Optimal regularity of the minimizer u: non-flat $\mathcal{M}$

- Straighten $\mathcal{M} \rightsquigarrow B_1' \coloneqq \{x_n = 0\} \cap B_1$ and consider the minimizer of
$$\int_{B_1} \nabla u \cdot \mathcal{A}(x) \nabla u, \quad u(\cdot, 0) \geq \phi \quad \text{on } B_1'$$
- Then $v(x) = u(x) - \phi(x', 0)$ satisfies the complementarity conditions
$$\begin{aligned} L_{\mathcal{A}} v = \operatorname{div}(\mathcal{A}\nabla v) = f \coloneqq -L_{\mathcal{A}}\phi \quad &\text{in } B_1^{\pm} \\ v \geq 0,\ \ \nu^{+}\cdot\mathcal{A}\nabla v + \nu^{-}\cdot\mathcal{A}\nabla v \geq 0,\ \ v(\nu^{+}\cdot\mathcal{A}\nabla v + \nu^{-}\cdot\mathcal{A}\nabla v) = 0 \quad &\text{on } B_1'. \end{aligned}$$
- Technical difficulty: co-normals $\mathcal{A}(x)\nu$ may not be aligned with normals ν on B_1'.

Optimal regularity of the minimizer u: non-flat $\mathcal{M}$

- Straighten $\mathcal{M} \rightsquigarrow B_1' := \{x_n = 0\} \cap B_1$ and consider the minimizer of
$$\int_{B_1} \nabla u \cdot \mathcal{A}(x) \nabla u, \quad u(\cdot, 0) \geq \phi \quad \text{on } B_1'$$
- Then $v(x) = u(x) - \phi(x', 0)$ satisfies the complementarity conditions

$$L_{\mathcal{A}} v = \operatorname{div}(\mathcal{A} \nabla v) = f := -L_{\mathcal{A}} \phi \quad \text{in } B_1^{\pm}$$
$$v \geq 0, \;\; \nu^+ \cdot \mathcal{A} \nabla v + \nu^- \cdot \mathcal{A} \nabla v \geq 0, \;\; v(\nu^+ \cdot \mathcal{A} \nabla v + \nu^- \cdot \mathcal{A} \nabla v) = 0 \quad \text{on } B_1'.$$

- Technical difficulty: co-normals $\mathcal{A}(x)\nu$ may not be aligned with normals ν on B_1'.
- Easily rectified with a diffeomorphism (as regular as $\mathcal{M}$). Thus, w.l.o.g. we may assume for $\mathcal{A}(x) = (a_{ij}(x))$ that

$$a_{in}(x', 0) = 0, \quad i = 1, \ldots, n-1.$$

Optimal regularity of the minimizer u: non-flat $\mathcal{M}$

- Minimizers $u \in C^{1,1/2}(B_1^{\pm} \cup B_1')$, under suitable assumptions on $\mathcal{A}$, ϕ.
- When $\mathcal{A} \in C^{1,\alpha}$ ($\mathcal{M} \in C^{2,\alpha}$), the truncated Almgren's formula still works [Guillen'09]

Optimal regularity of the minimizer u: non-flat $\mathcal{M}$

- Minimizers $u \in C^{1,1/2}(B_1^\pm \cup B_1')$, under suitable assumptions on $\mathcal{A}$, ϕ.
- When $\mathcal{A} \in C^{1,\alpha}$ ($\mathcal{M} \in C^{2,\alpha}$), the truncated Almgren's formula still works [Guillen'09]
- When $\mathcal{A} \in C^{0,1}$ ($\mathcal{M} \in C^{1,1}$), there are considerable difficulties but the truncation can be made to work [Garofalo-Smit Vega Garcia'14]:

$$\Phi(r) = \frac{\psi(r)}{r^{n-2}} e^{Kr^\sigma} \frac{d}{dr} \log\max\left\{\frac{1}{\psi(r)} \int_{\partial B_r} v^2\mu, r^{3+\delta}\right\},$$

where

$$\mu(x) = \frac{x\cdot\mathcal{A}(x)x}{|x|^2}, \quad \psi(r) = \begin{cases} e^{\int_0^r \frac{\int_{B_s} v^2 L_{\mathcal{A}}|x|}{\int_{\partial B_s} v^2\mu}} & v \not\equiv 0 \text{ on } B_r \\ r^{n-1}, & v \equiv 0 \text{ on } B_r \end{cases}$$

Optimal regularity of the minimizer u: non-flat $\mathcal{M}$

- Minimizers $u \in C^{1,1/2}(B_1^\pm \cup B_1')$, under suitable assumptions on $\mathcal{A}$, ϕ.
- When $\mathcal{A} \in C^{1,\alpha}$ ($\mathcal{M} \in C^{2,\alpha}$), the truncated Almgren's formula still works [Guillen'09]
- When $\mathcal{A} \in C^{0,1}$ ($\mathcal{M} \in C^{1,1}$), there are considerable difficulties but the truncation can be made to work [Garofalo-Smit Vega Garcia'14]:

$$\Phi(r) = \frac{\psi(r)}{r^{n-2}} e^{Kr^\sigma} \frac{d}{dr} \log \max \left\{ \frac{1}{\psi(r)} \int_{\partial B_r} v^2 \mu, r^{3+\delta} \right\},$$

where

$$\mu(x) = \frac{x \cdot \mathcal{A}(x) x}{|x|^2}, \quad \psi(r) = \begin{cases} e^{\int_0^r \frac{\int_{B_s} v^2 L_{\mathcal{A}} |x|}{\int_{\partial B_s} v^2 \mu}} & v \not\equiv 0 \text{ on } B_r \\ r^{n-1}, & v \equiv 0 \text{ on } B_r \end{cases}$$

- Very recently, [Koch-Ruland-Shi'15] have shown that $u \in C^{1,1/2}$ also when $\mathcal{A} \in W^{1,p}$, $p > 2n$

Optimal regularity of the minimizer u: non-flat $\mathcal{M}$

- Minimizers $u \in C^{1,1/2}(B_1^{\pm} \cup B_1')$, under suitable assumptions on $\mathcal{A}$, ϕ.
- When $\mathcal{A} \in C^{1,\alpha}$ ($\mathcal{M} \in C^{2,\alpha}$), the truncated Almgren's formula still works [Guillen'09]
- When $\mathcal{A} \in C^{0,1}$ ($\mathcal{M} \in C^{1,1}$), there are considerable difficulties but the truncation can be made to work [Garofalo-Smit Vega Garcia'14]:

$$\Phi(r) = \frac{\psi(r)}{r^{n-2}} e^{Kr^{\sigma}} \frac{d}{dr} \log \max \left\{ \frac{1}{\psi(r)} \int_{\partial B_r} v^2 \mu, r^{3+\delta} \right\},$$

where

$$\mu(x) = \frac{x \cdot \mathcal{A}(x) x}{|x|^2}, \quad \psi(r) = \begin{cases} e^{\int_0^r \frac{\int_{B_s} v^2 L_{\mathcal{A}} |x|}{\int_{\partial B_s} v^2 \mu}} & v \not\equiv 0 \text{ on } B_r \\ r^{n-1}, & v \equiv 0 \text{ on } B_r \end{cases}$$

- Very recently, [Koch-Ruland-Shi'15] have shown that $u \in C^{1,1/2}$ also when $\mathcal{A} \in W^{1,p}$, $p > 2n$
  - Uniform almost optimal regularity by using *Carleman estimates*, regularity of $\Gamma_{3/2}$, and then back to optimal regularity.

Structure of free boundary

- Truncated Almgren's formula Φ_k can be used to classify free boundary points

$$\Gamma = \bigcup_{\kappa} \Gamma_\kappa, \quad \Gamma_\kappa := \{x_0 \in \Gamma \mid \Phi_k^{x_0}(0+) = n - 1 + 2\kappa\}.$$

Structure of free boundary

- Truncated Almgren's formula Φ_k can be used to classify free boundary points

$$\Gamma = \bigcup_{\kappa} \Gamma_\kappa, \quad \Gamma_\kappa := \{x_0 \in \Gamma \mid \Phi_k^{x_0}(0+) = n - 1 + 2\kappa\}.$$

- Possible values of κ

$$\kappa = 3/2 \quad \text{or} \quad \kappa \in [2, k]$$

Structure of free boundary

- Truncated Almgren's formula Φ_k can be used to classify free boundary points

$$\Gamma = \bigcup_{\kappa} \Gamma_\kappa, \quad \Gamma_\kappa := \{x_0 \in \Gamma \mid \Phi_k^{x_0}(0+) = n - 1 + 2\kappa\}.$$

- Possible values of κ

$$\kappa = 3/2 \quad \text{or} \quad \kappa \in [2, k]$$

- $\Gamma_{3/2}$ is called the *regular set*. The gap of values between $3/2$ and 2 implies that $\Gamma_{3/2}$ is a relatively open subset of Γ.

Structure of free boundary

- Truncated Almgren's formula Φ_k can be used to classify free boundary points

$$\Gamma = \bigcup_{\kappa} \Gamma_\kappa, \quad \Gamma_\kappa \coloneqq \{x_0 \in \Gamma \mid \Phi_k^{x_0}(0+) = n - 1 + 2\kappa\}.$$

- Possible values of κ

$$\kappa = 3/2 \quad \text{or} \quad \kappa \in [2, k]$$

- $\Gamma_{3/2}$ is called the *regular set*. The gap of values between $3/2$ and 2 implies that $\Gamma_{3/2}$ is a relatively open subset of Γ.
- Equivalent characterization of $\Gamma_{3/2}$ is by *Almgren blowups*:

$$x_0 \in \Gamma_{3/2} \iff \tilde{u}_{x_0,r}(x) \coloneqq \frac{u(x_0 + rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{1/2}} \to c_n \operatorname{Re}(x \cdot e + i|x_n|)^{3/2}$$

over a sequence $r = r_j \to 0+$, for some unit vector $e \in \mathbb{R}^{n-1}$.

Regularity of $\Gamma_{3/2}$: flat $\mathcal{M}$ ($\mathcal{A} = I$)

- Let $x_0 = 0 \in \Gamma_{3/2}$ and $\tilde{u}_{x_0,r_j} \to c_n \operatorname{Re}(x_{n-1} + i|x_n|)^{3/2}$.

Regularity of $\Gamma_{3/2}$: flat $\mathcal{M}$ ($\mathcal{A} = I$)

- Let $x_0 = 0 \in \Gamma_{3/2}$ and $\tilde{u}_{x_0,r_j} \to c_n \operatorname{Re}(x_{n-1} + i|x_n|)^{3/2}$.

Theorem (Regularity of $\Gamma_{3/2}$)

If $\phi \in C^{2,1}$, them there exists $\delta = \delta_u > 0$ such that

$$\Gamma_{3/2} \cap B'_\delta = \{x_{n-1} = g(x'')\} \cap B'_\delta \quad \textit{for } g \in C^{1,\alpha}(B''_1)$$

Regularity of $\Gamma_{3/2}$: flat $\mathcal{M}$ ($\mathcal{A} = I$)

- Let $x_0 = 0 \in \Gamma_{3/2}$ and $\tilde{u}_{x_0,r_j} \to c_n \operatorname{Re}(x_{n-1} + i|x_n|)^{3/2}$.

Theorem (Regularity of $\Gamma_{3/2}$)

If $\phi \in C^{2,1}$, them there exists $\delta = \delta_u > 0$ such that

$$\Gamma_{3/2} \cap B'_\delta = \{x_{n-1} = g(x'')\} \cap B'_\delta \quad \textit{for } g \in C^{1,\alpha}(B''_1)$$

- **Step 1.** [Athanasopoulos-Caffarelli-Salsa'08]. When $\mathcal{M}$ is flat ($\mathcal{A} = I$) and $\phi \in C^{2,1}$, for unit $e \in \mathbb{R}^{n-1}$ close to e_{n-1} and $h = \partial_e \tilde{u}_{0,r}$ we have

$$\begin{aligned} h = 0 &\quad \text{on } \Lambda_r = \{\tilde{u}_r = \tilde{\phi}_r\} \subset \mathbb{R}^{n-1}, \\ |\Delta h| \le \epsilon_0 &\quad \text{in } B_1 \setminus \Lambda_r \\ h \ge -\epsilon_0 &\quad \text{in } B_1 \\ h \ge c_0 > 0 &\quad \text{on } B'_1 \times \{\pm c_n\} \end{aligned}$$

  which implies that $\partial_e \tilde{u}_r \ge 0 \Rightarrow \Gamma_{3/2}$ is a Lipschitz graph.

Regularity of $\Gamma_{3/2}$: flat $\mathcal{M}$ ($\mathcal{A} = I$)

- **Step 2.** Lipschitz $\Rightarrow C^{1,\alpha}$. Classical idea of [Athanasopoulos-Caffarelli'84]. Apply the *boundary Harnack principle* in a slit domain $B_\delta \setminus \Lambda$ to conclude

$$\frac{\partial_{e_j} u}{\partial_{e_{n-1}} u} \in C^\alpha(B_{\delta/2}), \quad j = 1, \ldots, n-1.$$

  implying that $\Gamma_{3/2} \cap B_{\delta/2}$ is $C^{1,\alpha}$.

Regularity of $\Gamma_{3/2}$: flat $\mathcal{M}$ ($\mathcal{A} = I$)

- **Step 2.** Lipschitz $\Rightarrow C^{1,\alpha}$. Classical idea of [Athanasopoulos-Caffarelli'84]. Apply the *boundary Harnack principle* in a slit domain $B_\delta \setminus \Lambda$ to conclude

$$\frac{\partial_{e_j} u}{\partial_{e_{n-1}} u} \in C^\alpha(B_{\delta/2}), \quad j = 1, \dots, n-1.$$

  implying that $\Gamma_{3/2} \cap B_{\delta/2}$ is $C^{1,\alpha}$.
- Recently, [De Silva-Savin'14] have shown that essentially

$$\frac{\partial_{e_j} u}{\partial_{e_{n-1}} u} \quad \text{"as regular as"} \quad \Gamma$$

  implying that $\Gamma_{3/2} \in C^\infty$ by a bootstrapping argument.

Regularity of $\Gamma_{3/2}$: flat $\mathcal{M}$ ($\mathcal{A} = I$)

- **Step 2.** Lipschitz $\Rightarrow C^{1,\alpha}$. Classical idea of [Athanasopoulos-Caffarelli'84]. Apply the *boundary Harnack principle* in a slit domain $B_\delta \setminus \Lambda$ to conclude

$$\frac{\partial_{e_j} u}{\partial_{e_{n-1}} u} \in C^\alpha(B_{\delta/2}), \quad j = 1, \dots, n-1.$$

  implying that $\Gamma_{3/2} \cap B_{\delta/2}$ is $C^{1,\alpha}$.
- Recently, [De Silva-Savin'14] have shown that essentially

$$\frac{\partial_{e_j} u}{\partial_{e_{n-1}} u} \quad \text{"as regular as"} \quad \Gamma$$

  implying that $\Gamma_{3/2} \in C^\infty$ by a bootstrapping argument.
- *Side result:* Real analyticity of $\Gamma_{3/2}$ can be shown with a hodograph-Legendre type transformation through subelliptic estimates for Baouendi-Grushin type operator [Koch-P.-Shi'14].

Regularity of $\Gamma_{3/2}$: non-flat $\mathcal{M}$ (variable $\mathcal{A}$)

- For variable $\mathcal{A}(x)$, directional derivatives $h = \partial_e u$ satisfy

$$\operatorname{div}(\mathcal{A}(x)\nabla h) = \partial_e f - \operatorname{div}(\partial_e \mathcal{A}(x)\nabla u) = \operatorname{div} F$$

Regularity of $\Gamma_{3/2}$: non-flat $\mathcal{M}$ (variable $\mathcal{A}$)

- For variable $\mathcal{A}(x)$, directional derivatives $h = \partial_e u$ satisfy

$$\operatorname{div}(\mathcal{A}(x)\nabla h) = \partial_e f - \operatorname{div}(\partial_e \mathcal{A}(x)\nabla u) = \operatorname{div} F$$

- When $\mathcal{A} \in C^{0,1}$ or worse, F is L^∞ at best, and the RHS is not as simple to deal with to conclude that $h = \partial_e u \geq 0$ near regular points.

Regularity of $\Gamma_{3/2}$: non-flat $\mathcal{M}$ (variable $\mathcal{A}$)

- For variable $\mathcal{A}(x)$, directional derivatives $h = \partial_e u$ satisfy

$$\operatorname{div}(\mathcal{A}(x)\nabla h) = \partial_e f - \operatorname{div}(\partial_e \mathcal{A}(x)\nabla u) = \operatorname{div} F$$

- When $\mathcal{A} \in C^{0,1}$ or worse, F is L^∞ at best, and the RHS is not as simple to deal with to conclude that $h = \partial_e u \geq 0$ near regular points.
- Very recently, this was actually shown to hold even for $\mathcal{A} \in W^{1,p}$, $p > 2n$, by [Koch-Ruland-Shi'15] with elaborate harmonic analysis techniques.

Regularity of $\Gamma_{3/2}$: non-flat $\mathcal{M}$ (variable $\mathcal{A}$)

- For variable $\mathcal{A}(x)$, directional derivatives $h = \partial_e u$ satisfy

$$\operatorname{div}(\mathcal{A}(x)\nabla h) = \partial_e f - \operatorname{div}(\partial_e \mathcal{A}(x)\nabla u) = \operatorname{div} F$$

- When $\mathcal{A} \in C^{0,1}$ or worse, F is L^∞ at best, and the RHS is not as simple to deal with to conclude that $h = \partial_e u \geq 0$ near regular points.
- Very recently, this was actually shown to hold even for $\mathcal{A} \in W^{1,p}$, $p > 2n$, by [Koch-Ruland-Shi'15] with elaborate harmonic analysis techniques.
- We will show however that there is a completely different technique, purely *energy based* that avoids directional differentiation completely and proves $C^{1,\alpha}$ regularity of $\Gamma_{3/2}$.

Epiperimetric inequality for **minimal surfaces**

Theorem (Epiperimetric inequality [Reifenberg'64a])

Let Y be a (polyhedral) orientable cone, with vertex at 0, of dimension m in $\mathbb{R}^n$, whose boundary lies on on the unit sphere. If Y lies sufficiently close to the diametral plane, then there exists a new surface Y^* with the same boundary such that

$$H^m Y^* \le (1-\eta) H^m Y + \eta H^m B^m,$$

where B^m is the m dimensional unit ball and $\eta = \eta(n, m) \in (0, 1)$.

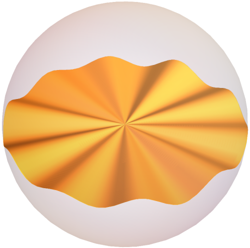
Epiperimetric inequality for **minimal surfaces**

Theorem (Epiperimetric inequality [Reifenberg'64a])

Let Y be a (polyhedral) orientable cone, with vertex at 0, of dimension m in $\mathbb{R}^n$, whose boundary lies on on the unit sphere. If Y lies sufficiently close to the diametral plane, then there exists a new surface Y^* with the same boundary such that

$$H^m Y^* \leq (1-\eta) H^m Y + \eta H^m B^m,$$

where B^m is the m dimensional unit ball and $\eta = \eta(n,m) \in (0,1)$.



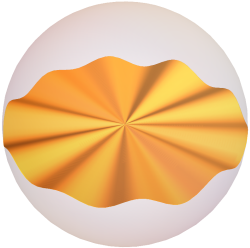
Epiperimetric inequality for **minimal surfaces**

Theorem (Epiperimetric inequality [Reifenberg'64a])

Let Y be a (polyhedral) orientable cone, with vertex at 0, of dimension m in $\mathbb{R}^n$, whose boundary lies on on the unit sphere. If Y lies sufficiently close to the diametral plane, then there exists a new surface Y^* with the same boundary such that

$$H^m Y^* \leq (1-\eta) H^m Y + \eta H^m B^m,$$

where B^m is the m dimensional unit ball and $\eta = \eta(n,m) \in (0,1)$.



- This then has been used to prove the real analyticity of flat minimal surfaces in [Reifenberg'64b]

Epiperimetric inequality for **classical obstacle problem**

- Let u be a solution of the normalized classical obstacle problem

$$\Delta u = \chi_{\{u>0\}}, \quad u \geq 0 \quad \text{in } B_1$$

Epiperimetric inequality for **classical obstacle problem**

- Let u be a solution of the normalized classical obstacle problem

$$\Delta u = \chi_{\{u>0\}}, \quad u \geq 0 \quad \text{in } B_1$$

- [Weiss'99] has proved that the following functional is monotone increasing:

$$W^{x_0}(u,r) = \frac{1}{r^{n+2}} \int_{B_r(x_0)} (|\nabla u|^2 + 2u) - \frac{2}{r^{n+3}} \int_{\partial B_r(x_0)} u^2,$$

for solutions of the classical obstacle problem.

Epiperimetric inequality for **classical obstacle problem**

- Let u be a solution of the normalized classical obstacle problem

$$\Delta u = \chi_{\{u>0\}}, \quad u \geq 0 \quad \text{in } B_1$$

- [Weiss'99] has proved that the following functional is monotone increasing:

$$W^{x_0}(u,r) = \frac{1}{r^{n+2}} \int_{B_r(x_0)} (|\nabla u|^2 + 2u) - \frac{2}{r^{n+3}} \int_{\partial B_r(x_0)} u^2,$$

for solutions of the classical obstacle problem.

- This functional can be used to classify free boundary points ($\Gamma = \partial\{u > 0\}$)

$$W^{x_0}(u, 0+) = \begin{cases} \alpha_n & \text{if } x_0 \text{ is regular} \\ 2\alpha_n & \text{if } x_0 \text{ is singular} \end{cases}$$

Epiperimetric inequality for **classical obstacle problem**

- There is also the following analogue of the epiperimetric inequality. Below, let $h(x) = \frac{1}{2}(x_n^+)^2$ be the halfspace solution.

Epiperimetric inequality for **classical obstacle problem**

- There is also the following analogue of the epiperimetric inequality. Below, let $h(x) = \frac{1}{2}(x_n^+)^2$ be the halfspace solution.

Theorem (Epiperimetric inequality [Weiss'99])

Let v be homogeneous of degree 2, $v(\lambda x) = \lambda^2 v(x)$, with $v \geq 0$. There exists $\delta > 0$ and $\eta \in (0,1)$ such that if $\|v - h\|_{W^{1,2}(B_1)} < \delta$, then there exists v^* with $v^* = v$ on ∂B_1 and $v^* \geq 0$ in B_1 such that

$$W(v^*, 1) \leq (1-\eta)W(v,1) + \eta W(h,1)$$

Epiperimetric inequality for **classical obstacle problem**

- There is also the following analogue of the epiperimetric inequality. Below, let $h(x) = \frac{1}{2}(x_n^+)^2$ be the halfspace solution.

Theorem (Epiperimetric inequality [Weiss'99])

Let v be homogeneous of degree 2, $v(\lambda x) = \lambda^2 v(x)$, with $v \geq 0$. There exists $\delta > 0$ and $\eta \in (0,1)$ such that if $\|v - h\|_{W^{1,2}(B_1)} < \delta$, then there exists v^* with $v^* = v$ on ∂B_1 and $v^* \geq 0$ in B_1 such that

$$W(v^*, 1) \leq (1 - \eta) W(v, 1) + \eta W(h, 1)$$

- Combining with the monotonicity of $W(v, r)$, [Weiss'99] then proves the $C^{1,\alpha}$ regularity of the free boundary.

Epiperimetric inequality for **classical obstacle problem**

- There is also the following analogue of the epiperimetric inequality. Below, let $h(x) = \frac{1}{2}(x_n^+)^2$ be the halfspace solution.

Theorem (Epiperimetric inequality [Weiss'99])

Let v be homogeneous of degree 2, $v(\lambda x) = \lambda^2 v(x)$, with $v \geq 0$. There exists $\delta > 0$ and $\eta \in (0,1)$ such that if $\|v - h\|_{W^{1,2}(B_1)} < \delta$, then there exists v^* with $v^* = v$ on ∂B_1 and $v^* \geq 0$ in B_1 such that

$$W(v^*, 1) \leq (1-\eta)W(v,1) + \eta W(h,1)$$

- Combining with the monotonicity of $W(v,r)$, [Weiss'99] then proves the $C^{1,\alpha}$ regularity of the free boundary.
- This approach turns out to be adaptable to the solutions of

$$\operatorname{div}(\mathcal{A}(x)\nabla u) = f(x)\chi_{\{x>0\}}, \quad u \geq 0$$

by [Focardi-Gelli-Spadaro'13] with $\mathcal{A} \in C^{0,1}$, $f \in C^{0,\alpha}$.

Epiperimetric inequality for **thin obstacle problem**

- [Garofalo-P.'09] have proved that the following Weiss-type formulas are monotone for solutions of the thin obstacle problem ($\mathcal{M} = B_1'$, $\phi = 0$).

$$W_\kappa^{x_0}(r) = \frac{1}{r^{n-2+2\kappa}} \int_{B_r(x_0)} |\nabla u|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r(x_0)} u^2, \quad x_0 \in \Gamma_\kappa.$$

Epiperimetric inequality for **thin obstacle problem**

- [Garofalo-P.'09] have proved that the following Weiss-type formulas are monotone for solutions of the thin obstacle problem ($\mathcal{M} = B_1'$, $\phi = 0$).

$$W_\kappa^{x_0}(r) = \frac{1}{r^{n-2+2\kappa}} \int_{B_r(x_0)} |\nabla u|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r(x_0)} u^2, \quad x_0 \in \Gamma_\kappa.$$

- The role of the "flat" solution is played by $h(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$. Note that $W_{3/2}(h, r) \equiv 0$.

Epiperimetric inequality for **thin obstacle problem**

- [Garofalo-P.'09] have proved that the following Weiss-type formulas are monotone for solutions of the thin obstacle problem ($\mathcal{M} = B_1'$, $\phi = 0$).

$$W_\kappa^{x_0}(r) = \frac{1}{r^{n-2+2\kappa}} \int_{B_r(x_0)} |\nabla u|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r(x_0)} u^2, \quad x_0 \in \Gamma_\kappa.$$

- The role of the "flat" solution is played by $h(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$. Note that $W_{3/2}(h, r) \equiv 0$.

Theorem (Epiper. ineq. [Garofalo-P.-Smit Vega Garcia'15])

Let v be homogeneous of degree $3/2$, $v(\lambda x) = \lambda^{3/2} v(x)$, and $v \geq 0$ on B_1'. There exists $\delta > 0$ and $\eta \in (0,1)$ such that if $\|v - h\|_{W^{1,2}(B_1)} < \delta$ then there is v^* with $v^* = v$ on ∂B_1, $v^* \geq 0$ on B_1', such that

$$W_{3/2}(v^*, 1) \leq (1-\eta) W_{3/2}(v, 1).$$

Epiperimetric inequality for **thin obstacle problem**

- [Garofalo-P.'09] have proved that the following Weiss-type formulas are monotone for solutions of the thin obstacle problem ($\mathcal{M} = B_1'$, $\phi = 0$).

$$W_\kappa^{x_0}(r) = \frac{1}{r^{n-2+2\kappa}} \int_{B_r(x_0)} |\nabla u|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r(x_0)} u^2, \quad x_0 \in \Gamma_\kappa.$$

- The role of the "flat" solution is played by $h(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$. Note that $W_{3/2}(h, r) \equiv 0$.

Theorem (Epiper. ineq. [Garofalo-P.-Smit Vega Garcia'15])

Let v be homogeneous of degree $3/2$, $v(\lambda x) = \lambda^{3/2} v(x)$, and $v \geq 0$ on B_1'. There exists $\delta > 0$ and $\eta \in (0,1)$ such that if $\|v - h\|_{W^{1,2}(B_1)} < \delta$ then there is v^* with $v^* = v$ on ∂B_1, $v^* \geq 0$ on B_1', such that

$$W_{3/2}(v^*, 1) \leq (1 - \eta) W_{3/2}(v, 1).$$

- Rediscovered by [Focardi-Spadaro'15].

How does the epiperimetric inequality work?

- Consider more general case of the thin obstacle problem with variable coefficients $\mathcal{A} \in C^{0,1}$, $\phi \in C^{1,1}$.

How does the epiperimetric inequality work?

- Consider more general case of the thin obstacle problem with variable coefficients $\mathcal{A} \in C^{0,1}$, $\phi \in C^{1,1}$.
- **Step 1**. The analogue of Weiss's formula for homogeneity 3/2:

$$W_{3/2}^{\mathcal{A}}(v, 1) = \frac{1}{r^{n+1}} \int_{B_r} (\nabla u \cdot \mathcal{A}(x) \nabla u + v f) - \frac{3/2}{r^{n+2}} \int_{\partial B_r} u^2 \mu,$$

$$\mu = \frac{x \cdot \mathcal{A}(x) x}{|x|^2}, \quad f = L_{\mathcal{A}} \phi.$$

How does the epiperimetric inequality work?

- Consider more general case of the thin obstacle problem with variable coefficients $\mathcal{A} \in C^{0,1}$, $\phi \in C^{1,1}$.
- **Step 1**. The analogue of Weiss's formula for homogeneity 3/2:

$$W^{\mathcal{A}}_{3/2}(v,1) = \frac{1}{r^{n+1}} \int_{B_r} (\nabla u \cdot \mathcal{A}(x) \nabla u + v f) - \frac{3/2}{r^{n+2}} \int_{\partial B_r} u^2 \mu,$$

$$\mu = \frac{x \cdot \mathcal{A}(x) x}{|x|^2}, \quad f = L_{\mathcal{A}} \phi.$$

- $r \mapsto W^{\mathcal{A}}_{3/2}(v,r) + Cr^{1/2}$ is monotone increasing for universal C.

How does the epiperimetric inequality work?

- Consider more general case of the thin obstacle problem with variable coefficients $\mathcal{A} \in C^{0,1}$, $\phi \in C^{1,1}$.
- **Step 1**. The analogue of Weiss's formula for homogeneity 3/2:

$$W^{\mathcal{A}}_{3/2}(v,1) = \frac{1}{r^{n+1}} \int_{B_r} (\nabla u \cdot \mathcal{A}(x) \nabla u + v f) - \frac{3/2}{r^{n+2}} \int_{\partial B_r} u^2 \mu,$$
$$\mu = \frac{x \cdot \mathcal{A}(x) x}{|x|^2}, \quad f = L_{\mathcal{A}} \phi.$$

- $r \mapsto W^{\mathcal{A}}_{3/2}(v,r) + Cr^{1/2}$ is monotone increasing for universal C.
- In particular this gives a bound from below

$$W^{\mathcal{A}}_{3/2}(v,r) \geq -Cr^{1/2}$$

How does the epiperimetric inequality work?

- Consider more general case of the thin obstacle problem with variable coefficients $\mathcal{A} \in C^{0,1}$, $\phi \in C^{1,1}$.
- **Step 1.** The analogue of Weiss's formula for homogeneity 3/2:

$$W^{\mathcal{A}}_{3/2}(v,1) = \frac{1}{r^{n+1}} \int_{B_r} (\nabla u \cdot \mathcal{A}(x)\nabla u + vf) - \frac{3/2}{r^{n+2}} \int_{\partial B_r} u^2\mu,$$
$$\mu = \frac{x \cdot \mathcal{A}(x)x}{|x|^2}, \quad f = L_{\mathcal{A}}\phi.$$

- $r \mapsto W^{\mathcal{A}}_{3/2}(v,r) + Cr^{1/2}$ is monotone increasing for universal C.
- In particular this gives a bound from below

$$W^{\mathcal{A}}_{3/2}(v,r) \geq -Cr^{1/2}$$

- We hope to get a bound from above from the epiperimetric inequality.

How does the epiperimetric inequality work?

- **Step 2.** Direct calculation shows

$$\frac{d}{dr} W^{\mathcal{A}}_{3/2}(v,r) \geq \frac{n+1}{r}\left[W^{I}_{3/2}(w_r,1) - W^{\mathcal{A}}_{3/2}(v,r)\right] + \frac{1}{r}\int_{\partial B_1} (\nu \cdot \nabla v_r - \tfrac{3}{2} v_r)^2 - C r^{-1/2},$$

  where

$$v_r(x) = \frac{v(rx)}{r^{3/2}}, \quad w_r(x) = |x|^{3/2} v_r(x/|x|)$$

How does the epiperimetric inequality work?

- **Step 2.** Direct calculation shows

$$\frac{d}{dr}W^{\mathcal{A}}_{3/2}(v,r) \geq \frac{n+1}{r}[W^{I}_{3/2}(w_r,1) - W^{\mathcal{A}}_{3/2}(v,r)] \\ + \frac{1}{r}\int_{\partial B_1}(\nu\cdot\nabla v_r - \tfrac{3}{2}v_r)^2 - Cr^{-1/2},$$

  where

$$v_r(x) = \frac{v(rx)}{r^{3/2}}, \quad w_r(x) = |x|^{3/2}v_r(x/|x|)$$

- Applying the epiperimetric inequality to w_r (there is a *catch*!) and using the minimality of v we arrive at

$$\frac{d}{dr}W^{\mathcal{A}}_{3/2}(v,r) \geq \frac{n+1}{r}\frac{\eta}{1-\eta}W^{\mathcal{A}}_{3/2}(v,r) - Cr^{-1/2}$$

  and integrating:

$$W^{\mathcal{A}}_{3/2}(v,r) \leq Cr^{\gamma}, \quad \text{with } \gamma = \tfrac{1}{2}\wedge(n+1)\tfrac{\eta}{1-\eta}$$

How does the epiperimetric inequality work?

- Note that we need the epiperimetric inequality only for the case $\mathcal{A} = I$.
- The *catch* in Step 2 above is that is that

$$v_r(x) = \frac{v(rx)}{r^{3/2}}$$

is not necessarily close to $h(x) = \operatorname{Re}(x_{n-1} + i|x_n|)^{3/2}$ but rather to a nonnegative multiple of its rotation: $a \operatorname{Re}(x' \cdot e + i|x_n|)^{3/2}$, were $a \geq 0$.

How does the epiperimetric inequality work?

- Note that we need the epiperimetric inequality only for the case $\mathcal{A} = I$.
- The *catch* in Step 2 above is that is that

$$v_r(x) = \frac{v(rx)}{r^{3/2}}$$

is not necessarily close to $h(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$ but rather to a nonnegative multiple of its rotation: $a\,\mathrm{Re}(x' \cdot e + i|x_n|)^{3/2}$, were $a \geq 0$.

- However, the Almgren scaling

$$\tilde{v}_r(x) = \frac{v(rx)}{\left(\frac{1}{\psi(r)} \int_{\partial B_r} v^2 \mu\right)^{1/2}}$$

is close to $c_n\,\mathrm{Re}(x' \cdot e + i|x_n|)^{3/2}$.

How does the epiperimetric inequality work?

- Note that we need the epiperimetric inequality only for the case $\mathcal{A} = I$.
- The *catch* in Step 2 above is that is that

$$v_r(x) = \frac{v(rx)}{r^{3/2}}$$

is not necessarily close to $h(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$ but rather to a nonnegative multiple of its rotation: $a\,\mathrm{Re}(x' \cdot e + i|x_n|)^{3/2}$, were $a \geq 0$.

- However, the Almgren scaling

$$\tilde{v}_r(x) = \frac{v(rx)}{\left(\frac{1}{\psi(r)} \int_{\partial B_r} v^2 \mu\right)^{1/2}}$$

is close to $c_n\,\mathrm{Re}(x' \cdot e + i|x_n|)^{3/2}$.

- We then notice that if the epiperimetric inequality holds for some function then it also holds for its nonnegative multiple.

How does the epiperimetric inequality work?

- **Step 3.** Control of the spinning of rescalings $v_r(x) = \dfrac{v(rx)}{r^{3/2}}$.

How does the epiperimetric inequality work?

- **Step 3.** Control of the spinning of rescalings $v_r(x) = \dfrac{v(rx)}{r^{3/2}}$.
- We claim that for $0 < s < t < r_0$ we have

$$\int_{\partial B_1} |v_t - v_s| \leq C t^{\gamma/2}$$

How does the epiperimetric inequality work?

- **Step 3.** Control of the spinning of rescalings $v_r(x) = \dfrac{v(rx)}{r^{3/2}}$.
- We claim that for $0 < s < t < r_0$ we have

$$\int_{\partial B_1} |v_t - v_s| \leq C t^{\gamma/2}$$

- We first have

$$\begin{aligned}
\int_{\partial B_1} |v_t - v_s| &\leq \int_{\partial B_1} \int_s^t \left| \frac{d}{dr} v_r(x) \right| \\
&\leq \int_{\partial B_1} r^{-1} |\nu \cdot \nabla v_r - \tfrac{3}{2} v_r| \\
&\leq \left(\int_s^t r^{-1} dr \right)^{1/2} \left(\int_s^t \frac{d}{dr} W^{\mathcal{A}}_{3/2}(v, r) + C r^{-1/2} \right)^{1/2} \\
&\leq C \left(\log \frac{t}{s} \right)^{1/2} t^{\gamma/2}
\end{aligned}$$

How does the epiperimetric inequality work?

- **Step 3.** Control of the spinning of rescalings $v_r(x) = \dfrac{v(rx)}{r^{3/2}}$.
- We claim that for $0 < s < t < r_0$ we have

$$\int_{\partial B_1} |v_t - v_s| \le C t^{\gamma/2}$$

- We first have

$$\begin{aligned}
\int_{\partial B_1} |v_t - v_s| &\le \int_{\partial B_1} \int_s^t \left| \frac{d}{dr} v_r(x) \right| \\
&\le \int_{\partial B_1} r^{-1} |\nu \cdot \nabla v_r - \tfrac{3}{2} v_r| \\
&\le \left(\int_s^t r^{-1} dr \right)^{1/2} \left(\int_s^t \frac{d}{dr} W^{\mathcal{A}}_{3/2}(v, r) + C r^{-1/2} \right)^{1/2} \\
&\le C \left(\log \frac{t}{s} \right)^{1/2} t^{\gamma/2}
\end{aligned}$$

- Then obtain the claim by a dyadic argument.

How does the epiperimetric inequality work?

- **Step 4.** As an immediate corollary, we obtain that for any *blowup* v_0 at $0 \in \Gamma_{3/2}$ (limit of rescalings v_{r_j}, $r_j \to 0+$) we have

$$\int_{\partial B_1} |v_r - v_0| \le C r^{\gamma/2}$$

How does the epiperimetric inequality work?

- **Step 4.** As an immediate corollary, we obtain that for any *blowup* v_0 at $0 \in \Gamma_{3/2}$ (limit of rescalings v_{r_j}, $r_j \to 0+$) we have

$$\int_{\partial B_1} |v_r - v_0| \le C r^{\gamma/2}$$

- This implies the uniqueness of blowup v_0 as well as the nondegeneracy $v_0 \not\equiv 0$.

How does the epiperimetric inequality work?

- **Step 4.** As an immediate corollary, we obtain that for any *blowup* v_0 at $0 \in \Gamma_{3/2}$ (limit of rescalings v_{r_j}, $r_j \to 0+$) we have

$$\int_{\partial B_1} |v_r - v_0| \le C r^{\gamma/2}$$

- This implies the uniqueness of blowup v_0 as well as the nondegeneracy $v_0 \not\equiv 0$.
- The blowups have the form

$$v_0(x) = a_0 \operatorname{Re}(x' \cdot e_{x_0} + i|x_n|)^{3/2},$$

$$a_0 > 0, \quad |e_0| = 1,\ e_0 \in \mathbb{R}^{n-1}$$

How does the epiperimetric inequality work?

- **Step 5.** By recentering at $x_0 \in \Gamma_{3/2}$ and considering

$$v_{x_0,r}(x) = \frac{v(x_0 + r\mathcal{A}^{1/2}(x_0)x) - rb_{x_0}x_n}{r^{3/2}},$$
$$b_{x_0} = e_n \cdot \mathcal{A}^{1/2}(x_0)\nabla v(x_0)$$

and choosing $r = |x_0 - y_0|^\sigma$ for close $x_0, y_0 \in \Gamma_{3/2}$, one can prove that

$$\int_{\partial B_1'} |v_{x_0,0} - v_{y_0,0}| \le C|x_0 - y_0|^\beta$$

for some $\beta > 0$.

How does the epiperimetric inequality work?

- **Step 5.** By recentering at $x_0 \in \Gamma_{3/2}$ and considering

$$v_{x_0,r}(x) = \frac{v(x_0 + r\mathcal{A}^{1/2}(x_0)x) - rb_{x_0}x_n}{r^{3/2}},$$

$$b_{x_0} = e_n \cdot \mathcal{A}^{1/2}(x_0)\nabla v(x_0)$$

  and choosing $r = |x_0 - y_0|^\sigma$ for close $x_0, y_0 \in \Gamma_{3/2}$, one can prove that

$$\int_{\partial B_1'} |v_{x_0,0} - v_{y_0,0}| \le C|x_0 - y_0|^\beta$$

  for some $\beta > 0$.
- If we now write for $x_0 \in \Gamma_{3/2}$

$$v_{x_0,0} = a_{x_0}\operatorname{Re}(x \cdot e_{x_0} + i|x_n|)^{3/2},$$

  we immediately obtain the β-Hölder continuity of the mappings

$$x_0 \mapsto a_{x_0}, \quad x_0 \mapsto e_{x_0}$$

  implying $C^{1,\beta}$ regularity of $\Gamma_{3/2}$.

Fractional obstacle problem

- For $s \in (0,1)$ and given the obstacle $\phi : \mathbb{R}^n \to \mathbb{R}$ consider the obstacle problem for the fractional Laplacian $(-\Delta)^s$:

$$(-\Delta)^s u \geq 0, \quad u \geq \phi, \quad (u-\phi)(-\Delta)^s u = 0 \quad \text{on } \mathbb{R}^n.$$

  Here

$$(-\Delta)^s u = c_{n,s}\, \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy.$$

Fractional obstacle problem

- For $s \in (0,1)$ and given the obstacle $\phi : \mathbb{R}^n \to \mathbb{R}$ consider the obstacle problem for the fractional Laplacian $(-\Delta)^s$:

$$(-\Delta)^s u \geq 0, \quad u \geq \phi, \quad (u-\phi)(-\Delta)^s u = 0 \quad \text{on } \mathbb{R}^n.$$

  Here

$$(-\Delta)^s u = c_{n,s}\, \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy.$$

- When $s = 1/2$, harmonically extending u to $\mathbb{R}^n \times \mathbb{R}_+$, we can recover $(-\Delta)^{1/2}$ as the Dirichlet-to-Neumann operator

$$(-\Delta)^{1/2} u = -\lim_{y\to 0+} \partial_y u(x,y).$$

Fractional obstacle problem

- For $s \in (0,1)$ and given the obstacle $\phi : \mathbb{R}^n \to \mathbb{R}$ consider the obstacle problem for the fractional Laplacian $(-\Delta)^s$:

$$(-\Delta)^s u \geq 0, \quad u \geq \phi, \quad (u-\phi)(-\Delta)^s u = 0 \quad \text{on } \mathbb{R}^n.$$

  Here

$$(-\Delta)^s u = c_{n,s}\,\text{p.v.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy.$$

- When $s = 1/2$, harmonically extending u to $\mathbb{R}^n \times \mathbb{R}_+$, we can recover $(-\Delta)^{1/2}$ as the Dirichlet-to-Neumann operator

$$(-\Delta)^{1/2} u = -\lim_{y \to 0+} \partial_y u(x,y).$$

- Then the obstacle problem for $(-\Delta)^{1/2}$ becomes a thin obstacle problem.

Fractional obstacle problem

- Similar *localization* works also for other fractional powers.

Fractional obstacle problem

- Similar *localization* works also for other fractional powers.
- Namely, for $a = 1 - 2s \in (-1, 1)$ consider the [Caffarelli-Silvestre'09] extension operator

$$L_a u = \mathrm{div}_{x,y}(|y|^a \nabla_{x,y} u) \quad \text{on } \mathbb{R}^{n+1}$$

and extend $u(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}_+$ by solving a Dirichlet problem

$$L_a u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \quad u(x, 0) = u(x)$$

Fractional obstacle problem

- Similar *localization* works also for other fractional powers.
- Namely, for $a = 1 - 2s \in (-1, 1)$ consider the [Caffarelli-Silvestre'09] extension operator

$$L_a u = \operatorname{div}_{x,y}(|y|^a \nabla_{x,y} u) \quad \text{on } \mathbb{R}^{n+1}$$

  and extend $u(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}_+$ by solving a Dirichlet problem

$$L_a u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \quad u(x, 0) = u(x)$$

- Then one can recover

$$(-\Delta)^s u = - \lim_{y \to 0+} y^a \partial_y u(x, y)$$

Fractional obstacle problem

- Similar *localization* works also for other fractional powers.
- Namely, for $a = 1 - 2s \in (-1, 1)$ consider the [Caffarelli-Silvestre'09] extension operator

$$L_a u = \mathrm{div}_{x,y}(|y|^a \nabla_{x,y} u) \quad \text{on } \mathbb{R}^{n+1}$$

and extend $u(x)$ from $\mathbb{R}^n$ to $\mathbb{R}^n \times \mathbb{R}_+$ by solving a Dirichlet problem

$$L_a u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+, \quad u(x, 0) = u(x)$$

- Then one can recover

$$(-\Delta)^s u = -\lim_{y \to 0+} y^a \partial_y u(x, y)$$

- This makes the fractional obstacle problem locally equivalent to the thin obstacle problem for L_a:

$$\int_{B_R} |\nabla u|^2 |y|^a \to \min, \quad u(x, 0) \geq \phi(x) \quad \text{on } B_R \cap \{y = 0\}$$

Fractional obstacle problem

- Many techniques work the same way as in the thin obstacle problem, when $\phi \in C^{1,1}$ [Caffarelli-Salsa-Silvestre'09]

Fractional obstacle problem

- Many techniques work the same way as in the thin obstacle problem, when $\phi \in C^{1,1}$ [Caffarelli-Salsa-Silvestre'09]
- When $\phi = 0$ the following Almgren's frequency function is monotone

$$N(r) = \frac{r \int_{B_r} |\nabla u|^2 |y|^a}{\int_{\partial B_r} u^2 |y|^a}$$

Fractional obstacle problem

- Many techniques work the same way as in the thin obstacle problem, when $\phi \in C^{1,1}$ [Caffarelli-Salsa-Silvestre'09]
- When $\phi = 0$ the following Almgren's frequency function is monotone
$$N(r) = \frac{r \int_{B_r} |\nabla u|^2 |y|^a}{\int_{\partial B_r} u^2 |y|^a}$$
- For $\phi \in C^{1,1}$ the truncated version is monotone:
$$\Phi(r) = re^{r^\sigma} \frac{d}{dr} \log \max \left\{ \int_{\partial B_r} v^2 |y|^a, r^{n+a+4} \right\}.$$

Fractional obstacle problem

- Many techniques work the same way as in the thin obstacle problem, when $\phi \in C^{1,1}$ [Caffarelli-Salsa-Silvestre'09]
- When $\phi = 0$ the following Almgren's frequency function is monotone

$$N(r) = \frac{r \int_{B_r} |\nabla u|^2 |y|^a}{\int_{\partial B_r} u^2 |y|^a}$$

- For $\phi \in C^{1,1}$ the truncated version is monotone:

$$\Phi(r) = r e^{r^\sigma} \frac{d}{dr} \log \max \left\{ \int_{\partial B_r} v^2 |y|^a, r^{n+a+4} \right\}.$$

- This allows to establish that $u \in C^{1,s}(\mathbb{R}^n)$.

Fractional obstacle problem

- Many techniques work the same way as in the thin obstacle problem, when $\phi \in C^{1,1}$ [Caffarelli-Salsa-Silvestre'09]
- When $\phi = 0$ the following Almgren's frequency function is monotone

$$N(r) = \frac{r \int_{B_r} |\nabla u|^2 |y|^a}{\int_{\partial B_r} u^2 |y|^a}$$

- For $\phi \in C^{1,1}$ the truncated version is monotone:

$$\Phi(r) = r e^{r^\sigma} \frac{d}{dr} \log \max \left\{ \int_{\partial B_r} v^2 |y|^a, r^{n+a+4} \right\}.$$

- This allows to establish that $u \in C^{1,s}(\mathbb{R}^n)$.
- For $\phi \in C^{2,1}$, the $C^{1,\alpha}$ regularity of the Γ_{1+s} (*regular set*) can be proved by taking the directional derivatives of u, as in the thin obstacle case.

Fractional obstacle problem with drift

- In applications to financial math, it is more appropriate to consider the fractional Laplacian with drift,

$$Lu = (-\Delta)^s u + b(x) \cdot \nabla u + c(x)u,$$

nonlocal version of *Black-Scholes operator* (here $x = \log S$, S underlying asset price)

Fractional obstacle problem with drift

- In applications to financial math, it is more appropriate to consider the fractional Laplacian with drift,

$$Lu = (-\Delta)^s u + b(x) \cdot \nabla u + c(x)u,$$

  nonlocal version of *Black-Scholes operator* (here $x = \log S$, S underlying asset price)

- For $\hat{\phi} : \mathbb{R}^n \to \mathbb{R}$, the solution to the obstacle problem

$$L\hat{u} \geq 0, \quad \hat{u} \geq \hat{\phi}, \quad (\hat{u} - \hat{\phi})L\hat{u} = 0 \quad \text{on } \mathbb{R}^n$$

  gives the valuation of perpetual American options.

Fractional obstacle problem with drift

- In applications to financial math, it is more appropriate to consider the fractional Laplacian with drift,

$$Lu = (-\Delta)^s u + b(x) \cdot \nabla u + c(x) u,$$

  nonlocal version of *Black-Scholes operator* (here $x = \log S$, S underlying asset price)
- For $\hat{\phi} : \mathbb{R}^n \to \mathbb{R}$, the solution to the obstacle problem

$$L\hat{u} \geq 0, \quad \hat{u} \geq \hat{\phi}, \quad (\hat{u} - \hat{\phi}) L\hat{u} = 0 \quad \text{on } \mathbb{R}^n$$

  gives the valuation of perpetual American options.
- When $s > 1/2$, long story short, the drift terms can be viewed as lower order terms and taken to the right hand side to prove that $u \in C^{1,\alpha}(\mathbb{R}^n)$ for all $\alpha < s$ first and then by a truncated Almgren's formula that $u \in C^{1,s}(\mathbb{R}^n)$, if $b, c \in C^s$ [P.-Pop'15]

Fractional obstacle problem with drift

- More precisely, if we consider

$$\phi = \hat{\phi} - (-\Delta)^{-s}(b \cdot \nabla u + cu)$$

$$v_{x_0}(x) = u(x,y) - \phi(x,y) - \tfrac{1}{2s}(-\Delta)^s \phi(x_0)|y|^a,$$

then v_{x_0} satisfies

$$\begin{aligned} L_a v_{x_0} &= 0 && \text{in } \mathbb{R}^n \times \mathbb{R}_\pm \\ v_{x_0} &\geq 0 && \text{on } \mathbb{R}^n \times \{0\} \\ L_a v_{x_0} &\leq f_{x_0} \mathcal{H}^n\big|_{y=0} && \text{on } \mathbb{R}^{n+1} \\ L_a v_{x_0} &= f_{x_0} \mathcal{H}^n\big|_{y=0} && \text{on } \mathbb{R}^{n+1} \setminus (\{y=0\} \cap \{v_{x_0} = 0\}). \end{aligned}$$

Fractional obstacle problem with drift

- More precisely, if we consider

$$\phi = \hat{\phi} - (-\Delta)^{-s}(b \cdot \nabla u + cu)$$

$$v_{x_0}(x) = u(x,y) - \phi(x,y) - \frac{1}{2s}(-\Delta)^s \phi(x_0)|y|^a,$$

then v_{x_0} satisfies

$$\begin{aligned} L_a v_{x_0} &= 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_\pm \\ v_{x_0} &\geq 0 \quad \text{on } \mathbb{R}^n \times \{0\} \\ L_a v_{x_0} &\leq f_{x_0} \mathcal{H}^n|_{y=0} \quad \text{on } \mathbb{R}^{n+1} \\ L_a v_{x_0} &= f_{x_0} \mathcal{H}^n|_{y=0} \quad \text{on } \mathbb{R}^{n+1} \setminus (\{y=0\} \cap \{v_{x_0} = 0\}). \end{aligned}$$

- Here $f_{x_0}(x) = 2((-\Delta)^s \phi(x) - (-\Delta)^s \phi(x_0))$ satisfies

$$|f_{x_0}(x)| \leq C|x - x_0|^s$$

Epiperimetric inequality for **fractional obstacle problem**

- Note that $\phi \in C^{3s}(\mathbb{R}^n)$ at best, even if $\hat{\phi}$ is smoother. The proof if the regularity of Γ_{1+s} by taking the directional derivatives becomes difficult (if possible at all).

Epiperimetric inequality for **fractional obstacle problem**

- Note that $\phi \in C^{3s}(\mathbb{R}^n)$ at best, even if $\hat{\phi}$ is smoother. The proof if the regularity of Γ_{1+s} by taking the directional derivatives becomes difficult (if possible at all).
- Luckily, there is an analogue of Weiss's monotonicity formula:

$$W^f_{1+s}(v_{x_0}, r) = \frac{1}{r^{n+2}} \int_{B_r} |\nabla v_{x_0}|^2 |y|^a + \int_{B'_r} v_{x_0} f_{x_0} - \frac{1+s}{r^{n+3}} \int_{\partial B_r} v^2_{x_0} |y|^a$$

Epiperimetric inequality for **fractional obstacle problem**

- Note that $\phi \in C^{3s}(\mathbb{R}^n)$ at best, even if $\hat{\phi}$ is smoother. The proof if the regularity of Γ_{1+s} by taking the directional derivatives becomes difficult (if possible at all).
- Luckily, there is an analogue of Weiss's monotonicity formula:

$$W^f_{1+s}(v_{x_0}, r) = \frac{1}{r^{n+2}} \int_{B_r} |\nabla v_{x_0}|^2 |y|^a + \int_{B'_r} v_{x_0} f_{x_0} - \frac{1+s}{r^{n+3}} \int_{\partial B_r} v^2_{x_0} |y|^a$$

- We have that $W^f_{1+s}(v_{x_0}, r) + Cr^{2s-1} \nearrow$

Epiperimetric inequality for **fractional obstacle problem**

- Note that $\phi \in C^{3s}(\mathbb{R}^n)$ at best, even if $\hat{\phi}$ is smoother. The proof if the regularity of Γ_{1+s} by taking the directional derivatives becomes difficult (if possible at all).
- Luckily, there is an analogue of Weiss's monotonicity formula:

$$W^f_{1+s}(v_{x_0}, r) = \frac{1}{r^{n+2}} \int_{B_r} |\nabla v_{x_0}|^2 |y|^a + \int_{B'_r} v_{x_0} f_{x_0} - \frac{1+s}{r^{n+3}} \int_{\partial B_r} v^2_{x_0} |y|^a$$

- We have that $W^f_{1+s}(v_{x_0}, r) + Cr^{2s-1} \nearrow$
- The blowups at $v_{x_0,0}$ at $x_0 \in \Gamma_{1+s}$ are then given by

$$v_{x_0,0}(x,y) = a_{x_0} h_{e_{x_0}}(x,y)$$
$$h_e(x,y) = \left(x \cdot e + \sqrt{(x \cdot e)^2 + y^2}\right)^s \left(x \cdot e - s\sqrt{(x \cdot e)^2 + y^2}\right)$$

Epiperimetric inequality for **fractional obstacle problem**

- When $f = 0$, Weiss-type formula has the form

$$W_{1+s}(v,r) = \frac{1}{r^{n+2}} \int_{B_r} |\nabla v|^2 |y|^a - \frac{1+s}{r^{n+3}} \int_{\partial B_r} v^2 |y|^a$$

Epiperimetric inequality for **fractional obstacle problem**

- When $f = 0$, Weiss-type formula has the form

$$W_{1+s}(v,r) = \frac{1}{r^{n+2}} \int_{B_r} |\nabla v|^2 |y|^a - \frac{1+s}{r^{n+3}} \int_{\partial B_r} v^2 |y|^a$$

Theorem (Epiper. ineq. [Garofalo-P.-Pop-Smit Vega Garcia'15])

Let v be homogeneous of degree $1+s$, $v(\lambda x) = \lambda^{1+s} v(x)$, and $v \geq 0$ on B_1'. There exists $\delta > 0$ and $\eta \in (0,1)$ such that if $\|v - h_{e_n}\|_{W^{1,2}(B_1,|y|^a)} < \delta$ then there exists v^* with $v^* = v$ on ∂B_1, $v^* \geq 0$ on B_1', such that

$$W_{1+s}(v^*,1) \leq (1-\eta) W_{1+s}(v,1).$$

Epiperimetric inequality for **fractional obstacle problem**

- When $f = 0$, Weiss-type formula has the form

$$W_{1+s}(v,r) = \frac{1}{r^{n+2}} \int_{B_r} |\nabla v|^2 |y|^a - \frac{1+s}{r^{n+3}} \int_{\partial B_r} v^2 |y|^a$$

Theorem (Epiper. ineq. [Garofalo-P.-Pop-Smit Vega Garcia'15])

Let v be homogeneous of degree $1+s$, $v(\lambda x) = \lambda^{1+s} v(x)$, and $v \geq 0$ on B_1'. There exists $\delta > 0$ and $\eta \in (0,1)$ such that if $\|v - h_{e_n}\|_{W^{1,2}(B_1,|y|^a)} < \delta$ then there exists v^* with $v^* = v$ on ∂B_1, $v^* \geq 0$ on B_1', such that

$$W_{1+s}(v^*, 1) \leq (1-\eta) W_{1+s}(v, 1).$$

- Arguing as in the case of thin obstacle problem, one can show that Γ_{1+s} is $C^{1,\alpha}$ in the fractional obstacle problem with drift.