# Parabolic Signorini Problem

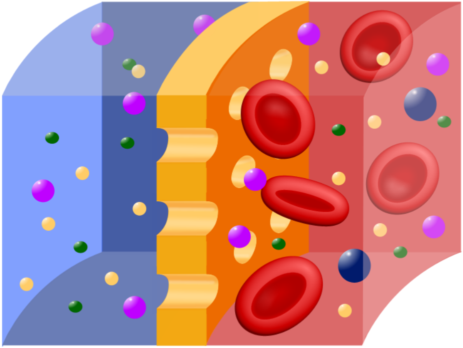
Arshak Petrosyan

PURDUE
UNIVERSITY

**Free Boundary Problems in Biology**
**Math Biosciences Institute, OSU**

November 14–18, 2011

# Semipermeable Membranes and Osmosis

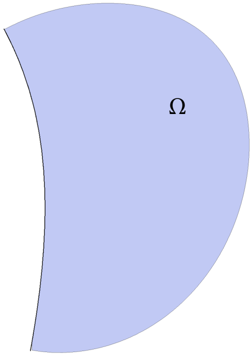


Picture Source: Wikipedia

- **Semipermeable membrane** is a membrane that is permeable only for a certain type of molecules (*solvents*) and blocks other molecules (*solutes*).
- Because of the chemical imbalance, the solvent flows through the membrane from the region of smaller concentration of solute to the region of higher concentation (*osmotic pressure*).
- The flow occurs in one direction. The flow continues until a sufficient pressure builds up on the other side of the membrane (to compensate for osmotic pressure), which then shuts the flow. This process is known as **osmosis**.

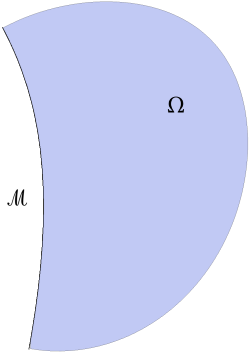
# Mathematical Formulation: Unilateral Problem

- Given open $\Omega \subset \mathbb{R}^n$



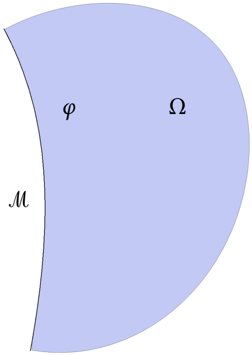
# Mathematical Formulation: Unilateral Problem

- Given open $\Omega \subset \mathbb{R}^n$
- $\mathcal{M} \subset \partial\Omega$ *semipermeable* part of the boundary



# Mathematical Formulation: Unilateral Problem

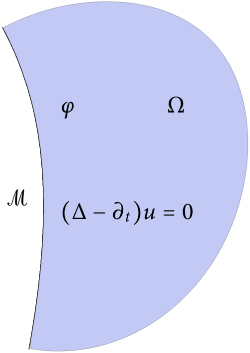
- Given open $\Omega \subset \mathbb{R}^n$
- $\mathcal{M} \subset \partial\Omega$ *semipermeable* part of the boundary
- $\varphi : \mathcal{M}_T := \mathcal{M} \times (0, T] \to \mathbb{R}$ *osmotic pressure*



# Mathematical Formulation: Unilateral Problem

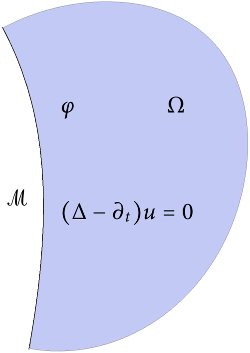
- Given open $\Omega \subset \mathbb{R}^n$
- $\mathcal{M} \subset \partial\Omega$ *semipermeable* part of the boundary
- $\varphi : \mathcal{M}_T := \mathcal{M} \times (0, T] \to \mathbb{R}$ *osmotic pressure*
- $u : \Omega_T := \Omega \times (0, T] \to \mathbb{R}$ the *pressure* of the chemical solution, that satisfies a diffusion equation (*slightly compressible fluid*)

$$\Delta u - \partial_t u = 0 \quad \text{in } \Omega_T$$



# Mathematical Formulation: Unilateral Problem

- Given open $\Omega \subset \mathbb{R}^n$
- $\mathcal{M} \subset \partial\Omega$ *semipermeable* part of the boundary
- $\varphi : \mathcal{M}_T := \mathcal{M} \times (0, T] \to \mathbb{R}$ *osmotic pressure*
- $u : \Omega_T := \Omega \times (0, T] \to \mathbb{R}$ the *pressure* of the chemical solution, that satisfies a diffusion equation (*slightly compressible fluid*)



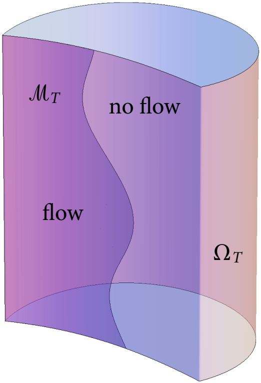

$$\Delta u - \partial_t u = 0 \quad \text{in } \Omega_T$$

- On $\mathcal{M}_T$ we have the following boundary conditions (*finite permeability*)

$$\begin{aligned} u > \varphi \quad &\Rightarrow \quad \partial_\nu u = 0 && \textbf{(no flow)} \\ u \le \varphi \quad &\Rightarrow \quad \partial_\nu u = \lambda(u - \varphi) && \textbf{(flow)} \end{aligned}$$

# Mathematical Formulation: Unilateral Problem

- Given open $\Omega \subset \mathbb{R}^n$
- $\mathcal{M} \subset \partial\Omega$ *semipermeable* part of the boundary
- $\varphi : \mathcal{M}_T := \mathcal{M} \times (0, T] \to \mathbb{R}$ *osmotic pressure*
- $u : \Omega_T := \Omega \times (0, T] \to \mathbb{R}$ the *pressure* of the chemical solution, that satisfies a diffusion equation (*slightly compressible fluid*)




$$\Delta u - \partial_t u = 0 \quad \text{in } \Omega_T$$

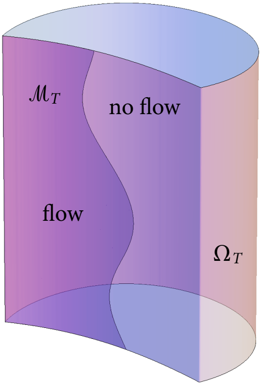
- On $\mathcal{M}_T$ we have the following boundary conditions (*finite permeability*)

$$\begin{aligned} u > \varphi \quad &\Rightarrow \quad \partial_\nu u = 0 && \textbf{(no flow)} \\ u \le \varphi \quad &\Rightarrow \quad \partial_\nu u = \lambda(u - \varphi) && \textbf{(flow)} \end{aligned}$$

# Parabolic Signorini Problem

- Letting $\lambda \to \infty$ we obtain the following conditions on $\mathcal{M}_T$ (*infinite permeability*)

$$
\begin{aligned}
u &\geq \varphi \\
\partial_\nu u &\geq 0 \\
(u-\varphi)\partial_\nu u &= 0
\end{aligned}
$$

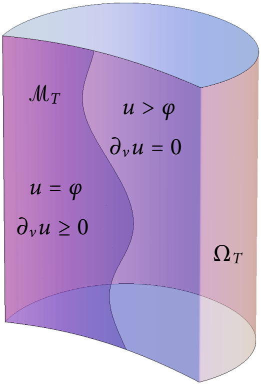


# Parabolic Signorini Problem

- Letting $\lambda \to \infty$ we obtain the following conditions on $\mathcal{M}_T$ (*infinite permeability*)

$$
\begin{aligned}
u &\geq \varphi \\
\partial_\nu u &\geq 0 \\
(u-\varphi)\partial_\nu u &= 0
\end{aligned}
$$

- These are known as the **Signorini boundary conditions**

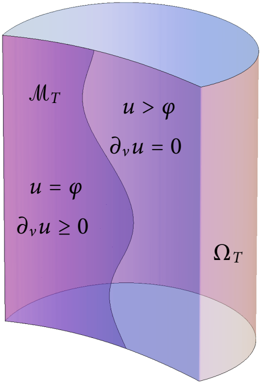


# Parabolic Signorini Problem

- Letting $\lambda \to \infty$ we obtain the following conditions on $\mathcal{M}_T$ (*infinite permeability*)

$$\begin{aligned} u &\geq \varphi \\ \partial_\nu u &\geq 0 \\ (u - \varphi)\partial_\nu u &= 0 \end{aligned}$$

- These are known as the **Signorini boundary conditions**
- Since $u$ should stay above $\varphi$ on $\mathcal{M}_T$, $\varphi$ is known as the **thin obstacle**. The problem is known as *Parabolic Signorini Problem* or *Parabolic Thin Obstacle Problem*.



# Parabolic Signorini Problem

- The function $u(x,t)$ the solves the following variational inequality:

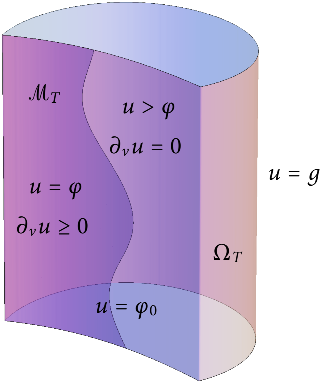
$$\int_\Omega \nabla u \cdot \nabla(u-v) + \partial_t u(u-v) \geq 0$$

$$u \in \mathfrak{K}, \quad \partial_t u \in L^2(\Omega)$$

$$\text{for all } v \in \mathfrak{K}$$

where

$$\mathfrak{K} = \{v \in W^{1,2}(\Omega) : v\big|_{\mathcal{M}} \geq \varphi,\ v\big|_{\partial\Omega\setminus\mathcal{M}} = g\}$$



# Parabolic Signorini Problem

- The function $u(x,t)$ the solves the following variational inequality:

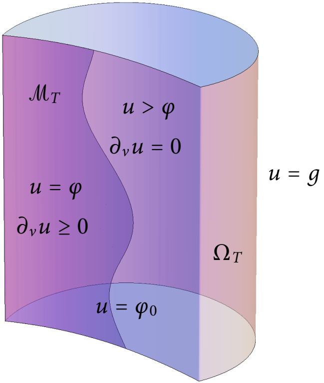
$$\int_\Omega \nabla u \cdot \nabla(u-v) + \partial_t u(u-v) \geq 0$$

$$u \in \mathfrak{K}, \quad \partial_t u \in L^2(\Omega)$$

$$\text{for all } v \in \mathfrak{K}$$

where

$$\mathfrak{K} = \{v \in W^{1,2}(\Omega) : v\big|_{\mathcal{M}} \geq \varphi,\ v\big|_{\partial\Omega\setminus\mathcal{M}} = g\}$$



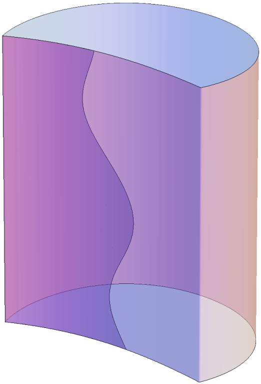

- Then for any (reasonable) initial condition

$$u = \varphi_0 \quad \text{on } \Omega_0 = \Omega \times \{0\}$$

the solution exist and unique. See [Duvaut-Lions 1986].

# Free Boundary Problem

- The parabolic Signorini problem is a free boundary problem.

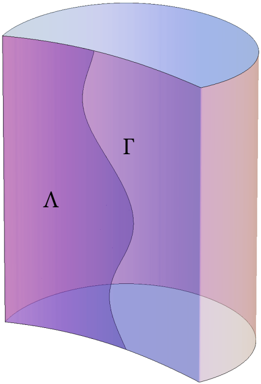


# Free Boundary Problem

- The parabolic Signorini problem is a free boundary problem.
- Let $\Lambda := \{(x,t) \in \mathcal{M}_T : u = \varphi\}$ be the so-called *coincidence set*. Then

$$\Gamma := \partial_{\mathcal{M}} \Lambda$$

is the *free boundary*.

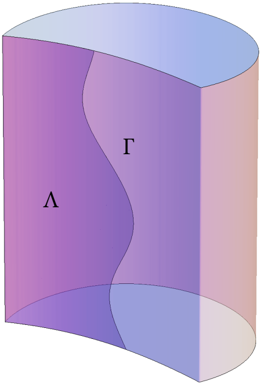


# Free Boundary Problem

- The parabolic Signorini problem is a free boundary problem.
- Let $\Lambda := \{(x,t) \in \mathcal{M}_T : u = \varphi\}$ be the so-called *coincidence set*. Then

$$\Gamma := \partial_{\mathcal{M}} \Lambda$$

  is the *free boundary*.
- One then interested in the structure, geometric properties and the regularity of the free boundary.

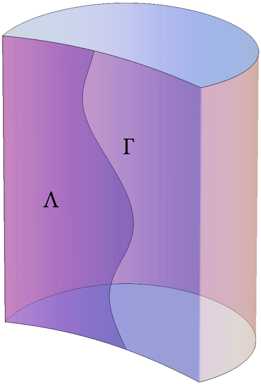


# Free Boundary Problem

- The parabolic Signorini problem is a free boundary problem.
- Let $\Lambda := \{(x,t) \in \mathcal{M}_T : u = \varphi\}$ be the so-called *coincidence set*. Then

$$\Gamma := \partial_{\mathcal{M}} \Lambda$$

  is the *free boundary*.
- One then interested in the structure, geometric properties and the regularity of the free boundary.




- In order to do so one has to know the *optimal regularity* of the solution $u$ in $\Omega_T$ up to $\mathcal{M}_T$.

# Parabolic Signorini Problem: Known Results

### Theorem (“$C^{1,\alpha}$-regularity” [Ural'tseva 1985])

*Let $u$ be a solution of the Parabolic Signorini Problem with $\varphi \in C_x^{1,1} \cap C_t^{0,1}(\mathcal{M}_T)$, $\varphi_0 \in \mathrm{Lip}(\Omega_0)$, and $g \in L^2(\mathcal{S}_T)$. Then $\nabla u \in C_{x,t}^{\alpha,\alpha/2}(K)$ for any $K \Subset \Omega_T \cup \mathcal{M}_T$ and*

$$\|\nabla u\|_{C_{x,t}^{\alpha,\alpha/2}(K)} \leq C_K\big(\|\varphi\|_{C_x^{1,1}\cap C_t^{0,1}(\mathcal{M}_T)} + \|\varphi_0\|_{\mathrm{Lip}(\Omega_0)} + \|g\|_{L^2(\mathcal{S}_T)}\big)$$

# Parabolic Signorini Problem: Known Results

**Theorem ("$C^{1,\alpha}$-regularity" [Ural'tseva 1985])**

*Let $u$ be a solution of the Parabolic Signorini Problem with $\varphi \in C_x^{1,1} \cap C_t^{0,1}(\mathcal{M}_T)$, $\varphi_0 \in \mathrm{Lip}(\Omega_0)$, and $g \in L^2(\mathcal{S}_T)$. Then $\nabla u \in C_{x,t}^{\alpha,\alpha/2}(K)$ for any $K \Subset \Omega_T \cup \mathcal{M}_T$ and*

$$\|\nabla u\|_{C_{x,t}^{\alpha,\alpha/2}(K)} \leq C_K\big(\|\varphi\|_{C_x^{1,1}\cap C_t^{0,1}(\mathcal{M}_T)} + \|\varphi_0\|_{\mathrm{Lip}(\Omega_0)} + \|g\|_{L^2(\mathcal{S}_T)}\big)$$

- In the elliptic case a similar result has been proved by [Caffarelli 1979]

# Parabolic Signorini Problem: Known Results

**Theorem ("$C^{1,\alpha}$-regularity" [Ural'tseva 1985])**

*Let $u$ be a solution of the Parabolic Signorini Problem with $\varphi \in C_x^{1,1} \cap C_t^{0,1}(\mathcal{M}_T)$, $\varphi_0 \in \mathrm{Lip}(\Omega_0)$, and $g \in L^2(\mathcal{S}_T)$. Then $\nabla u \in C_{x,t}^{\alpha,\alpha/2}(K)$ for any $K \Subset \Omega_T \cup \mathcal{M}_T$ and*

$$\|\nabla u\|_{C_{x,t}^{\alpha,\alpha/2}(K)} \leq C_K \left( \|\varphi\|_{C_x^{1,1} \cap C_t^{0,1}(\mathcal{M}_T)} + \|\varphi_0\|_{\mathrm{Lip}(\Omega_0)} + \|g\|_{L^2(\mathcal{S}_T)} \right)$$

- In the elliptic case a similar result has been proved by [Caffarelli 1979]
- Proof in [Ural'tseva 1985] in the elliptic case worked also for nonhomogeneous equation $\Delta u = f$, $f \in L^\infty(\Omega)$, with Signorini boundary conditions. That fact then implies the regularity in the parabolic case.

# Parabolic Signorini Problem: Known Results

**Theorem ("$C^{1,\alpha}$-regularity" [Ural'tseva 1985])**

*Let $u$ be a solution of the Parabolic Signorini Problem with $\varphi \in C_x^{1,1} \cap C_t^{0,1}(\mathcal{M}_T)$, $\varphi_0 \in \mathrm{Lip}(\Omega_0)$, and $g \in L^2(\mathcal{S}_T)$. Then $\nabla u \in C_{x,t}^{\alpha,\alpha/2}(K)$ for any $K \Subset \Omega_T \cup \mathcal{M}_T$ and*

$$\|\nabla u\|_{C_{x,t}^{\alpha,\alpha/2}(K)} \leq C_K\left(\|\varphi\|_{C_x^{1,1}\cap C_t^{0,1}(\mathcal{M}_T)} + \|\varphi_0\|_{\mathrm{Lip}(\Omega_0)} + \|g\|_{L^2(\mathcal{S}_T)}\right)$$

- In the elliptic case a similar result has been proved by [Caffarelli 1979]
- Proof in [Ural'tseva 1985] in the elliptic case worked also for nonhomogeneous equation $\Delta u = f$, $f \in L^\infty(\Omega)$, with Signorini boundary conditions. That fact then implies the regularity in the parabolic case.
- Except some specific cases, no general results have been known for the free boundary.

# Parabolic Signorini Problem: Optimal Regularity

In the case when $\mathcal{M}$ is flat, we have the following theorem.

# Parabolic Signorini Problem: Optimal Regularity

In the case when $\mathcal{M}$ is flat, we have the following theorem.

**Theorem ("$C^{1,1/2}$-regularity" [Danielli-Garofalo-$\mathcal{P}$.-To 2011])**

*Let $u$ be a solution of the Parabolic Signorini Problem with flat $\mathcal{M}$ and $\varphi \in C^{1,1}_x \cap C^{0,1}_t(\mathcal{M}_T)$, $\varphi_0 \in \mathrm{Lip}(\Omega_0)$, and $g \in L^2(\mathcal{S}_T)$. Then $\nabla u \in C^{1/2,1/4}_{x,t}(K)$ for any $K \Subset \Omega_T \cup \mathcal{M}_T$ and*

$$\|\nabla u\|_{C^{1/2,1/4}_{x,t}(K)} \leq C_K\big(\|\varphi\|_{C^{1,1}_{x,t}\cap C^{0,1}_t(M_T)} + \|\varphi_0\|_{\mathrm{Lip}(\Omega_0)} + \|g\|_{L^2(\mathcal{S}_T)}\big)$$

# Parabolic Signorini Problem: Optimal Regularity

In the case when $\mathcal{M}$ is flat, we have the following theorem.

**Theorem ("$C^{1,1/2}$-regularity" [Danielli-Garofalo-$\mathscr{P}$.-To 2011])**

*Let $u$ be a solution of the Parabolic Signorini Problem with flat $\mathcal{M}$ and $\varphi \in C_x^{1,1} \cap C_t^{0,1}(\mathcal{M}_T)$, $\varphi_0 \in \mathrm{Lip}(\Omega_0)$, and $g \in L^2(\mathcal{S}_T)$. Then $\nabla u \in C_{x,t}^{1/2,1/4}(K)$ for any $K \Subset \Omega_T \cup \mathcal{M}_T$ and*

$$\|\nabla u\|_{C_{x,t}^{1/2,1/4}(K)} \leq C_K \left( \|\varphi\|_{C_{x,t}^{1,1} \cap C_t^{0,1}(M_T)} + \|\varphi_0\|_{\mathrm{Lip}(\Omega_0)} + \|g\|_{L^2(\mathcal{S}_T)} \right)$$

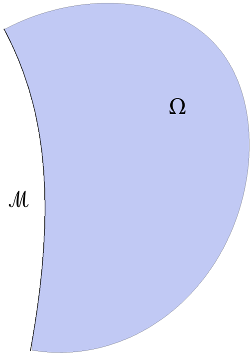
- This theorem is precise in the sense that it gives the best regularity possible, even in time-independendet case:

$$u(x,t) = \mathrm{Re}(x_1 + i x_n)^{3/2}$$

solves the Signorini problem in $\mathbb{R}_+^n \times \mathbb{R}$ with $\mathcal{M} = \mathbb{R}^{n-1}$.

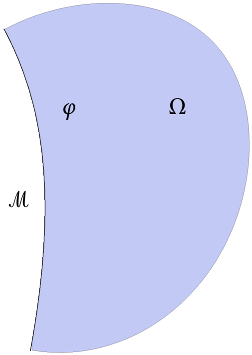
# Elliptic Case: Thin Obstacle Problem

- Given $\Omega \subset \mathbb{R}^n$, $\mathcal{M} \subset \partial\Omega$



# Elliptic Case: Thin Obstacle Problem

- Given $\Omega \subset \mathbb{R}^n$, $\mathcal{M} \subset \partial\Omega$
- $\varphi : \mathcal{M} \to \mathbb{R}$ **(thin obstacle)**
  $g : \partial\Omega \setminus \mathcal{M} \to \mathbb{R}$, $g > \varphi$ on $\mathcal{M} \cap \partial\Omega$.



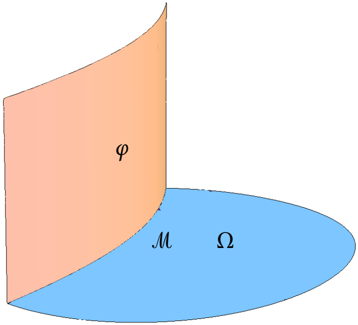
# Elliptic Case: Thin Obstacle Problem

- Given $\Omega \subset \mathbb{R}^n$, $\mathcal{M} \subset \partial\Omega$
- $\varphi : \mathcal{M} \to \mathbb{R}$ **(thin obstacle)**
  $g : \partial\Omega \setminus \mathcal{M} \to \mathbb{R}$, $g > \varphi$ on $\mathcal{M} \cap \partial\Omega$.
- Minimize the Dirichlet integral

$$D_\Omega(u) = \int_\Omega |\nabla u|^2 dx$$

on the closed convex set

$$\mathfrak{K} = \{u \in W^{1,2}(\Omega) \mid u\big|_{\mathcal{M}} \geq \varphi,\ u\big|_{\partial\Omega\setminus\mathcal{M}} = g\}$$



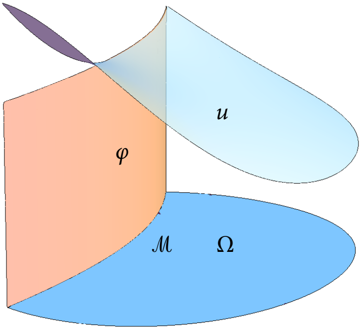
# Elliptic Case: Thin Obstacle Problem

- Given $\Omega \subset \mathbb{R}^n$, $\mathcal{M} \subset \partial\Omega$
- $\varphi : \mathcal{M} \to \mathbb{R}$ **(thin obstacle)**
  $g : \partial\Omega \setminus \mathcal{M} \to \mathbb{R}$, $g > \varphi$ on $\mathcal{M} \cap \partial\Omega$.
- Minimize the Dirichlet integral

$$D_\Omega(u) = \int_\Omega |\nabla u|^2 dx$$

on the closed convex set

$$\mathfrak{K} = \{u \in W^{1,2}(\Omega) \mid u\big|_{\mathcal{M}} \geq \varphi,\ u\big|_{\partial\Omega\setminus\mathcal{M}} = g\}$$




- The minimizer $u$ satisfies

$$\Delta u = 0 \quad \text{in } \Omega$$
$$u \geq \varphi, \quad \partial_\nu u \geq 0, \quad (u - \varphi)\partial_\nu u = 0 \quad \text{on } \mathcal{M}$$

# Elliptic Case: Optimal Regularity

- The progress in parabolic case was motivated by the breakthrough result of [Athanasopoulos-Caffarelli 2000] establishing the $C^{1,1/2}$ regularity in the elliptic thin obstacle problem.

**Theorem**

*Let $u$ be a solution of the thin obstacle problem for flat $\mathcal{M}$, with $\varphi \in C^{1,1}(\mathcal{M})$ and $g \in L^2(\partial\Omega \setminus \mathcal{M})$. Then $u \in C^{1,1/2}(K)$ for any $K \Subset \Omega \cup \mathcal{M}$ and*

$$\|u\|_{C^{1,1/2}(K)} \leq C_K \left( \|\varphi\|_{C^{1,1}(\mathcal{M})} + \|g\|_{L^2} \right).$$

# Elliptic Case: Optimal Regularity

- The progress in parabolic case was motivated by the breakthrough result of [Athanasopoulos-Caffarelli 2000] establishing the $C^{1,1/2}$ regularity in the elliptic thin obstacle problem.

**Theorem**

*Let $u$ be a solution of the thin obstacle problem for flat $\mathcal{M}$, with $\varphi \in C^{1,1}(\mathcal{M})$ and $g \in L^2(\partial\Omega \setminus \mathcal{M})$. Then $u \in C^{1,1/2}(K)$ for any $K \Subset \Omega \cup \mathcal{M}$ and*

$$\|u\|_{C^{1,1/2}(K)} \le C_K \left( \|\varphi\|_{C^{1,1}(\mathcal{M})} + \|g\|_{L^2} \right).$$

- $\varphi = 0$: [Athanasopoulos-Caffarelli 2000]

# Elliptic Case: Optimal Regularity

- The progress in parabolic case was motivated by the breakthrough result of [Athanasopoulos-Caffarelli 2000] establishing the $C^{1,1/2}$ regularity in the elliptic thin obstacle problem.

**Theorem**

*Let $u$ be a solution of the thin obstacle problem for flat $\mathcal{M}$, with $\varphi \in C^{1,1}(\mathcal{M})$ and $g \in L^2(\partial\Omega \setminus \mathcal{M})$. Then $u \in C^{1,1/2}(K)$ for any $K \Subset \Omega \cup \mathcal{M}$ and*

$$\|u\|_{C^{1,1/2}(K)} \leq C_K \left( \|\varphi\|_{C^{1,1}(\mathcal{M})} + \|g\|_{L^2} \right).$$

- $\varphi = 0$: [Athanasopoulos-Caffarelli 2000]
- $\varphi \in C^{2,1}$: [Athanasopoulos-Caffarelli-Salsa 2007]

# Elliptic Case: Optimal Regularity

- The progress in parabolic case was motivated by the breakthrough result of [Athanasopoulos-Caffarelli 2000] establishing the $C^{1,1/2}$ regularity in the elliptic thin obstacle problem.

**Theorem**

*Let $u$ be a solution of the thin obstacle problem for flat $\mathcal{M}$, with $\varphi \in C^{1,1}(\mathcal{M})$ and $g \in L^2(\partial\Omega \setminus \mathcal{M})$. Then $u \in C^{1,1/2}(K)$ for any $K \Subset \Omega \cup \mathcal{M}$ and*

$$\|u\|_{C^{1,1/2}(K)} \leq C_K \left( \|\varphi\|_{C^{1,1}(\mathcal{M})} + \|g\|_{L^2} \right).$$

- $\varphi = 0$: [Athanasopoulos-Caffarelli 2000]
- $\varphi \in C^{2,1}$: [Athanasopoulos-Caffarelli-Salsa 2007]
- $\varphi \in C^{1,1}$: [$\mathcal{P}$-To 2010]

# Zero Obstacle $\varphi$: Normalization

- Assume $\mathcal{M}$ is flat: $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, $\varphi = 0$

# Zero Obstacle $\varphi$: Normalization

- Assume $\mathcal{M}$ is flat: $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, $\varphi = 0$
- If $u$ solves Signorini problem, after translation, rotation, and scaling, we may normalize $u$ as follows:

# Zero Obstacle $\varphi$: Normalization

- Assume $\mathcal{M}$ is flat: $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, $\varphi = 0$
- If $u$ solves Signorini problem, after translation, rotation, and scaling, we may normalize $u$ as follows:

## Definition (Class $\mathfrak{S}$)

We say $u$ is a **normalized solution** of Signorini problem iff

$$\Delta u = 0 \quad \text{in } B_1^+$$
$$u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\,\partial_{x_n} u = 0 \quad \text{on } B_1'$$
$$0 \in \Gamma(u) = \partial\Lambda(u) = \partial\{u = 0\}.$$

We denote the class of normalized solutions by $\mathfrak{S}$.

# Zero Obstacle $\varphi$: Normalization

- Assume $\mathcal{M}$ is flat: $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$, $\varphi = 0$
- If $u$ solves Signorini problem, after translation, rotation, and scaling, we may normalize $u$ as follows:

## Definition (Class $\mathfrak{S}$)

We say $u$ is a **normalized solution** of Signorini problem iff

$$\Delta u = 0 \quad \text{in } B_1^+$$
$$u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\,\partial_{x_n} u = 0 \quad \text{on } B_1'$$
$$0 \in \Gamma(u) = \partial\Lambda(u) = \partial\{u = 0\}.$$

We denote the class of normalized solutions by $\mathfrak{S}$.

- *Notation*: $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, +\infty)$, $\quad B_1^+ := B_1 \cap \mathbb{R}^n_+$, $\quad B_1' := B_1 \cap (\mathbb{R}^{n-1} \times \{0\})$

# Zero Obstacle $\varphi$: Normalization

- Every $u \in \mathfrak{S}$ can be extended from $B_1^+$ to $B_1$ by even symmetry

$$u(x', -x_n) := u(x', x_n).$$

# Zero Obstacle $\varphi$: Normalization

- Every $u \in \mathfrak{S}$ can be extended from $B_1^+$ to $B_1$ by even symmetry

$$u(x', -x_n) := u(x', x_n).$$

- The resulting function will satisfy

$$\begin{aligned} \Delta u &\le 0 \quad \text{in } B_1 \\ \Delta u &= 0 \quad \text{in } B_1 \setminus \Lambda(u) \\ u\,\Delta u &= 0 \quad \text{in } B_1. \end{aligned}$$

Here $\Lambda(u) = \{u = 0\} \subset B_1'$.

# Zero Obstacle $\varphi$: Normalization

- Every $u \in \mathfrak{S}$ can be extended from $B_1^+$ to $B_1$ by even symmetry

$$u(x', -x_n) := u(x', x_n).$$

- The resulting function will satisfy

$$\begin{aligned} \Delta u &\le 0 \quad \text{in } B_1 \\ \Delta u &= 0 \quad \text{in } B_1 \setminus \Lambda(u) \\ u\,\Delta u &= 0 \quad \text{in } B_1. \end{aligned}$$

  Here $\Lambda(u) = \{u = 0\} \subset B_1'$.

- More specifically:

$$\Delta u = 2(\partial_{x_n} u)\,\mathcal{H}^{n-1}\big|_{\Lambda(u)} \quad \text{in } \mathcal{D}'(B_1).$$

# Almgren's Frequency Function

**Theorem (Monotonicity of the frequency)**

*Let $u \in \mathfrak{S}$. Then the* **frequency function**

$$r \mapsto N(r,u) := \frac{r \int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2} \nearrow \quad \textit{for} \quad 0 < r < 1.$$

*Moreover, $N(r,u) \equiv \kappa \iff x \cdot \nabla u - \kappa u = 0$ in $B_1$, i.e. $u$ is homogeneous of degree $\kappa$ in $B_1$.*

- [Almgren 1979] for harmonic $u$

# Almgren's Frequency Function

## Theorem (Monotonicity of the frequency)

*Let* $u \in \mathfrak{S}$. *Then the* **frequency function**

$$r \mapsto N(r,u) := \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2} \nearrow \quad \textit{for} \quad 0 < r < 1.$$

*Moreover,* $N(r,u) \equiv \kappa \iff x \cdot \nabla u - \kappa u = 0$ *in* $B_1$, *i.e.* $u$ *is homogeneous of degree* $\kappa$ *in* $B_1$.

- [Almgren 1979] for harmonic $u$
- [Garofalo-Lin 1986-87] for divergence form elliptic operators

# Almgren's Frequency Function

**Theorem (Monotonicity of the frequency)**

*Let $u \in \mathfrak{S}$. Then the* **frequency function**

$$r \mapsto N(r,u) := \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2} \nearrow \quad \text{for} \quad 0 < r < 1.$$

*Moreover, $N(r,u) \equiv \kappa \iff x \cdot \nabla u - \kappa u = 0$ in $B_1$, i.e. $u$ is homogeneous of degree $\kappa$ in $B_1$.*

- [Almgren 1979] for harmonic $u$
- [Garofalo-Lin 1986-87] for divergence form elliptic operators
- [Athanasopoulos-Caffarelli-Salsa 2007] for thin obstacle problem

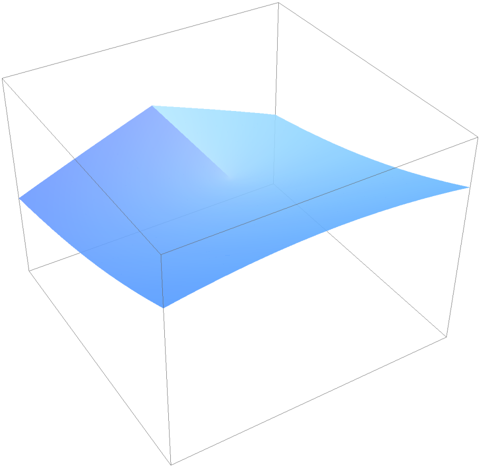


Figure: Solution of the thin obstacle problem $\mathrm{Re}(x_1 + i|x_2|)^{3/2}$

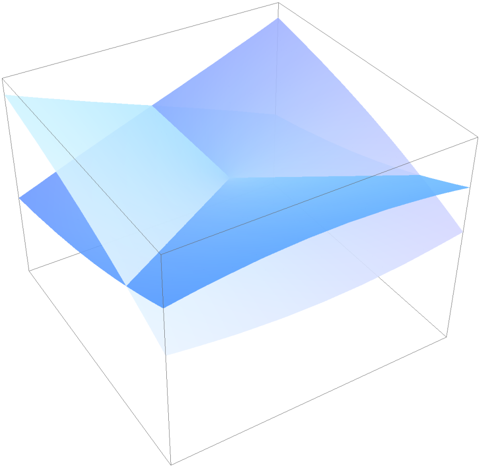


Figure: Multi-valued harmonic function $\mathrm{Re}(x_1 + ix_2)^{3/2}$

# Rescalings and Blowups

- For a solution $u \in \mathfrak{S}$ and $r > 0$ consider **rescalings**

$$u_r(x) := \frac{u(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{\frac{1}{2}}}.$$

# Rescalings and Blowups

- For a solution $u \in \mathfrak{S}$ and $r > 0$ consider **rescalings**

$$u_r(x) := \frac{u(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{\frac{1}{2}}}.$$

- The rescaling is normalized so that

$$\|u_r\|_{L^2(\partial B_1)} = 1.$$

# Rescalings and Blowups

- For a solution $u \in \mathfrak{S}$ and $r > 0$ consider **rescalings**

$$u_r(x) := \frac{u(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{\frac{1}{2}}}.$$

- The rescaling is normalized so that

$$\|u_r\|_{L^2(\partial B_1)} = 1.$$

- Limits of subsequences $\{u_{r_j}\}$ for some $r_j \to 0+$ are known as **blowups**.

# Rescalings and Blowups

- For a solution $u \in \mathfrak{S}$ and $r > 0$ consider **rescalings**

$$u_r(x) := \frac{u(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{\frac{1}{2}}}.$$

- The rescaling is normalized so that

$$\|u_r\|_{L^2(\partial B_1)} = 1.$$

- Limits of subsequences $\{u_{r_j}\}$ for some $r_j \to 0+$ are known as **blowups**.
- Generally the blowups may be different over different subsequences $r = r_j \to 0+$.

# Homogeneity of Blowups

- Uniform estimates on rescalings $\{u_r\}$:

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) = N(r, u) \le N(1, u).$$

# Homogeneity of Blowups

- Uniform estimates on rescalings $\{u_r\}$:

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) = N(r, u) \leq N(1, u).$$

- Hence, $\exists$ blowup $u_0$ over a sequence $r_j \to 0+$

$$u_{r_j} \to u_0 \quad \text{in } W^{1,2}(B_1)$$

# Homogeneity of Blowups

- Uniform estimates on rescalings $\{u_r\}$:

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) = N(r, u) \leq N(1, u).$$

- Hence, $\exists$ blowup $u_0$ over a sequence $r_j \to 0+$

$$u_{r_j} \to u_0 \quad \text{in } L^2(\partial B_1)$$

# Homogeneity of Blowups

- Uniform estimates on rescalings $\{u_r\}$:

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) = N(r, u) \le N(1, u).$$

- Hence, $\exists$ blowup $u_0$ over a sequence $r_j \to 0+$

$$u_{r_j} \to u_0 \quad \text{in } C^1_{\mathrm{loc}}(B'_1 \cup B^\pm_1)$$

# Homogeneity of Blowups

- Uniform estimates on rescalings $\{u_r\}$:

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) = N(r, u) \leq N(1, u).$$

- Hence, $\exists$ blowup $u_0$ over a sequence $r_j \to 0+$

$$u_{r_j} \to u_0 \quad \text{in } C^1_{\text{loc}}(B_1' \cup B_1^{\pm})$$

**Proposition (Homogeneity of blowups)**

*Let $u \in \mathfrak{S}$ and the blowup $u_0$ be as above. Then, $u_0$ is homogeneous of degree $\kappa = N(0+, u)$.*

# Homogeneity of Blowups

- Uniform estimates on rescalings $\{u_r\}$:

$$\int_{B_1} |\nabla u_r|^2 = N(1, u_r) = N(r, u) \leq N(1, u).$$

- Hence, $\exists$ blowup $u_0$ over a sequence $r_j \to 0+$

$$u_{r_j} \to u_0 \quad \text{in } C^1_{\mathrm{loc}}(B_1' \cup B_1^\pm)$$

**Proposition (Homogeneity of blowups)**

*Let $u \in \mathfrak{S}$ and the blowup $u_0$ be as above. Then, $u_0$ is homogeneous of degree $\kappa = N(0+, u)$.*

**Proof.**

$N(r, u_0) = \lim_{r_j \to 0+} N(r, u_{r_j}) = \lim_{r_j \to 0+} N(r r_j, u) = N(0+, u)$ □

# Proof of $C^{1,1/2}$ regularity

**Lemma ([Athanasopoulos-Caffarelli 2000])**

*Let $u_0$ be a homogeneous global solution of the thin obstacle problem with homogeneity $\kappa$. Then $\kappa \geq 3/2$.*

- Explicit solution for which $\kappa = 3/2$ is achieved is $\mathrm{Re}(x_1 + i|x_n|)^{3/2}$

# Proof of $C^{1,1/2}$ regularity

**Lemma ([Athanasopoulos-Caffarelli 2000])**

*Let $u_0$ be a homogeneous global solution of the thin obstacle problem with homogeneity $\kappa$. Then $\kappa \geq 3/2$.*

- Explicit solution for which $\kappa = 3/2$ is achieved is $\mathrm{Re}(x_1 + i|x_n|)^{3/2}$
- From Lemma we obtain that $N(0+, u) = \kappa \geq 3/2$ for any $u \in \mathfrak{S}$.

# Proof of $C^{1,1/2}$ regularity

**Lemma ([Athanasopoulos-Caffarelli 2000])**

*Let $u_0$ be a homogeneous global solution of the thin obstacle problem with homogeneity $\kappa$. Then $\kappa \geq 3/2$.*

- Explicit solution for which $\kappa = 3/2$ is achieved is $\mathrm{Re}(x_1 + i|x_n|)^{3/2}$
- From Lemma we obtain that $N(0+, u) = \kappa \geq 3/2$ for any $u \in \mathfrak{S}$.
- From here one can show that

$$\int_{\partial B_r} u^2 \leq C r^{n+2}, \quad 0 < r < 1$$

and consequently that

$$u \in C^{1,1/2}(B^{\pm}_{1/2} \cup B'_{1/2}).$$

# Nonzero Obstacle $\varphi$: Normalization

- Let now $u$ solve the thin obstacle problem with nonzero obstacle $\varphi \in C^{1,1}$:

$$\Delta u = 0 \quad \text{in } B_1^+$$
$$u \geq \varphi, \quad -\partial_{x_n} u \geq 0, \quad (u - \varphi)\, \partial_{x_n} u = 0 \quad \text{on } B_1'.$$

# Nonzero Obstacle $\varphi$: Normalization

- Let now $u$ solve the thin obstacle problem with nonzero obstacle $\varphi \in C^{1,1}$:

$$\Delta u = 0 \quad \text{in } B_1^+$$
$$u \geq \varphi, \quad -\partial_{x_n} u \geq 0, \quad (u - \varphi)\, \partial_{x_n} u = 0 \quad \text{on } B_1'.$$

- Consider the difference

$$v(x) = u(x) - \varphi(x').$$

# Nonzero Obstacle $\varphi$: Normalization

- Let now $u$ solve the thin obstacle problem with nonzero obstacle $\varphi \in C^{1,1}$:

$$\Delta u = 0 \quad \text{in } B_1^+$$
$$u \geq \varphi, \quad -\partial_{x_n} u \geq 0, \quad (u - \varphi)\,\partial_{x_n} u = 0 \quad \text{on } B_1'.$$

- Consider the difference

$$v(x) = u(x) - \varphi(x').$$

- Then it will be the class $\mathfrak{S}^f$ with $f = \Delta' \varphi \in L^\infty(B_1^+)$.

### Definition (Class $\mathfrak{S}^f$)

We say that $v \in \mathfrak{S}^f$ for some $f \in L^\infty(B_1^+)$ if

$$\Delta v = f \quad \text{in } B_1^+$$
$$v \geq 0, \quad -\partial_{x_n} v \geq 0, \quad v\,\partial_{x_n} v = 0 \quad \text{on } B_1'.$$

# Nonzero Obstacle $\varphi$: Truncated Frequency Function

## Theorem (Monotonicity of truncated frequency)

*Let* $v \in \mathfrak{S}^f$. *Then for any* $\delta > 0$ *there exists* $C = C(\|f\|_{L^\infty}, \delta) > 0$ *such that*

$$r \mapsto \Phi(r, v) = r e^{Cr^\delta} \frac{d}{dr} \log \max\left\{ \int_{\partial B_r} v^2, r^{n+3-2\delta} \right\} + 3(e^{Cr^\delta} - 1) \nearrow$$

*for* $0 < r < 1$.

# Nonzero Obstacle $\varphi$: Truncated Frequency Function

**Theorem (Monotonicity of truncated frequency)**

*Let $v \in \mathfrak{S}^f$. Then for any $\delta > 0$ there exists $C = C(\|f\|_{L^\infty}, \delta) > 0$ such that*

$$r \mapsto \Phi(r, v) = r e^{Cr^\delta} \frac{d}{dr} \log \max \left\{ \int_{\partial B_r} v^2, r^{n+3-2\delta} \right\} + 3(e^{Cr^\delta} - 1) \nearrow$$

*for* $0 < r < 1$.

- Originally due to [Caffarelli-Salsa-Silvestre 2008] in the thin obstacle problem, under the additional assumption $|f(x)| \leq C|x'|$.

# Nonzero Obstacle $\varphi$: Truncated Frequency Function

### Theorem (Monotonicity of truncated frequency)

*Let $v \in \mathfrak{S}^f$. Then for any $\delta > 0$ there exists $C = C(\|f\|_{L^\infty}, \delta) > 0$ such that*

$$r \mapsto \Phi(r, v) = re^{Cr^\delta} \frac{d}{dr} \log \max \left\{ \int_{\partial B_r} v^2, r^{n+3-2\delta} \right\} + 3(e^{Cr^\delta} - 1) \nearrow$$

*for $0 < r < 1$.*

- Originally due to [Caffarelli-Salsa-Silvestre 2008] in the thin obstacle problem, under the additional assumption $|f(x)| \leq C|x'|$.
- In this form, essentially in [$\mathscr{P}$.-To 2010].

# Nonzero Obstacle $\varphi$: Truncated Frequency Function

**Theorem (Monotonicity of truncated frequency)**

*Let $v \in \mathfrak{S}^f$. Then for any $\delta > 0$ there exists $C = C(\|f\|_{L^\infty}, \delta) > 0$ such that*

$$r \mapsto \Phi(r, v) = re^{Cr^\delta} \frac{d}{dr} \log \max\left\{ \int_{\partial B_r} v^2, r^{n+3-2\delta} \right\} + 3(e^{Cr^\delta} - 1) \nearrow$$

*for $0 < r < 1$.*

- Originally due to [Caffarelli-Salsa-Silvestre 2008] in the thin obstacle problem, under the additional assumption $|f(x)| \le C|x'|$.
- In this form, essentially in [$\mathscr{P}$.-To 2010].
- Proof consists in estimating the error terms. The truncation of the growth is needed to absorb those terms.

# Parabolic Case: Poon's Monotonicity Formula

- The optimal regularity in the elliptic case was obtained with the help of Almgren's Frequency Function. So we need a parabolic analogue of the frequency.

# Parabolic Case: Poon's Monotonicity Formula

- The optimal regularity in the elliptic case was obtained with the help of Almgren's Frequency Function. So we need a parabolic analogue of the frequency.

**Theorem ([Poon 1996])**

*Let $u$ be a caloric function (solution of the heat equation) in the strip $S_R = \mathbb{R}^n \times (-R^2, 0]$. Then*

$$N(r,u) = \frac{r^2 \int_{t=-r^2} |\nabla u|^2 G(x,r^2)\,dx}{\int_{t=-r^2} u^2 G(x,r^2)\,dx} \quad \nearrow \quad \text{for } 0 < r < R.$$

*Moreover, $N(r,u) \equiv \kappa \iff u$ is parabolically homogeneous of degree $\kappa$, i.e. $u(\lambda x, \lambda^2 t) = \lambda^\kappa u(x,t)$.*

# Parabolic Case: Poon's Monotonicity Formula

- The optimal regularity in the elliptic case was obtained with the help of Almgren's Frequency Function. So we need a parabolic analogue of the frequency.

**Theorem ([Poon 1996])**

*Let $u$ be a caloric function (solution of the heat equation) in the strip $S_R = \mathbb{R}^n \times (-R^2, 0]$. Then*

$$N(r,u) = \frac{r^2 \int_{t=-r^2} |\nabla u|^2 G(x,r^2)\,dx}{\int_{t=-r^2} u^2 G(x,r^2)\,dx} \quad \nearrow \quad \text{for } 0 < r < R.$$

*Moreover, $N(r,u) \equiv \kappa \iff u$ is parabolically homogeneous of degree $\kappa$, i.e. $u(\lambda x, \lambda^2 t) = \lambda^\kappa u(x,t)$.*

- Here $G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$, $t > 0$ is the heat (Gaussian) kernel.

# Parabolic Case: Normalization

- Suppose now $u$ solves the Parabolic Signorini Problem in $Q_1^+ = B_1^+ \times (-1, 0]$ with $\mathcal{M} = B_1'$ and $\varphi \in C_x^{1,1} \cap C_t^{0,1}$.

# Parabolic Case: Normalization

- Suppose now $u$ solves the Parabolic Signorini Problem in $Q_1^+ = B_1^+ \times (-1, 0]$ with $\mathcal{M} = B_1'$ and $\varphi \in C_x^{1,1} \cap C_t^{0,1}$.
- We want to "extend" $v$ to the half-strip $S_1^+ = \mathbb{R}_+^n \times (-1, 0]$ in the following way.

# Parabolic Case: Normalization

- Suppose now $u$ solves the Parabolic Signorini Problem in $Q_1^+ = B_1^+ \times (-1, 0]$ with $\mathcal{M} = B_1'$ and $\varphi \in C_x^{1,1} \cap C_t^{0,1}$.
- We want to "extend" $v$ to the half-strip $S_1^+ = \mathbb{R}_+^n \times (-1, 0]$ in the following way.
- Let $\eta \in C_0^\infty(B_1)$ be a cutoff function such that

$$\eta = \eta(|x|), \quad 0 \le \eta \le 1, \quad \eta\Big|_{B_{1/2}} = 1, \quad \operatorname{supp} \eta \subset B_{3/4}$$

  and consider

$$v(x,t) = [u(x,t) - \varphi(x', 0, t)]\eta(x).$$

# Parabolic Case: Normalization

- Suppose now $u$ solves the Parabolic Signorini Problem in $Q_1^+ = B_1^+ \times (-1, 0]$ with $\mathcal{M} = B_1'$ and $\varphi \in C_x^{1,1} \cap C_t^{0,1}$.
- We want to "extend" $v$ to the half-strip $S_1^+ = \mathbb{R}_+^n \times (-1, 0]$ in the following way.
- Let $\eta \in C_0^\infty(B_1)$ be a cutoff function such that

$$\eta = \eta(|x|), \quad 0 \le \eta \le 1, \quad \eta\big|_{B_{1/2}} = 1, \quad \operatorname{supp} \eta \subset B_{3/4}$$

  and consider

$$v(x, t) = [u(x, t) - \varphi(x', 0, t)]\eta(x).$$

- Then $v$ solves the Signorini problem in the half-strip $S_1^+ = \mathbb{R}_+^n \times (-1, 0]$ with a nonzero right-hand side

$$\Delta v - \partial_t v = f := \eta(x)[-\Delta' \varphi + \partial_t \varphi] + [u - \varphi(x', t)]\Delta \eta + 2\nabla u \nabla \eta$$

# Parabolic Case: Normalization

- Suppose now $u$ solves the Parabolic Signorini Problem in $Q_1^+ = B_1^+ \times (-1, 0]$ with $\mathcal{M} = B_1'$ and $\varphi \in C_x^{1,1} \cap C_t^{0,1}$.
- We want to "extend" $v$ to the half-strip $S_1^+ = \mathbb{R}_+^n \times (-1, 0]$ in the following way.
- Let $\eta \in C_0^\infty(B_1)$ be a cutoff function such that

$$\eta = \eta(|x|), \quad 0 \le \eta \le 1, \quad \eta\big|_{B_{1/2}} = 1, \quad \operatorname{supp} \eta \subset B_{3/4}$$

  and consider

$$v(x, t) = [u(x, t) - \varphi(x', 0, t)]\eta(x).$$

- Then $v$ solves the Signorini problem in the half-strip $S_1^+ = \mathbb{R}_+^n \times (-1, 0]$ with a nonzero right-hand side

$$\Delta v - \partial_t v = f := \eta(x)[-\Delta'\varphi + \partial_t \varphi] + [u - \varphi(x', t)]\Delta\eta + 2\nabla u \nabla \eta$$

- Important note: the right-hand side $f$ is nonzero even if $\varphi \equiv 0$.

# Averaged and Truncated Poon's Formula

- For the extended $u$ define

$$h_u(t) = \int_{\mathbb{R}^n_+} u(x,t)^2 G(x,-t)\,dx$$

$$i_u(t) = -t \int_{\mathbb{R}^n_+} |\nabla u(x,t)|^2 G(x,-t)\,dx,$$

# Averaged and Truncated Poon's Formula

- For the extended $u$ define

$$h_u(t) = \int_{\mathbb{R}^n_+} u(x,t)^2 G(x,-t)\,dx$$
$$i_u(t) = -t \int_{\mathbb{R}^n_+} |\nabla u(x,t)|^2 G(x,-t)\,dx,$$

- Poon's frequency is now given by

$$N(r,u) = \frac{i_u(-r^2)}{h_u(-r^2)}.$$

# Averaged and Truncated Poon's Formula

- For the extended $u$ define

$$h_u(t) = \int_{\mathbb{R}^n_+} u(x,t)^2 G(x,-t)\,dx$$

$$i_u(t) = -t \int_{\mathbb{R}^n_+} |\nabla u(x,t)|^2 G(x,-t)\,dx,$$

- Poon's frequency is now given by

$$N(r,u) = \frac{i_u(-r^2)}{h_u(-r^2)}.$$

- For our generalization, however, $i_u$ and $h_u$ are too irregular and we have to **average** them to regain missing regularity:

$$H_u(r) = \frac{1}{r^2}\int_{-r^2}^0 h_u(t)\,dt = \frac{1}{r^2}\int_{S_r^+} u(x,t)^2 G(x,-t)\,dx\,dt$$

$$I_u(r) = \frac{1}{r^2}\int_{-r^2}^0 i_u(t)\,dt = \frac{1}{r^2}\int_{S_r^+} |t||\nabla u(x,t)|^2 G(x,-t)\,dx\,dt$$

# Averaged and Truncated Poon's Formula

**Theorem ([Danielli-Garofalo-$\mathscr{P}$.-To 2011])**

*Let* $v \in \mathfrak{S}^f(S_1^+)$. *Then for any* $\delta > 0$ *there exist* $C$ *such that*

$$\Phi(r, v) = \frac{1}{2} r e^{Cr^\delta} \frac{d}{dr} \log \max\{H_v(r), r^{4-2\delta}\} + \frac{3}{2}(e^{Cr^\delta} - 1) \quad \nearrow$$

*for* $0 < r < 1$.

# Averaged and Truncated Poon's Formula

**Theorem ([Danielli-Garofalo-$\mathscr{P}$.-To 2011])**

*Let $v \in \mathfrak{S}^f(S_1^+)$. Then for any $\delta > 0$ there exist $C$ such that*

$$\Phi(r,v) = \frac{1}{2} r e^{Cr^\delta} \frac{d}{dr} \log \max\{H_v(r), r^{4-2\delta}\} + \frac{3}{2}(e^{Cr^\delta} - 1) \quad \nearrow$$

*for $0 < r < 1$.*

- Using this generalized frequency formula, as well as an estimation on parabolic homogeneity of blowups we obtain the optimal regularity.

# Parabolic Rescalings and Blowups

- As in the elliptic case, we consider the rescalings

$$u_r(x,t) = \frac{u(rx, r^2 t)}{H_u(r)^{1/2}}, \quad f_r(x,t) = \frac{r^2 f(rx, r^2 t)}{H_u(r)^{1/2}},$$

for $(x,t) \in S^+_{1/r} = \mathbb{R}^n_+ \times (-1/r^2, 0]$

# Parabolic Rescalings and Blowups

- As in the elliptic case, we consider the rescalings

$$u_r(x,t) = \frac{u(rx, r^2 t)}{H_u(r)^{1/2}}, \quad f_r(x,t) = \frac{r^2 f(rx, r^2 t)}{H_u(r)^{1/2}},$$

  for $(x,t) \in S^+_{1/r} = \mathbb{R}^n_+ \times (-1/r^2, 0]$
- If $\Phi_u(0+) < 4 - 2\delta$ then one can show that the family $\{u_r\}$ is convergent in suitable sense on $\mathbb{R}^n_+ \times (-\infty, 0]$ to a parabolically homogeneous solution $u_0$ of the Parabolic Signorini Problem

$$\Delta u_0 - \partial_t u_0 = 0 \quad \text{in } \mathbb{R}^n_+ \times (-\infty, 0]$$
$$u_0 \geq 0, \quad -\partial_{x_n} u_0 \geq 0, \quad u_0 \partial_{x_n} u_0 = 0 \quad \text{on } \mathbb{R}^{n-1} \times (-\infty, 0]$$

# Parabolic Rescalings and Blowups

- As in the elliptic case, we consider the rescalings

$$u_r(x,t) = \frac{u(rx, r^2 t)}{H_u(r)^{1/2}}, \quad f_r(x,t) = \frac{r^2 f(rx, r^2 t)}{H_u(r)^{1/2}},$$

for $(x,t) \in S^+_{1/r} = \mathbb{R}^n_+ \times (-1/r^2, 0]$

- If $\Phi_u(0+) < 4 - 2\delta$ then one can show that the family $\{u_r\}$ is convergent in suitable sense on $\mathbb{R}^n_+ \times (-\infty, 0]$ to a parabolically homogeneous solution $u_0$ of the Parabolic Signorini Problem

$$\Delta u_0 - \partial_t u_0 = 0 \quad \text{in } \mathbb{R}^n_+ \times (-\infty, 0]$$
$$u_0 \geq 0, \quad -\partial_{x_n} u_0 \geq 0, \quad u_0 \partial_{x_n} u_0 = 0 \quad \text{on } \mathbb{R}^{n-1} \times (-\infty, 0]$$

- Parabolic homogeneity $u_0$ is $\kappa = \frac{1}{2}\Phi_0(0+) < 2 - \delta < 2$. Besides, because of $C^{1,\alpha}$-regularity, also $\kappa \geq 1 + \alpha > 1$. Thus:

$$1 < \kappa < 2.$$

# Parabolically Homogeneous Global Solutions

**Lemma ([Danielli-Garofalo-$\mathscr{P}$.-To 2011])**

*Let $u_0$ be a parabolically homogeneous solution of the Parabolic Signorini Problem in $\mathbb{R}^n_+ \times (-\infty, 0]$ with homogeneity $1 < \kappa < 2$. Then necessarily $\kappa = 3/2$ and*

$$u_0(x,t) = C\,\mathrm{Re}(x_1 + i x_n)^{3/2},$$

*after a possible rotation in $\mathbb{R}^{n-1}$.*

# Parabolically Homogeneous Global Solutions

**Lemma ([Danielli-Garofalo-𝒫.-To 2011])**

*Let $u_0$ be a parabolically homogeneous solution of the Parabolic Signorini Problem in $\mathbb{R}^n_+ \times (-\infty, 0]$ with homogeneity $1 < \kappa < 2$. Then necessarily $\kappa = 3/2$ and*

$$u_0(x,t) = C\,\mathrm{Re}(x_1 + ix_n)^{3/2},$$

*after a possible rotation in $\mathbb{R}^{n-1}$.*

- The proof is based on a rather deep monotonicity formula of Caffarelli to reduce it to dimension $n = 2$ and then analysing of the principal eigenvalues of the Ornstein-Uhlenbeck operator $-\Delta + \frac{1}{2}x\cdot\nabla$ in $\mathbb{R}^2$ for the slit planes

$$\Omega_a := \mathbb{R}^2 \setminus ((-\infty, a] \times \{0\}).$$

# Proof of Optimal Regularity

- From Lemma we obtain that $\Phi_u(0+) \geq 3$, if $\Phi_u(0+) < 4 - 2\delta$. Thus, always $\Phi_u(0+) \geq 3$.

# Proof of Optimal Regularity

- From Lemma we obtain that $\Phi_u(0+) \geq 3$, if $\Phi_u(0+) < 4 - 2\delta$. Thus, always $\Phi_u(0+) \geq 3$.
- This implies $H_u(r) = \int_{\mathbb{R}^n_+} u^2 G(x, -t)\,dx\,dt \leq C r^3$

# Proof of Optimal Regularity

- From Lemma we obtain that $\Phi_u(0+) \geq 3$, if $\Phi_u(0+) < 4 - 2\delta$. Thus, always $\Phi_u(0+) \geq 3$.
- This implies $H_u(r) = \int_{\mathbb{R}^n_+} u^2 G(x, -t)\,dx\,dt \leq Cr^3$
- This further implies that

$$\sup_{Q^+_{r/2}(x_0,t_0)} |u| \leq Cr^{3/2}$$

for any $(x_0, t_0) \in Q'_{1/2}$ such that $u(x_0, t_0) = 0$.

# Proof of Optimal Regularity

- From Lemma we obtain that $\Phi_u(0+) \geq 3$, if $\Phi_u(0+) < 4 - 2\delta$. Thus, always $\Phi_u(0+) \geq 3$.
- This implies $H_u(r) = \int_{\mathbb{R}^n_+} u^2 G(x, -t)\, dx\, dt \leq Cr^3$
- This further implies that

$$\sup_{Q^+_{r/2}(x_0,t_0)} |u| \leq Cr^{3/2}$$

for any $(x_0, t_0) \in Q'_{1/2}$ such that $u(x_0, t_0) = 0$.

- Using interior parabolic estimates one then obtains

$$\nabla u \in C^{1/2,1/4}_{x,t}(Q^+_{1/4}).$$

# Finer Study of the Free Boundary

- Assume now $\varphi$ is in parabolic Hölder class $H^{\ell,\ell/2}_{x',t}(B'_1)$, with $\ell = k + \gamma \geq 2$, $k \in \mathbb{N}$, $0 < \gamma \leq 1$.

# Finer Study of the Free Boundary

- Assume now $\varphi$ is in parabolic Hölder class $H^{\ell,\ell/2}_{x',t}(B'_1)$, with $\ell = k + \gamma \geq 2$, $k \in \mathbb{N}$, $0 < \gamma \leq 1$.
- Then the parabolic Taylor polynomial $q_k(x', t)$ of degree $k$ satisfies

$$|\varphi(x', t) - q_k(x', t)| \leq C(|x'|^2 + t)^{\ell/2}.$$

# Finer Study of the Free Boundary

- Assume now $\varphi$ is in parabolic Hölder class $H^{\ell,\ell/2}_{x',t}(B_1')$, with $\ell = k + \gamma \geq 2$, $k \in \mathbb{N}$, $0 < \gamma \leq 1$.
- Then the parabolic Taylor polynomial $q_k(x', t)$ of degree $k$ satisfies

$$|\varphi(x', t) - q_k(x', t)| \leq C(|x'|^2 + t)^{\ell/2}.$$

- Extend $q_k$ from $\mathbb{R}^{n-1} \times \mathbb{R}$ to a caloric polynomial $Q_k(x, t)$ on $\mathbb{R}^n \times \mathbb{R}$:

$$\Delta Q_k - \partial_t Q_k = 0, \quad Q_k(x', 0, t) = q_k(x', t).$$

# Finer Study of the Free Boundary

- Assume now $\varphi$ is in parabolic Hölder class $H^{\ell,\ell/2}_{x',t}(B'_1)$, with $\ell = k + \gamma \geq 2$, $k \in \mathbb{N}$, $0 < \gamma \leq 1$.
- Then the parabolic Taylor polynomial $q_k(x', t)$ of degree $k$ satisfies

$$|\varphi(x', t) - q_k(x', t)| \leq C(|x'|^2 + t)^{\ell/2}.$$

- Extend $q_k$ from $\mathbb{R}^{n-1} \times \mathbb{R}$ to a caloric polynomial $Q_k(x, t)$ on $\mathbb{R}^n \times \mathbb{R}$:

$$\Delta Q_k - \partial_t Q_k = 0, \quad Q_k(x', 0, t) = q_k(x', t).$$

- Consider then

$$v_k(x, t) = [u(x, t) - Q_k(x, t) - \varphi(x', t) - q_k(x', t)]\eta(x)$$

with a cutoff function $\eta(x)$.

# Finer Study of the Free Boundary

- Then $v_k$ solves the Signorini problem in $\mathfrak{S}_1^+$ with nonzero right-hand-side

$$\Delta v_k - \partial_t v_k = f_k \quad \text{in } S_1^+$$

with

$$|f_k(x,t)| \leq C(|x|^2 + |t|)^{(\ell-2)/2}.$$

# Finer Study of the Free Boundary

- Then $v_k$ solves the Signorini problem in $\mathfrak{S}_1^+$ with nonzero right-hand-side

$$\Delta v_k - \partial_t v_k = f_k \quad \text{in } S_1^+$$

with

$$|f_k(x,t)| \leq C(|x|^2 + |t|)^{(\ell-2)/2}.$$

**Theorem (Better Truncated Monotonicity Formula,[D-G-$\mathcal{P}$-T 2011])**

*For $v_k$ as above and $\delta < \gamma$ there exist $C = C_\delta$ such that*

$$\Phi^{(\ell)}(r, v_k) = \frac{1}{2} r e^{Cr^\delta} \frac{d}{dr} \log \max\{H_{v_k}(r), r^{2\ell-2\delta}\} + \frac{3}{2}(e^{Cr^\delta} - 1) \quad \nearrow$$

*for $0 < r < 1$.*

# Classification of Free Boundary Points

**Definition**

We say $(0,0) \in \Gamma_\kappa^{(\ell)}$ iff $\Phi^{(\ell)}(0+, v_k) = \kappa$.

- One can show that $3/2 \le \kappa \le \ell$ and therefore we have a foliation

$$\Gamma = \bigcup_{3/2 \le \kappa \le \ell} \Gamma_\kappa^{(\ell)}$$

# Classification of Free Boundary Points

## Definition

We say $(0,0) \in \Gamma_\kappa^{(\ell)}$ iff $\Phi^{(\ell)}(0+, v_k) = \kappa$.

- One can show that $3/2 \leq \kappa \leq \ell$ and therefore we have a foliation

$$\Gamma = \bigcup_{3/2 \leq \kappa \leq \ell} \Gamma_\kappa^{(\ell)}$$

- The more regular is the thin obstacle $\varphi$, the larger is $\ell$, the finer we can classify free boundary points.

# Classification of Free Boundary Points

## Definition

We say $(0,0) \in \Gamma_\kappa^{(\ell)}$ iff $\Phi^{(\ell)}(0+, v_k) = \kappa$.

- One can show that $3/2 \le \kappa \le \ell$ and therefore we have a foliation

$$\Gamma = \bigcup_{3/2 \le \kappa \le \ell} \Gamma_\kappa^{(\ell)}$$

- The more regular is the thin obstacle $\varphi$, the larger is $\ell$, the finer we can classify free boundary points.
- It is conjectured that $\kappa$ can take only discrete values $\kappa = 3/2, 7/2, \ldots, 2m - 1/2, \ldots$.

# Classification of Free Boundary Points

## Definition

We say $(0,0) \in \Gamma_\kappa^{(\ell)}$ iff $\Phi^{(\ell)}(0+, v_k) = \kappa$.

- One can show that $3/2 \leq \kappa \leq \ell$ and therefore we have a foliation

$$\Gamma = \bigcup_{3/2 \leq \kappa \leq \ell} \Gamma_\kappa^{(\ell)}$$

- The more regular is the thin obstacle $\varphi$, the larger is $\ell$, the finer we can classify free boundary points.
- It is conjectured that $\kappa$ can take only discrete values $\kappa = 3/2, 7/2, \ldots, 2m - 1/2, \ldots$.
- As of now it is known only that there is no $\kappa$ in $(3/2, 2)$, so $\kappa = 3/2$ is isolated.

# Regularity of the Free Boundary

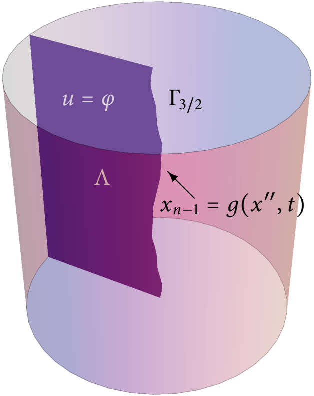
- The set $\Gamma_{3/2}$ is known as the *Regular Set*.

**Theorem ([Danielli-Garofalo-$\mathscr{P}$-To 2011])**

*Let* $\varphi \in H^{3,3/2}(Q_1')$. *If* $(0,0) \in \Gamma_{3/2}$ *then there exists* $\delta > 0$ *such that*

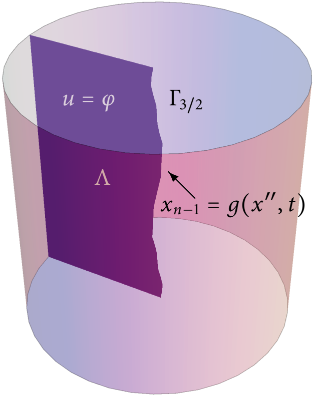
$$\Gamma \cap Q_\delta = \Gamma_{3/2} \cap Q_\delta = \{x_{n-1} = g(x'', t)\} \cap Q_\delta,$$

*where* $g$ *is such that* $\nabla g \in C^{\alpha,\alpha/2}_{x'',t}$.



# Regularity of the Free Boundary

- The set $\Gamma_{3/2}$ is known as the *Regular Set*.




**Theorem ([Danielli-Garofalo-𝒫-To 2011])**

*Let $\varphi \in H^{3,3/2}(Q_1')$. If $(0,0) \in \Gamma_{3/2}$ then there exists $\delta > 0$ such that*

$$\Gamma \cap Q_\delta = \Gamma_{3/2} \cap Q_\delta = \{x_{n-1} = g(x'', t)\} \cap Q_\delta,$$

*where $g$ is such that $\nabla g \in C^{\alpha,\alpha/2}_{x'',t}$.*

- One first shows that $g \in \mathrm{Lip}(1, 1/2)$

# Regularity of the Free Boundary

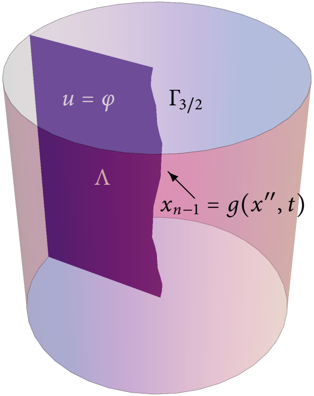
- The set $\Gamma_{3/2}$ is known as the *Regular Set*.

**Theorem ([Danielli-Garofalo-𝒫-To 2011])**

*Let $\varphi \in H^{3,3/2}(Q_1')$. If $(0,0) \in \Gamma_{3/2}$ then there exists $\delta > 0$ such that*

$$\Gamma \cap Q_\delta = \Gamma_{3/2} \cap Q_\delta = \{x_{n-1} = g(x'', t)\} \cap Q_\delta,$$

*where $g$ is such that $\nabla g \in C^{\alpha,\alpha/2}_{x'',t}$.*




- One first shows that $g \in \mathrm{Lip}(1, 1/2)$
- $\mathrm{Lip} \Rightarrow C^{1,\alpha}$ follows from a special version of the Parabolic Boundary Harnack Principle by [Shi 2011].