Higher regularity of the free boundary in the thin obstacle problem

Arshak Petrosyan
joint with Herbert Koch and Wenhui Shi

Purdue University

Nicola's 50+10 Birthday
May 26–27, 2014

Isaac Newton Institute for Mathematical Sciences

Classical obstacle problem

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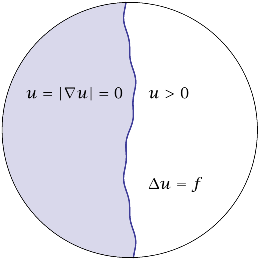

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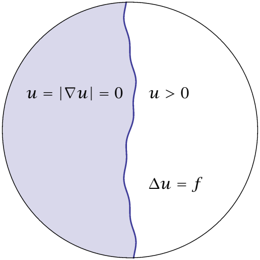

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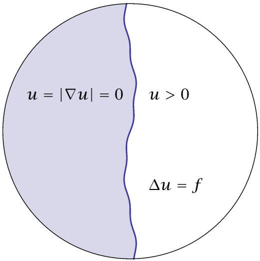

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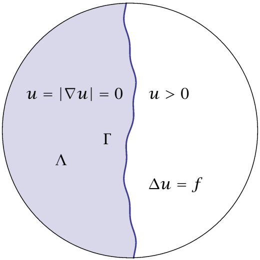

- Main objects of study

$$\textbf{Coincidence set}: \quad \Lambda(u) := \{x \in D : u = 0\}$$

$$\textbf{Free Boundary}: \quad \Gamma(u) := \partial\Lambda(u)$$

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Known results:

- If f is smooth (enough), then $u \in C^{1,1}_{\mathrm{loc}}(D)$ [Frehse'79], see also [Caffarelli'98]

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- For $x_0 \in \Sigma$, the rescalings converge to *polynomial solutions*

  $$u_{x_0,r}(x) \to p_{x_0}(x) = x \cdot A_{x_0} x$$

  over $r = r_j \to 0^+$, where A_{x_0} is a positive matrix with $\mathrm{tr}(A_{x_0}) = C_{x_0}$.

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- Follows from continuous dependence of blowups on $x_0 \in \Sigma$ and Whitney's extension theorem [CAFFARELLI'98]

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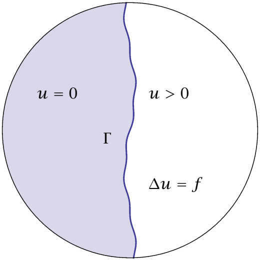

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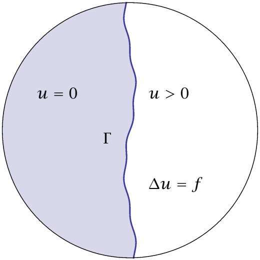

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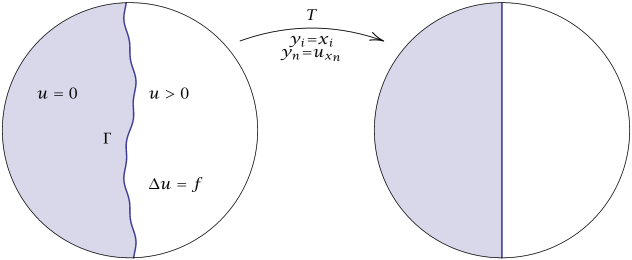

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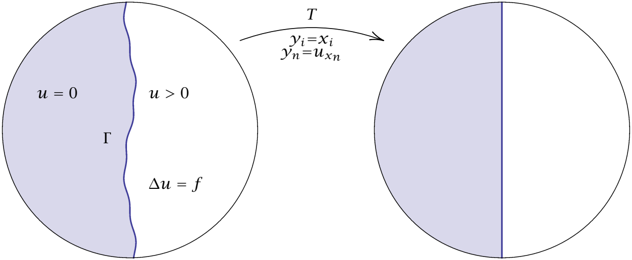

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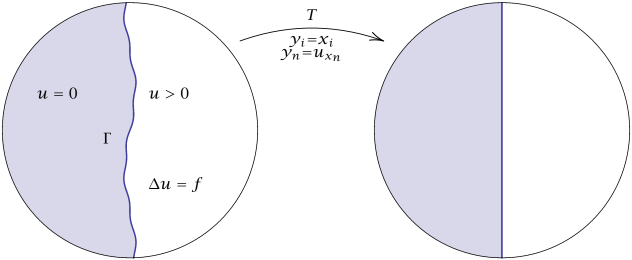

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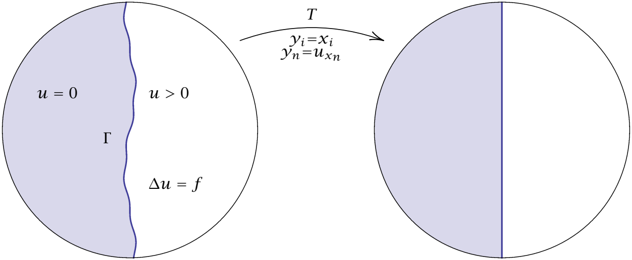

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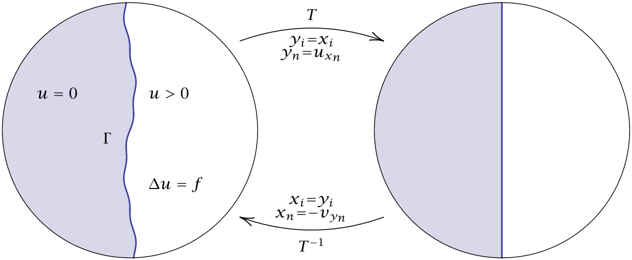

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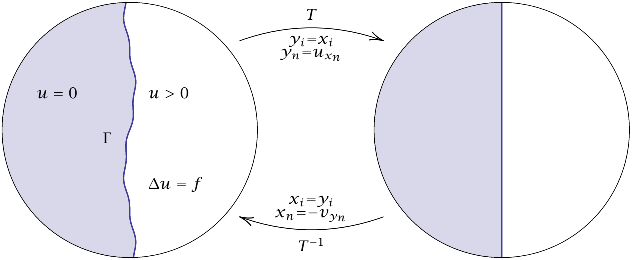

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- Thus $\Gamma : x_n = -v_{y_n}(x', 0)$ and the regularity of Γ is related to that of v.

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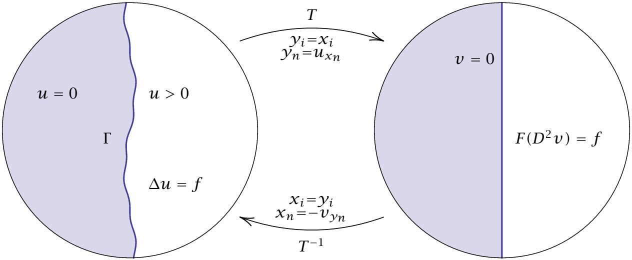

- By direct computation, for $j, k = 1, \dots, n-1$

$$u_{x_n x_n} = -\frac{1}{v_{y_n y_n}}, \quad u_{x_n x_j} = -\frac{v_{y_n y_j}}{v_{y_n y_n}}, \quad u_{x_j x_k} = v_{y_j y_k} - \frac{v_{y_n y_j} v_{y_n y_k}}{v_{y_n y_n}}$$

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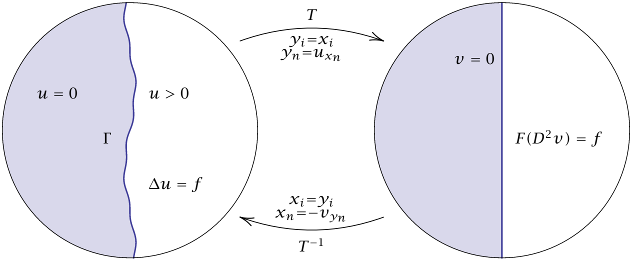

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- v will satisfy fully nonlinear elliptic equation

$$F(D^2 v) =: \sum_{i=1}^{n-1} v_{y_i y_i} - \frac{1}{v_{y_n y_n}} - \frac{1}{v_{y_n y_n}} \sum_{i=1}^{n-1} v_{y_n y_i}^2 = f(y', -v_{y_n})$$

in $B_\delta \cap \{y_n > 0\}$

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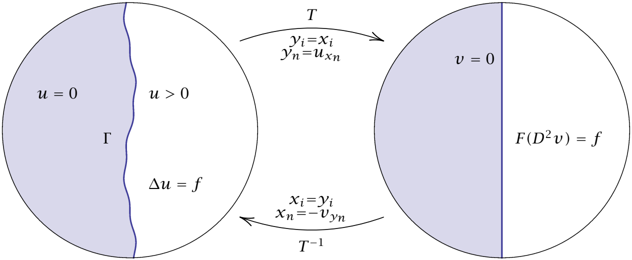

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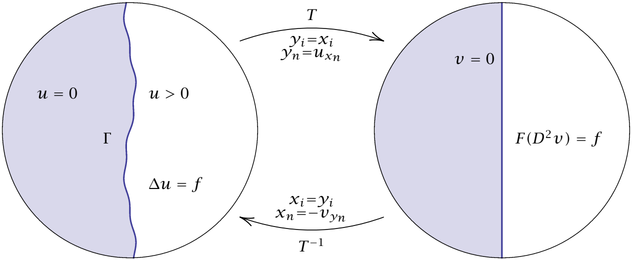

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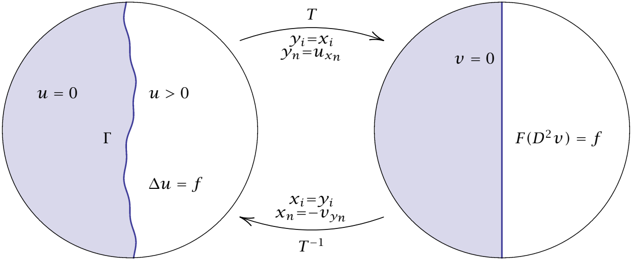

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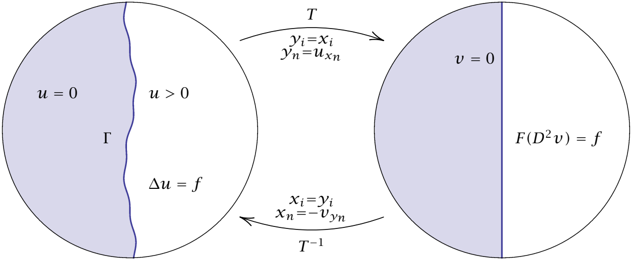

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- Recalling that Γ is parametrized by

$$\Gamma : x_n = -v_{y_n}(x', 0)$$

we obtain that Γ is real analytic.

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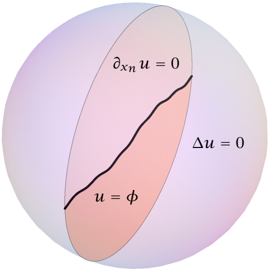

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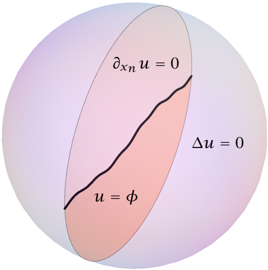

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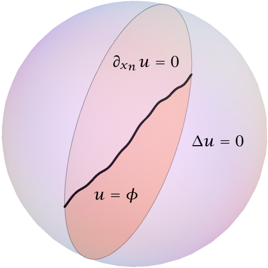

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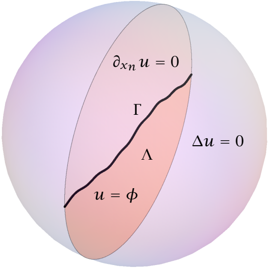

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  is **Almgren's frequency formula**, which is *monotone increasing* in r [Almgren'00]
- $x_0 \in \Gamma_\kappa \iff$ blowups at x_0 are homogeneous of degree κ

$$u_{x_0,r}(x) = \frac{u(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r(x)} u^2\right)^{1/2}} \to u_0(x), \quad u_0(\lambda x) = \lambda^\kappa u_0(x).$$

Thin obstacle (Signorini) problem

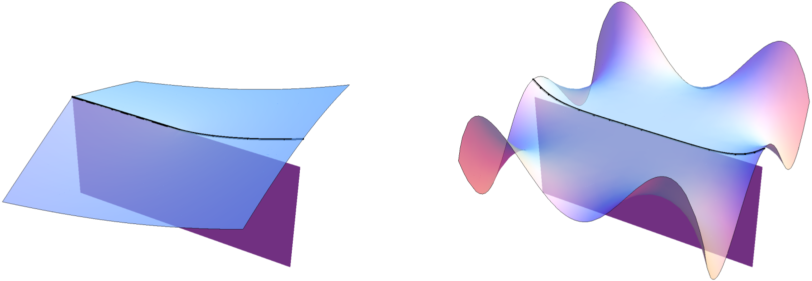

Figure: Graphs of $\mathrm{Re}(x_1 + i\,|x_2|)^{3/2}$ and $\mathrm{Re}(x_1 + i\,|x_2|)^6$

- When $\kappa = 3/2$, the only blowups are $u_0(x) = C_n \,\mathrm{Re}(x' \cdot e' + i|x_n|)^{3/2}$, for $|e'| = 1$.

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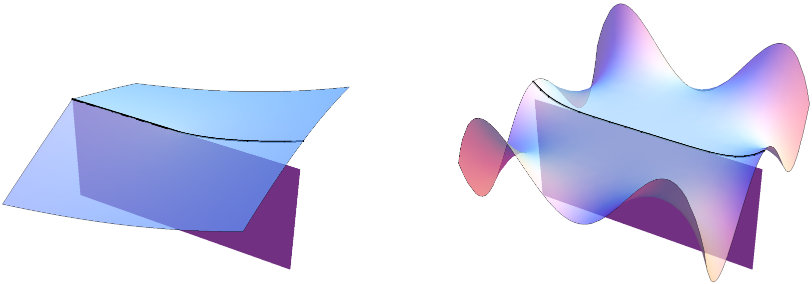

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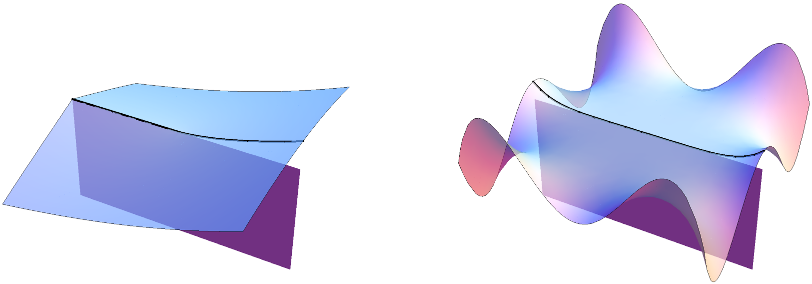

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- For other values of κ, the blowups are not classified, except when $n = 2$ (simple exercise).

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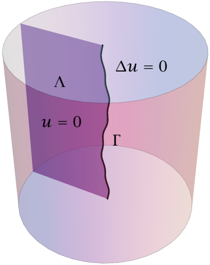
Theorem (Structure of Σ [Garofalo-$\mathcal{P}$'09])

$\Sigma = \bigcup_{m\in\mathbb{N}} \Gamma_{2m}$. *Moreover, Σ is contained in a countable union of C^1 manifolds.*

Partial hodograph transform



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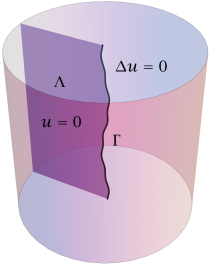

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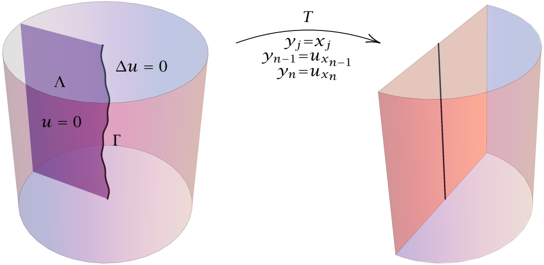

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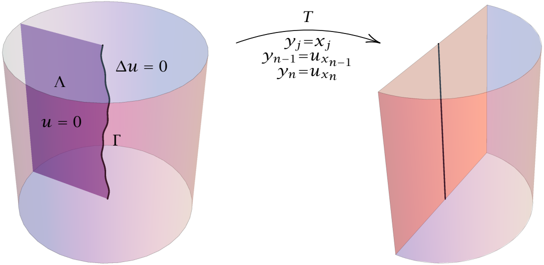

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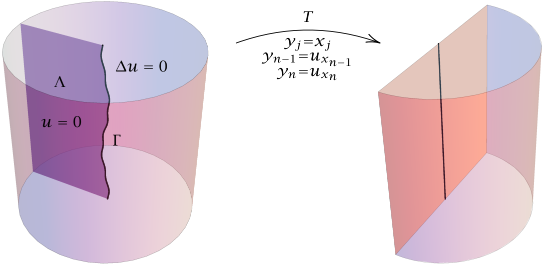
$$T : (x'', x_{n-1}, x_n) \mapsto (x'', u_{x_{n-1}}, u_{x_n})$$

- $T(B_1 \setminus \Lambda) \subset \{y_{n-1} > 0\}$, $T(\Lambda^{\pm}) \subset \{y_{n-1} = 0\}$, $T(\Gamma) \subset \{y_{n-1} = 0, y_n = 0\}$

Partial hodograph transform



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$$y_{n-1} - i y_n = \frac{3}{2}(x_{n-1} + i x_n)^{1/2}$$

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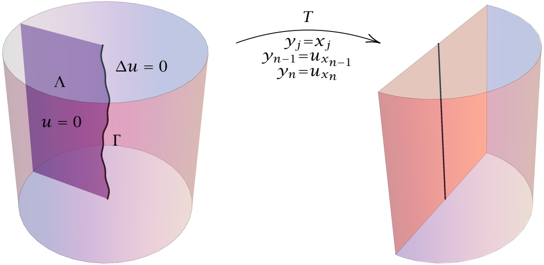

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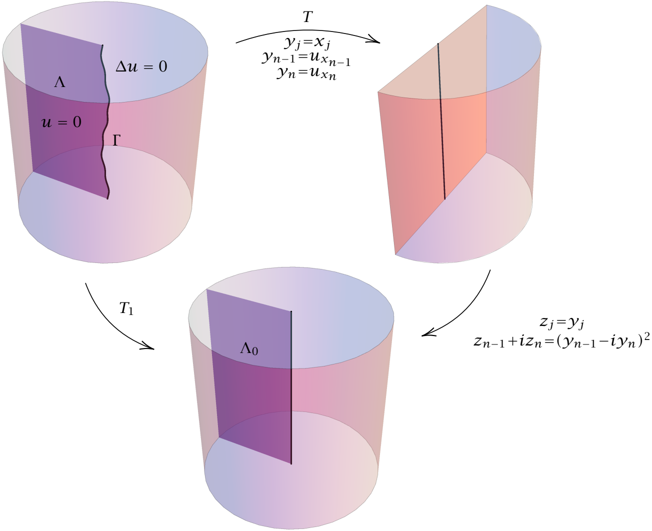
- To better visualize the transformation T, we compose it with

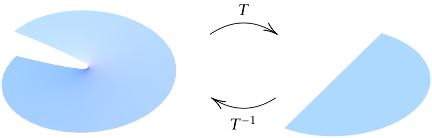
$$y \mapsto z : z_j = y_j, \quad z_{n-1} + iz_n = (y_{n-1} - iy_n)^2$$

Partial hodograph transform



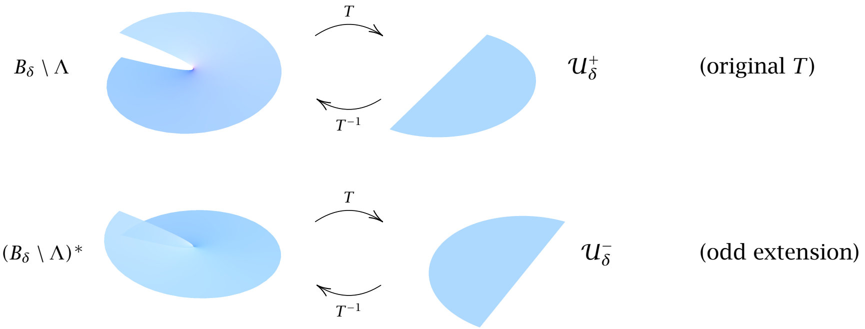
Partial hodograph transform

$B_\delta \setminus \Lambda$

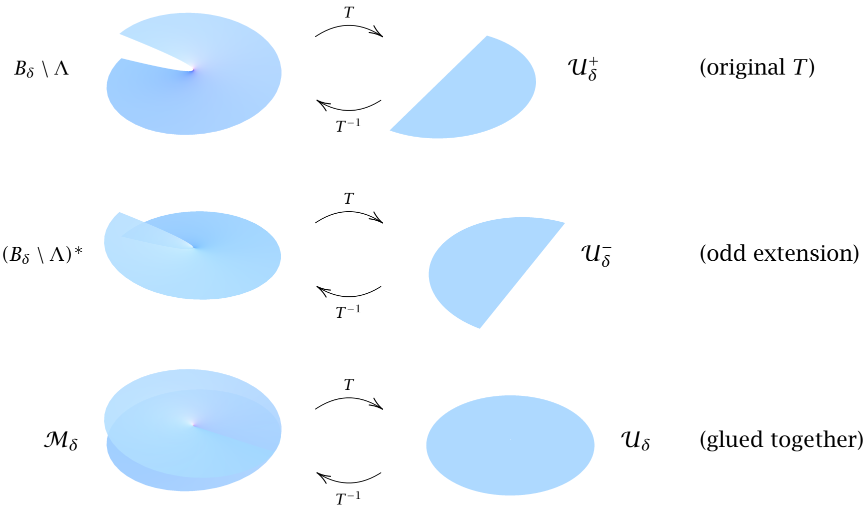


$\mathcal{U}_\delta^+$ (original T)

Partial hodograph transform



Partial hodograph transform



Partial hodograph transform

Theorem (Invertibility of hodograph transform, [Koch-$\mathcal{P}$-Shi'14])

There exists a $\delta > 0$ *such that* $T : \mathcal{M}_\delta \to \mathcal{U}_\delta = T(\mathcal{M}_\delta)$ *is invertible.*

Lemma (Homogeneous blowups)

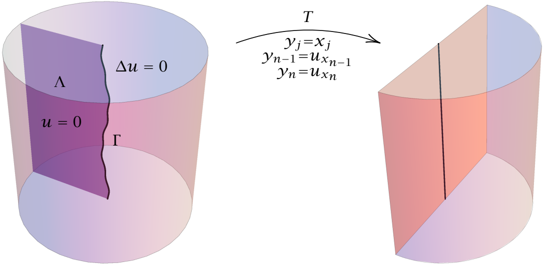
For every $x_0 \in \Gamma$*,*

$$u_{x_0,r}(x) = \frac{u(x_0 + rx)}{r^{3/2}} \to C_{x_0} \operatorname{Re}(x \cdot \nu_{x_0} + i x_n)^{3/2},$$

as $r \to 0$*, where* ν_{x_0} *is the normal to* Γ *at* x_0*. Moreover,* $x_0 \mapsto C_{x_0}$ *is continuous and the above convergence is uniform on compact subsets of* Γ*.*

- Proof uses **Weiss** and **Monneau type monotonicity formulas**, as well as a new **boundary Hopf-type principle** for domains of the type $B_1 \setminus \Lambda$

Partial Legendre transform



- Assume now that T is invertible and define the **partial Legendre transform**

$$v(y) = u(x) - x_{n-1}y_{n-1} - x_n y_n$$

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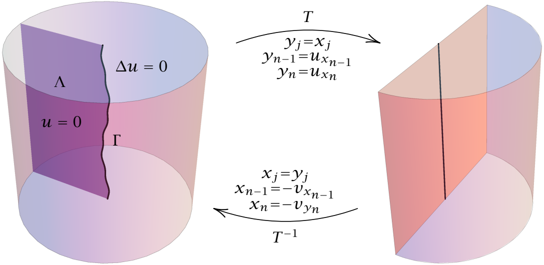

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$$T^{-1} : (y'', y_{n-1}, y_n) \mapsto (y'', -v_{y_{n-1}}, -v_{y_n})$$

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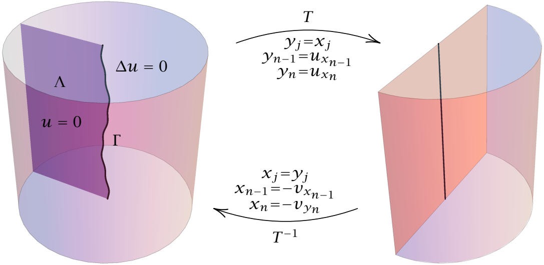

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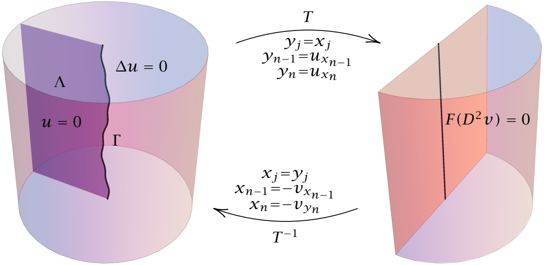

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- By direct computation, we have the following differentiation formulas:

$$\partial_{x_j} u = \partial_{y_j} v, \quad \partial_{x_{n-1}} u = -y_{n-1}, \quad \partial_{x_n} u = -y_n$$

$$\partial_{x_i x_i} u = \partial_{y_i y_i} v - (\partial_{y_{n-1} y_i} v, \partial_{y_n y_i} v) \begin{pmatrix} \partial_{y_{n-1} y_{n-1}} v & \partial_{y_{n-1} y_n} v \\ \partial_{y_{n-1} y_n} v & \partial_{y_n y_n} v \end{pmatrix}^{-1} \begin{pmatrix} \partial_{y_i y_{n-1}} v \\ \partial_{y_i y_n} v \end{pmatrix}$$

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- Hence v satisfies a fully nonlinear equation

$$\begin{aligned} \tilde{F}(D_y^2 v) = \sum_{i=1}^{n-2} \partial_{y_i y_i} v - \operatorname{tr} \begin{pmatrix} \partial_{y_{n-1} y_{n-1}} v & \partial_{y_{n-1} y_n} v \\ \partial_{y_{n-1} y_n} v & \partial_{y_n y_n} v \end{pmatrix}^{-1} \\ - \sum_{i=1}^{n-2} (\partial_{y_i y_{n-1}} v, \partial_{y_i y_n} v) \begin{pmatrix} \partial_{y_{n-1} y_{n-1}} v & \partial_{y_{n-1} y_n} v \\ \partial_{y_{n-1} y_n} v & \partial_{y_n y_n} v \end{pmatrix}^{-1} \begin{pmatrix} \partial_{y_i y_{n-1}} v \\ \partial_{y_i y_n} v \end{pmatrix} = 0. \end{aligned}$$

Partial Legendre transform

- Multiplying both sides by

$$J(v) = \det \begin{pmatrix} \partial_{y_{n-1}y_{n-1}} v & \partial_{y_{n-1}y_n} v \\ \partial_{y_{n-1}y_n} v & \partial_{y_n y_n} v \end{pmatrix},$$

we can write it in the form

$$F(D^2 v) = \partial_{y_{n-1}y_{n-1}} v + \partial_{y_n y_n} v - \sum_{i=1}^{n-2} \det(V^i) = 0,$$

where V^i, $i = 1, \ldots, n-2$, is the 3×3 matrix

$$V^i = \begin{pmatrix} \partial_{y_i y_i} v & \partial_{y_i y_{n-1}} v & \partial_{y_i y_n} v \\ \partial_{y_{n-1} y_i} v & \partial_{y_{n-1} y_{n-1}} v & \partial_{y_{n-1} y_n} v \\ \partial_{y_n y_i} v & \partial_{y_n y_{n-1}} v & \partial_{y_n y_n} v \end{pmatrix}_{3\times 3}$$

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- The equation is degenerate, since $J(v) \sim -\frac{64}{81}(y_{n-1}^2 + y_n^2)$

Subelliptic structure

- Consider the linearization of $F(D^2u)$. Let

$$F_{ij}(M) = \partial_{m_{ij}} F(M), \quad M = (m_{ij})_{n\times n}$$

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- By direct computation $(F_{ij}(D^2v))_{i,j} = ABA^t$, where

$$A = \begin{pmatrix} Y & 0 & 0 & \cdots & 0 \\ 0 & Y & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & Y & 0 \\ 0 & 0 & 0 & \cdots & I_2 \end{pmatrix}_{n\times 2(n-1)} \quad B = (b_{ij}) = \begin{pmatrix} B_0 & 0 & 0 & \cdots & B_1 \\ 0 & B_0 & 0 & \cdots & B_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & B_0 & B_{n-2} \\ B_1^t & B_2^t & \cdots & \cdots & B_{n-1} \end{pmatrix}_{2(n-1)\times 2(n-1)}$$

with

$$Y = \frac{8}{9}(y_{n-1}, y_n), \quad B_0 = \begin{pmatrix} b_0(y) & 0 \\ 0 & \tilde{b}_0(y) \end{pmatrix}, \quad B_i = \begin{pmatrix} b_{i,1}(y) & b_{i,2}(y) \\ \tilde{b}_{i,1}(y) & \tilde{b}_{i,2}(y) \end{pmatrix}.$$

Subelliptic structure

- Entries of the matrix B are smooth in $\mathcal{U}_\delta \setminus \{y_{n-1} = y_n = 0\}$ and continuous up to $\{y_{n-1} = y_n = 0\}$. Based on the study of blowups of v and T^{-1}.

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$$AB(0)A^t = \begin{pmatrix} \frac{64}{81}(y_{n-1}^2 + y_n^2)I_{n-2} & 0 \\ 0 & I_2 \end{pmatrix}$$

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- Up to a constant, this corresponds to the **Baouendi-Grushin type operator**:

$$\mathcal{L}_0 = (y_{n-1}^2 + y_n^2)\sum_{i=1}^{n-2} \partial_{i,i}^2 + \partial_{n-1,n-1}^2 + \partial_{n,n}^2.$$

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Subelliptic structure

- Entries of the matrix B are smooth in $\mathcal{U}_\delta \setminus \{y_{n-1} = y_n = 0\}$ and continuous up to $\{y_{n-1} = y_n = 0\}$. Based on the study of blowups of v and T^{-1}.
- At $y = 0$ we have $B(0) = I_{2(n-1)}$, which makes

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- $\mathcal{L}_0$ is a well-studied subelliptic operator and can be written as Hörmander type sum of squares operator

$$\mathcal{L}_0 = \sum_{k=1}^{2(n-1)} Y_k^2, \quad \{Y_k\} = \{y_\alpha \partial_j, \partial_\beta\}, \quad \alpha, \beta = n-1, n, \quad j = 1, \dots, n-2$$

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Theorem (Smoothness of $\mathcal{R}$ [KOCH-$\mathcal{P}$-SHI'14])

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- Recall that the latter follows from the parametrization
$$\Gamma : x_{n-1} = -\partial_{n-1} v(x'', 0, 0),\ x_n = 0.$$

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- Consequently, Γ is also real analytic.