Elliptic and parabolic obstacle problems with thin and Lipschitz obstacles

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PURDUE
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Multi-asset American options

1. Let $S_1, S_2, \dots S_n$ denote the prices of n risky dividend paying assets that satisfy the stochastic differential equations

$$dS_i(t) = (\mu_i - \delta_i)S_i(t)dt + \sigma_i S_i(t)dW_i,$$

where $dW_i(t)$ are standard Brownian motions such that

$$\mathrm{E}(dW_i) = 0, \quad \mathrm{Var}(dW_i) = dt, \quad \mathrm{Cov}(dW_i, dW_j) = \rho_{ij}dt.$$

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2. If $V(S_1, \dots, S_n, t)$ is the price of the European style option derived from these assets, with *payoff function* $\Phi(S_1, \dots, S_n)$ at time T, then V must satisfy the *Black-Scholes equation*

$$\mathscr{L}V := \frac{\partial}{\partial t}V + \frac{1}{2}\sum_{i,j=1}^{n} \alpha_{ij} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r - \delta_i)S_i \frac{\partial V}{\partial S_i} - rV = 0 \quad (t < T)$$

$$V(S_1, \dots, S_n, T) = \Phi(S_1, \dots, S_n).$$

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$$\mathscr{L}V \le 0, \quad V \ge \Phi, \quad \mathscr{L}V(V-\Phi)=0 \quad \text{on } (\mathbb{R}_+)^n \times (-\infty, T)$$

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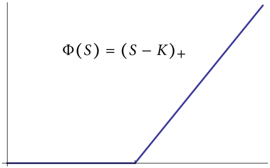
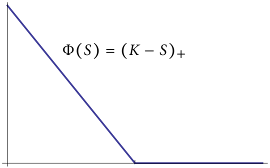
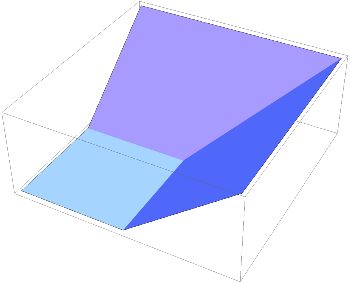
3. Typically $\Phi(S)$ is only Lipschitz continuous
   - $n = 1$: $\Phi(S) = (S-K)_+$ American call option
   - $n = 1$: $\Phi(S) = (K-S)_+$ American put option
   - $n = 2$: $\Phi(S) = (\max\{S_1,S_2\} - K)_+$ American call max-options
   - $n = 2$: $\Phi(S) = (\min\{S_1,S_2\} - K)_+$ American call min-options

   Not that these Φ's are also piecewise smooth (important!)

Multi-asset American options







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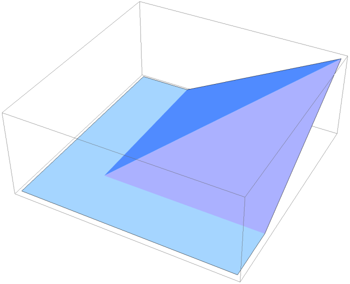

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Parabolic obstacle problem

1. With an appropriate transformation of variables (including $x_i = \log S_i$), this can be rewritten as a variational inequality for the *heat operator* for a function $v = v(x,t)$

$$(\Delta - \partial_t)v \le 0, \quad v - \varphi \ge 0, \quad (\Delta - \partial_t)v(v - \varphi) = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

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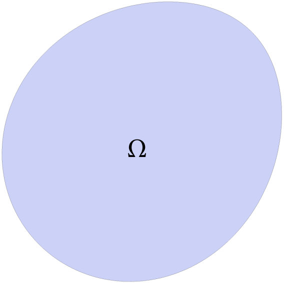
3. The solutions of the obstacle problem are well understood when φ is smooth. However, the general theory of free boundaries with nonsmooth (say Lipschitz) obstacles φ is still lacking. We will discuss what complication arise when φ is piecewise-smooth.

Classical obstacle problem

- Given
  - Ω domain in $\mathbb{R}^n$



Classical obstacle problem

- Given
  - Ω domain in $\mathbb{R}^n$
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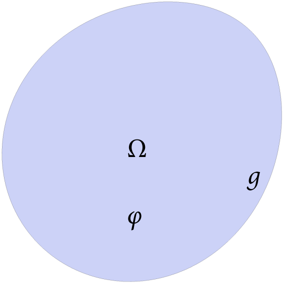

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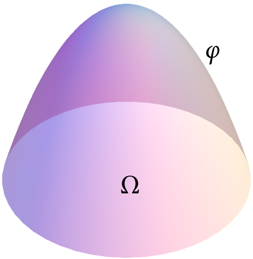

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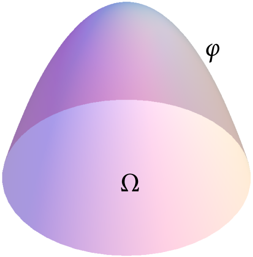
- Minimize the Dirichlet integral

$$D_\Omega(u) = \int_\Omega |\nabla u|^2 dx$$

on the closed convex set

$$\mathfrak{K} = \{u \in W^{1,2}(\Omega) \mid u = g \text{ on } \partial\Omega, u \geq \varphi \text{ on } \Omega\}.$$



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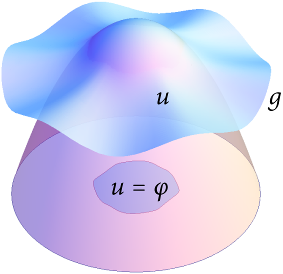

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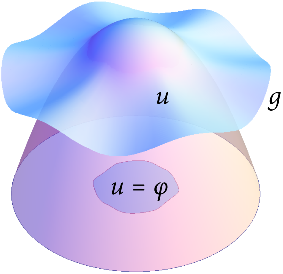

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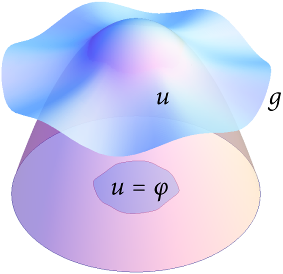

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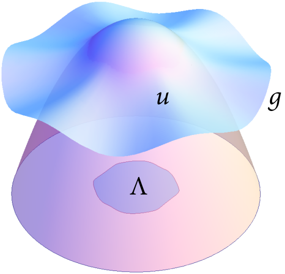
is known as the *coincidence set*.



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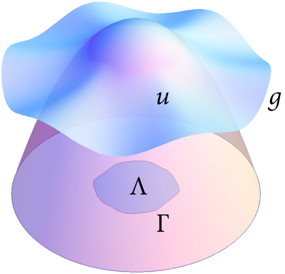

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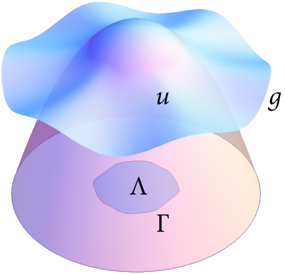

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- The regularity properties of u and Γ are fairly well studied when $\varphi \in C^{1,1}$ and $\Delta\varphi < 0$.

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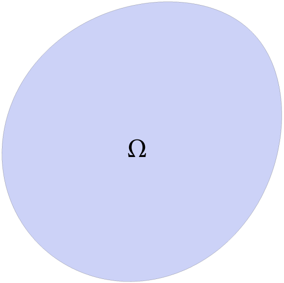

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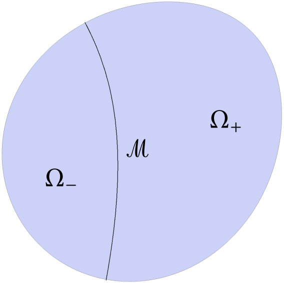

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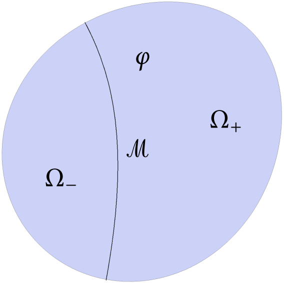
$$\partial_{\nu+}\varphi + \partial_{\nu_-}\varphi \geq 0 \quad \text{on } \mathcal{M}$$

We call it a **rooftop-like obstacle**



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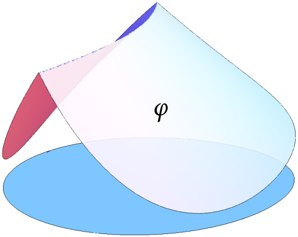

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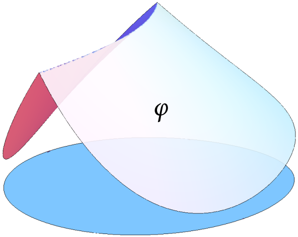

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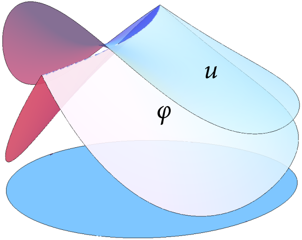

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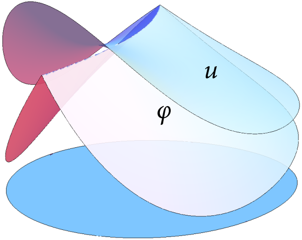

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- This generally cannot be improved if φ is only Lipschitz, but our extra structure allows an improvement.

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The minimizer u satisfies

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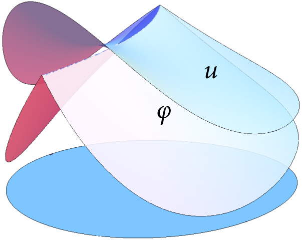

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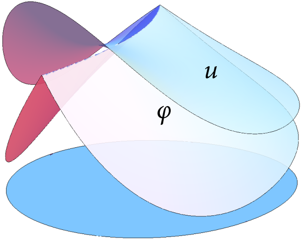

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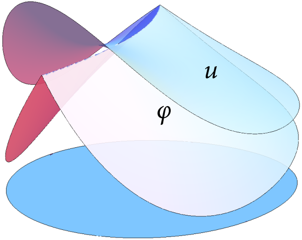

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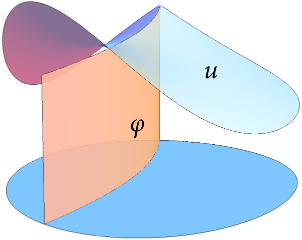

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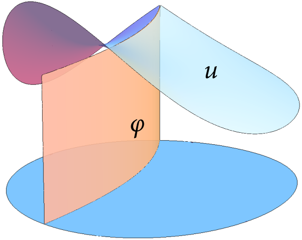

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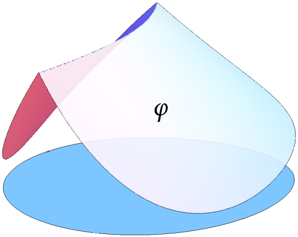
- Many of our techniques has been developed first for this problem: [Athanasopoulos-Caffarelli 2006], [Caffarelli-Silvestre-Salsa 2008], [Garofalo-$\mathcal{P}$. 2009], etc.

Piecewise smooth Lipschitz obstacles

- The condition

$$\partial_{\nu+}\varphi + \partial_{\nu-}\varphi \geq 0$$

allows to have a contact over the "ridge" of φ, which makes the problem more difficult.



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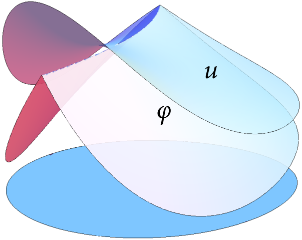

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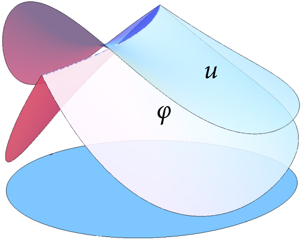

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- On the other hand, if

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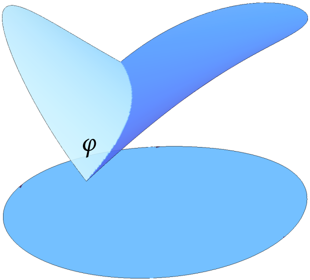

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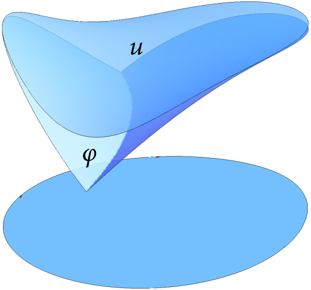

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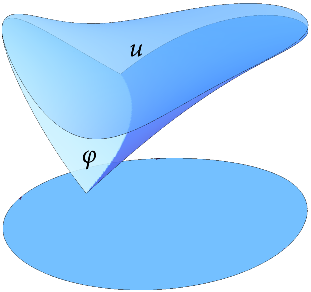

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$C^{1,1/2}$ regularity

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Theorem ([$\mathcal{P}$.-To 2010])

If u is a solution of the obstacle problem with rooftop-like obstacle in Ω, then

$$u \in C^{1,1/2}_{\mathrm{loc}}(\Omega_\pm \cup \mathcal{M}).$$

- This is the best possible regularity. The function

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- The regularity is the same as in the thin obstacle problem [Athanasopoulos-Caffarelli 2006]

Normalization: class $\mathfrak{S}_M$

- Assume $\mathcal{M}$ is flat: $\mathcal{M} = \mathbb{R}^{n-1} \times \{0\}$.

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We say $u \in \mathfrak{S}_M$ if $\|u\|_{\mathrm{Lip}(B_1)} \le M$

- $\Delta u = f$ in $B_1^\pm$ with $\|f\|_{L^\infty(B_1)} \le M$
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- *Notation*: $\mathbb{R}^n_\pm = \{\pm x_n > 0\}, \quad B_1^\pm := B_1 \cap \mathbb{R}^n_\pm, \quad B_1' := B_1 \cap (\mathbb{R}^{n-1} \times \{0\})$

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If $u \in \mathfrak{S}_M$ *then there exists* $\alpha = \alpha_M \in (0,1)$ *and* $C_M > 0$ *such that*

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- Originally by [Caffarelli 1979] when $f = 0$ then by [Ural'tseva 1985] for bounded f.

Almgren's monotonicity of the frequency

Theorem (Monotonicity of the frequency, [Almgren 1979])

Let u be harmonic in B_1. Then the **frequency function**

$$r \mapsto N(r,u) := \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2} \nearrow \quad \textit{for} \quad 0 < r < 1.$$

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- [Athanasopoulos-Caffarelli-Salsa 2007] for the thin obstacle problem

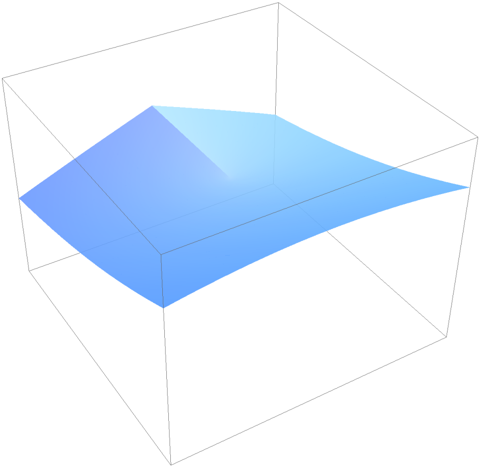

Figure: Solution of the thin obstacle problem $\mathrm{Re}(x_1 + i|x_2|)^{3/2}$

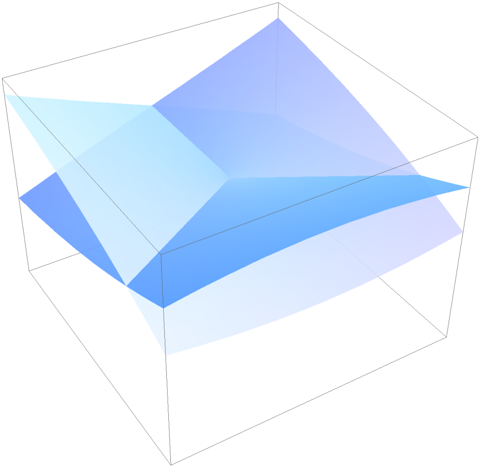

Figure: Multi-valued harmonic function $\mathrm{Re}(x_1 + ix_2)^{3/2}$

Truncated frequency function

Theorem (Monotonicity of truncated frequency, [𝒫.-To 2010])

Let $u \in \mathfrak{S}_M$. *Then for any* $\delta > 0$ *there exists* $C = C(M, \delta) > 0$ *such that*

$$r \mapsto \Phi(r, u) = re^{Cr^\delta} \frac{d}{dr} \log \max \left\{ \int_{\partial B_r} u^2, r^{n+3-2\delta} \right\} + 3(e^{Cr^\delta} - 1) \nearrow$$

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- Proof consists in estimating the error terms. The truncation of the growth is needed to absorb those terms. $C^{1,\alpha}$ regularity is used in an essential way.

Blowups at the origin

- Let $u \in \mathfrak{S}_M$ and for $r > 0$ consider the **rescalings**

$$u_r(x) := \frac{u(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{1/2}}, \quad f_r(x) := \frac{r^2 f(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{1/2}}$$

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- Moreover, if $r > 0$ is such that $\int_{\partial B_r} u^2 \geq r^{n+3-2\delta}$ (above truncation), then

$$|f_r(x)| \leq M r^{\delta} \to 0, \quad x \in B_{1/r}$$

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- Using the monotonicity of the truncated frequency it can be shown consequently that $\{u_r\}$ is uniformly bounded
$$W^{1,2}(B_1), \quad \mathrm{Lip}(B_{1/2}), \quad C^{1,\alpha}(B_{1/4})$$
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- Moreover, the degree of homogeneity κ of u_0 is such that
$$\Phi(0+,u) = n-1+2\kappa.$$

Proof of $C^{1,1/2}$ regularity

Lemma ([Athanasopoulos-Caffarelli 2000])

Let u_0 be a homogeneous global solution of the thin obstacle problem with homogeneity κ. Then $\kappa \geq 3/2$.

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- From Lemma we obtain that $\Phi(0+, u) = n - 1 + 2\kappa \geq n + 2$ for any $u \in \mathfrak{S}_M$.
- From here one can show that

$$\int_{\partial B_r} u^2 \leq C r^{n+2}, \quad 0 < r < 1$$

and consequently that

$$u \in C^{1,1/2}(B^{\pm}_{1/2} \cup B'_{1/2}).$$

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- In particular, this includes (locally) the parabolic obstacle problem with piecewise smooth rooftop-like obstacles with

$$f = \Delta\varphi \chi_{\{u=\varphi\}} \in L^\infty(\Omega_T).$$

Parabolic Signorini problem: known results

Theorem ([Ural'tseva 1985])

Let v be a solution of the Parabolic Signorini Problem with $\varphi \in C^{2,1}_{x,t}(\mathcal{M}_T)$, $\varphi_0 \in \mathrm{Lip}(\Omega_0)$, and $f \in L^\infty(\Omega_T)$. Then $\nabla v \in C^{\alpha,\alpha/2}_{x,t}(K)$ for any $K \Subset \Omega_T \cup \mathcal{M}_T$ and

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- This corresponds to $C^{1,\alpha}$ regularity of solutions in the elliptic case
- Our goal is to extend the optimal regularity result in the elliptic case to the time dependent case.

Parabolic Signorini problem: optimal regularity

Theorem ([Danielli-Garofalo-$\mathscr{P}$.-To 2011])

Let v be a solution of the Parabolic Signorini Problem with flat $\mathcal{M}$ and $\varphi \in C^{2,1}_{x,t}(\mathcal{M}_T)$, $\varphi_0 \in \mathrm{Lip}(\Omega_0)$, and $f \in L^\infty(\Omega_T)$. Then $\nabla v \in C^{1/2,1/4}_{x,t}(K)$ for any $K \Subset \Omega_T \cup \mathcal{M}_T$ and

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- This theorem is precise in the sense that it gives the same optimal regularity of $C^{1,1/2}$ in the time-independent case.

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- The optimal regularity in the elliptic case was obtained with the help of Almgren's Frequency Function. So we need a parabolic analogue of the frequency.

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Let u be a caloric function (solution of the heat equation) in the strip $S_R = \mathbb{R}^n \times (-R^2, 0]$. Then

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- Here $G(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$, $t > 0$ is the heat (Gaussian) kernel.

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- The new right-hand side F is nonzero even if $f \equiv 0$.

Averaged and truncated Poon's formula

- For the extended u define

$$h_u(t) = \int_{\mathbb{R}^n_+} u(x,t)^2 G(x,-t)dx$$

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$$N(r,u) = \frac{i_u(-r^2)}{h_u(-r^2)}.$$
- For our generalization, however, i_u and h_u are too irregular and we have to **average** them to regain missing regularity:
$$H_u(r) = \frac{1}{r^2}\int_{-r^2}^0 h_u(t)dt = \frac{1}{r^2}\int_{S_r^+} u(x,t)^2 G(x,-t)dxdt$$
$$I_u(r) = \frac{1}{r^2}\int_{-r^2}^0 i_u(t)dt = \frac{1}{r^2}\int_{S_r^+} |t||\nabla u(x,t)|^2 G(x,-t)dxdt$$

Averaged and truncated Poon's formula

Theorem ([Danielli-Garofalo-$\mathscr{P}$.-To 2011])

Let u be obtained from the solution of the Parabolic Signorini Problem in Q_1^+ *as described. Then for any* $\delta > 0$ *there exist C such that*

$$\Phi_u(r) = \frac{1}{2} r e^{Cr^\delta} \frac{d}{dr} \log \max\{H_u(r), r^{4-2\delta}\} + \frac{3}{2}(e^{Cr^\delta} - 1) \quad \nearrow$$

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- Using this generalized frequency formula, as well as an estimation on parabolic homogeneity of blowups we obtain the optimal regularity.

Rescalings and blowups

- As in the elliptic case, we consider the rescalings

$$u_r(x,t) = \frac{u(rx, r^2 t)}{H_u(r)^{1/2}}, \quad F_r(x,t) = \frac{r^2 F(rx, r^2 t)}{H_u(r)^{1/2}},$$

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- If $\Phi_u(0+) < 4 - 2\delta$ then one can show that the family $\{u_r\}$ is convergent in suitable sense on $\mathbb{R}^n_+ \times (-\infty, 0]$ to a parabolically homogeneous solution u_0 of the Parabolic Signorini Problem

$$\Delta u_0 - \partial_t u_0 = 0 \quad \text{in } \mathbb{R}^n_+ \times (-\infty, 0]$$
$$u_0 \geq 0, \quad -\partial_{x_n} u_0 \geq 0, \quad u_0 \partial_{x_n} u_0 = 0 \quad \text{on } \mathbb{R}^{n-1} \times (-\infty, 0]$$

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- Parabolic homogeneity u_0 is $\kappa = \frac{1}{2}\Phi_0(0+) < 2 - \delta < 2$. Besides, because of $C^{1,\alpha}$-regularity, also $\kappa \geq 1 + \alpha > 1$. Thus:

$$1 < \kappa < 2.$$

Homogeneous global solutions

Lemma ([Danielli-Garofalo-$\mathscr{P}$.-To 2011])

Let u_0 be a parabolically homogeneous solution of the Parabolic Signorini Problem in $\mathbb{R}^n_+ \times (-\infty, 0]$ with homogeneity $1 < \kappa < 2$. Then necessarily $\kappa = 3/2$ and

$$u_0(x,t) = C\,\mathrm{Re}(x_1 + i x_n)^{3/2},$$

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- The proof is based on a rather deep monotonicity formula of Caffarelli to reduce it to dimension $n = 2$ and then analysing of the principal eigenvalues of the Ornstein-Uhlenbeck operator $-\Delta + \frac{1}{2}x \cdot \nabla$ in $\mathbb{R}^2$ for the slit planes

$$\Omega_a := \mathbb{R}^2 \setminus ((-\infty, a] \times \{0\}).$$

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- This further implies that

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  for any $(x_0, t_0) \in Q'_{1/2}$ such that $u(x_0, t_0) = 0$.
- Using interior parabolic estimates one then obtains

$$\nabla u \in C^{1/2,1/4}_{x,t}(Q^+_{1/4}).$$