Almost Minimizers for the Thin Obstacle Problem

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Table of contents

The Thin Obstacle Problem

The thin obstacle (Signorini) problem

- Given

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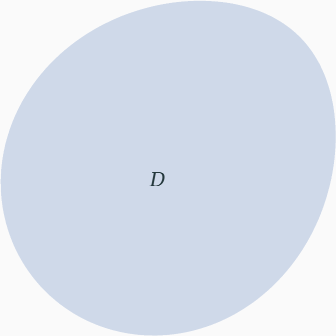

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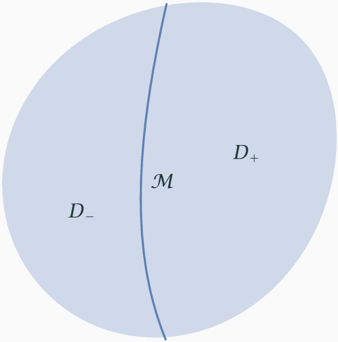

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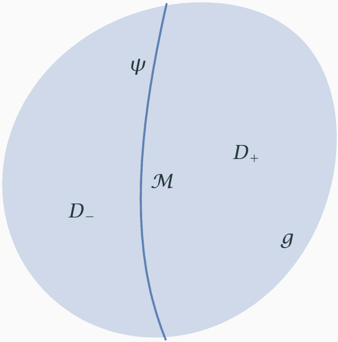

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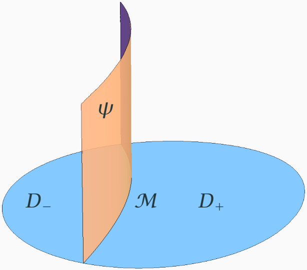

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$$\mathcal{J}_D(u) := \int_D |\nabla u|^2 dx$$

on the closed convex set

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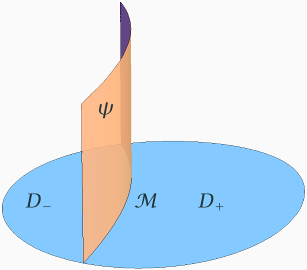

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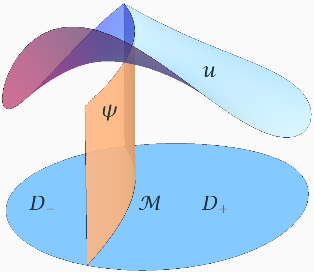

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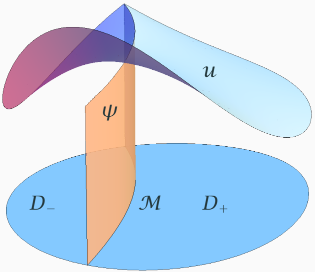

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- *Signorini (complementarity) conditions* on $\mathcal{M}$

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u - \psi &\geq 0 \\
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(u - \psi)(\partial_{\nu^+} u + \partial_{\nu^-} u) &= 0
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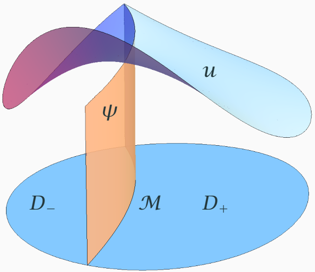

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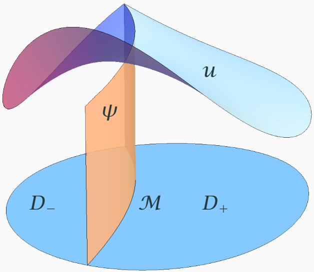

- Main objectives of study
  - Regularity of u
  - Structure and regularity of the **free boundary**

$$\Gamma(u) \coloneqq \partial_{\mathcal{M}}\{x \in \mathcal{M} \mid u = \psi\}$$

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- Obstacle problem for the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$

$$u - \phi \geq 0, \quad (-\Delta)^s u \geq 0, \quad (u - \phi)(-\Delta)^s u = 0 \quad \text{in } \mathbb{R}^n.$$

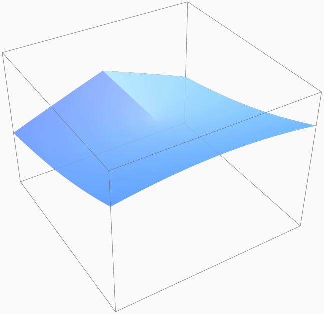
The thin obstacle problem corresponds to $s = 1/2$.

Regularity of the minimizer u

- Generally, it is easy to realize that u is not smooth in D, as its graph may develop a Lipschitz corner across $\mathcal{M}$.

  Explicit example: $\mathcal{M} = \{x_n = 0\}$, $\psi = 0$,

$$u(x) = \mathrm{Re}(x_{n-1} + i|x_n|)^{3/2}$$



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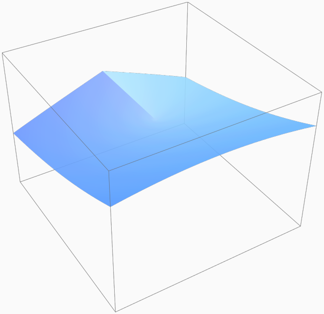

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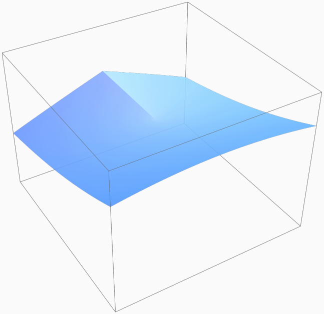

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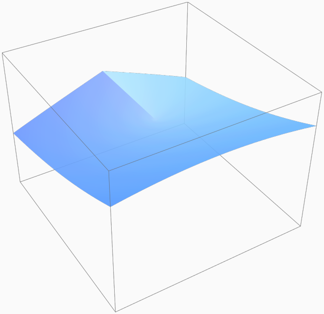

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  - [ATHANASOPOULOS-CAFFARELLI-SALSA'08] ($\mathcal{M}$ flat, nonzero ψ)
  - [GUILLEN'09], [GAROFALO-SMIT VEGA GARCIA'14] (nonflat $\mathcal{M}$)
  - [CAFFARELLI-SILVESTRE-SALSA'09] (fractional obstacle problem)
  - [DANIELLI-GAROFALO-P.-TO'17] (parabolic case)
  - ...

Structure of the free boundary $\Gamma(u)$

- Assume for simplicity $\mathcal{M} = \{x_n = 0\}$, $\psi = 0$, D symmetric w.r.t. $\mathcal{M}$, and $u(x', -x_n) = u(x', x_n)$

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$$r \mapsto N(r) = N(r, u, x_0) := \frac{r \int_{B_r(x_0)} |\nabla u|^2}{\int_{\partial B_r(x_0)} u^2} \nearrow, \quad 0 < r < \operatorname{dist}(x_0, \partial D)$$

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- Another characterization with *Almgren blowups*:

$$u^A_{x_0,0} = \lim_{r = r_j \to 0} u^A_{x_0,r}, \quad u^A_{x_0,r}(x) \coloneqq \frac{u(x_0 + rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2 \right)^{1/2}}$$

  are homogeneous of degree $\kappa = \kappa(x_0)$: $u^A_{x_0,0}(\lambda x) = \lambda^\kappa u^A_{x_0,0}(x)$, $\lambda > 0$.

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- $\Gamma_{3/2}(u)$ is called the *regular set*.

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- $\Sigma(u) \coloneqq \bigcup_{m=1}^{\infty} \Gamma_{2m}(u)$ is called *singular set*.

$$x_0 \in \Sigma(u) \iff \lim_{r \to 0} \frac{|\{u(\cdot, 0) = 0\} \cap B'_r(x_0)|}{|B'_r(x_0)|} = 0 \iff u^A_{x_0,0} \text{ is polynomial.}$$

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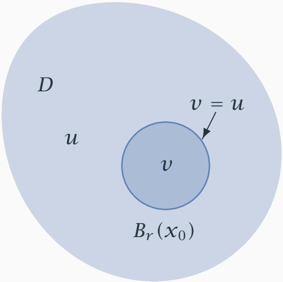

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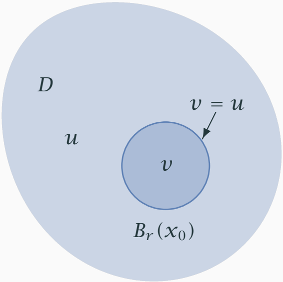

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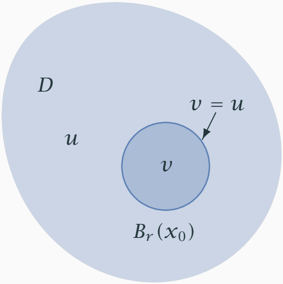

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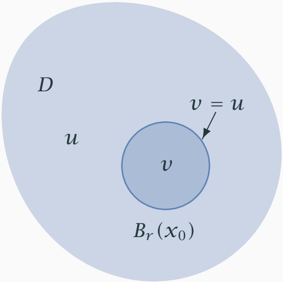

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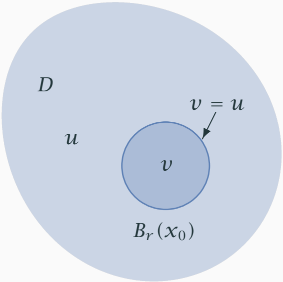

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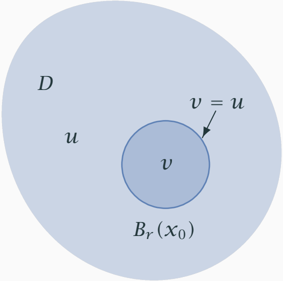

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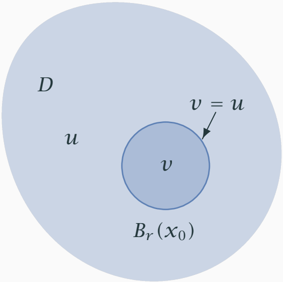

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- Similar notions were used in Geometric Measure Theory e.g. by [ALMGREN'76], [BOMBIERI'82].

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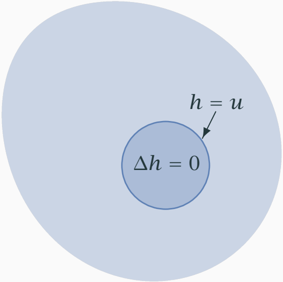

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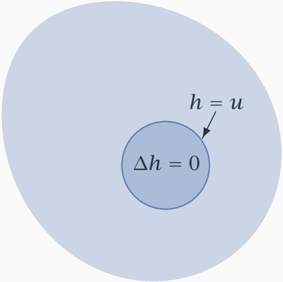

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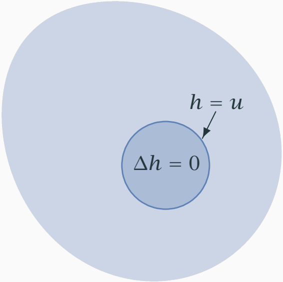

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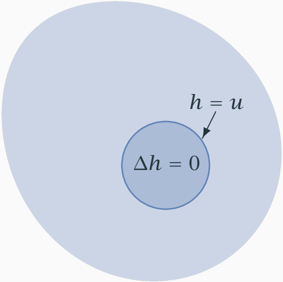

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- This allows to treat almost minimizers as perturbations of harmonic functions.

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If u is an almost minimizer with $\omega(r) = r^\alpha$, $0 < \alpha < 2$, then $u \in C^{1,\alpha/2}(D)$.

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Almost Minimizers for the Thin Obstacle Problem

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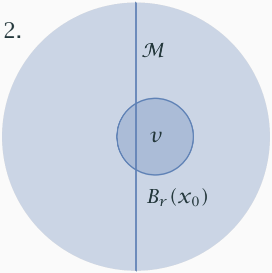

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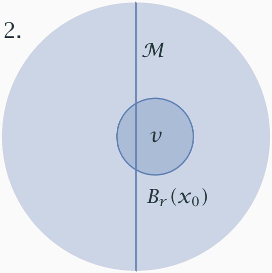

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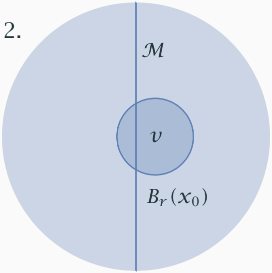

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- Almost minimizer were considered earlier also for other free boundary problems: Alt-Caffarelli-Friedman type energy functional

$$\int_D |\nabla u|^2 + q_+(x)\chi_{\{u>0\}} + q_-(x)\chi_{\{u<0\}}$$

by [David-Toro'15], [David-Engelstein-Toro'17], [DeSilva-Savin'19], and its "thin" one-phase counterpart [DeSilva-Savin'18]

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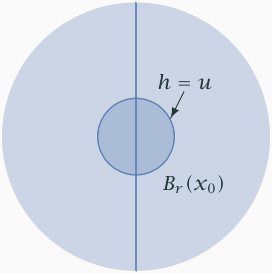

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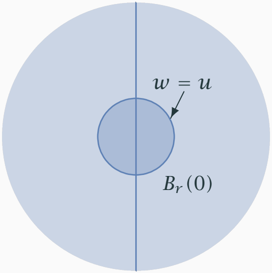

- ***κ*-homogeneous replacement:** for $x_0 = 0$ and $r > 0$ small, replace u in B_r with a κ-homogeneous function

$$w(x) = \left(\frac{|x|}{r}\right)^{\kappa} u\left(r\frac{x}{|x|}\right).$$

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- Hereafter u is an almost minimizer for the thin obstacle problem.
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which is sub-mean value for subharmonic function $|\nabla h|^2$.

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Theorem (JEON-P.'19)

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Monotonicity Formulas

Weiss-type monotonicity formula

- As we saw, for solutions h of the Signorini problem, *Almgren's frequency formula*

$$N(r, h, x_0) = \frac{r \int_{B_r(x_0)} |\nabla h|^2}{\int_{\partial B_r(x_0)} h^2}$$

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  which also tells about the case of equality.

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For $x_0 \in \Gamma(u) \cap B_{1/2}$, $0 < \kappa < \kappa_0$, *there are* $a_\kappa = a(\kappa)$ *and* $b = b(\kappa_0)$ *s.t.*

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- With algebraic manipulations, we get

$$\frac{d}{dr} W_\kappa(r) \geq \frac{e^{a_\kappa r^\alpha}}{r^{n+2\kappa-2}} \int_{\partial B_r} \left(\partial_\nu u - \frac{\kappa(1 - br^\alpha)}{r} u\right)^2, \quad 0 < r < r_0(\alpha, n, \kappa_0).$$

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- Then from monotonicity of W_κ

$$s < r < r_0 \quad \Rightarrow \quad W_\kappa(s) \leq W_\kappa(r) < 0 \quad \Rightarrow \quad \frac{N(s)}{1 - bs^\alpha} < \kappa.$$

Analysis of the Free Boundary

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  3. Now, if $N(0+, u) \geq \kappa$, without further assumptions, we can conclude only
  $$\int_{\partial B_r} u^2 \leq C\left(\log\frac{1}{r}\right)r^{n+2\kappa-1}$$
  and cannot really say if
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$$\frac{d}{dr}\|u_r^\phi\|_{L^2(\partial B_1)} \le \frac{e^{cr^\alpha}}{r^{1/2}}\left(\frac{d}{dr}W_\kappa(r)\right)^{1/2}$$
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$$\left|\|u_r^\phi\|_{L^2(\partial B_1)} - \|u_s^\phi\|_{L^2(\partial B_1)}\right| \le C\left(\log\frac{r}{s}\right)^{1/2}[W_\kappa(r)-W_\kappa(s)]^{1/2}, \quad 0<s<r$$
  2. Similarly,
$$\int_{\partial B_1}|u_r^\phi - u_s^\phi| \le C\left(\log\frac{r}{s}\right)^{1/2}[W_\kappa(r)-W_\kappa(s)]^{1/2}, \quad 0<s<r$$
  3. However, if we know
$$W_\kappa(r)\le Cr^\delta, \quad \text{for some } \delta>0$$
then by a *dyadic* argument we can get rid of log
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  3. Or, we if know
  $$W_\kappa(r) \leq C\left(\log\frac{1}{r}\right)^{-\delta}, \quad \text{for some } \delta > 1,$$
  then by an *exponential dyadic* argument we can also get rid of log
  $$\int_{\partial B_r} u^2 \leq Cr^{n+2\kappa-1}$$
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Epiperimetric inequality, $\kappa = 3/2$

Theorem (GAROFALO-SMIT VEGA GARCIA-P.'16, FOCARDI-SPADARO'16)

If $w \in W^{1,2}(B_1)$, 3/2-homogeneous, $w \geq 0$ on B_1' and h solves the Signorini problem with boundary values w, then for some $\eta = \eta(n) > 0$

$$W^0_{3/2}(h) \leq (1 - \eta) W^0_{3/2}(w).$$

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- Thus, we have the control of rescalings!

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- **Regular set** of the free boundary is defined as the set where the homogeneity $\kappa(x_0) = 3/2$:

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$$x_0 \mapsto u^{\phi}_{x_0,0} = a_{x_0} \operatorname{Re}(x \cdot \nu_{x_0} + i x_n)^{3/2} \quad \text{is } C^{\gamma} \text{ on } \Gamma_{3/2}$$

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- Then $x_0 \mapsto \nu_{x_0}$ is C^{γ}, implying that $\Gamma_{3/2}$ is locally $C^{1,\gamma}$.

Log epiperimetric inequality, $\kappa = 2m$, $m \in \mathbb{N}$

Theorem (COLOMBO-SPOLAOR-VELICHKOV'17)

If $w \in W^{1,2}(B_1)$, κ-homogeneous, $w \geq 0$ on B_1', $\int_{\partial B_1} w^2 \leq 1$, $|W_\kappa^0(w)| \leq 1$, there $\varepsilon = \varepsilon(n,\kappa) > 0$ such that if h solves the Signorini problem with $h = w$ on ∂B_1 then

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$$\int_{\partial B_r} u^2 \leq C\left(\log\frac{1}{r}\right)^\sigma r^{n+2\kappa-1} \quad \Rightarrow \quad W_\kappa(r) \leq C\left(\log\frac{1}{r}\right)^{\sigma - \frac{n}{n-2}}$$

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- In the last step of iteration we arrive at

$$W_\kappa(r) \leq C \left(\log \frac{1}{r}\right)^{-\frac{n}{n-2}}$$

and thus, we have the control of rescalings!

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- For minimizers, this goes back to [Garofalo-P.'09] with C^1 manifolds, and to [Colombo-Spolaor-Velichkov'17] with manifolds of class $C^{1,\log}$.

Thank you!

Stay safe!