Almost monotonicity formulas for elliptic and parabolic operators with variable coefficients

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Original Elliptic Monotonicity Formula

Theorem (Alt-Caffarelli-Friedman 1984)

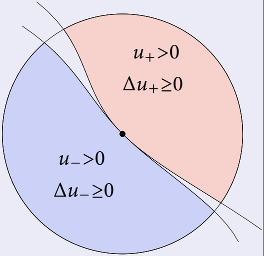
Let $u_\pm$ be two continuous functions in B_1 in $\mathbb{R}^n$ such that

$$u_\pm \geq 0, \quad \Delta u_\pm \geq 0, \quad u_+ \cdot u_- = 0 \quad in \quad B_1$$

then the functional

$$\varphi(r) = \varphi(r, u_+, u_-) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u_+|^2}{|x|^{n-2}} dx \int_{B_r} \frac{|\nabla u_-|^2}{|x|^{n-2}} dx$$

is monotone nondecreasing in $r \in (0, 1]$.



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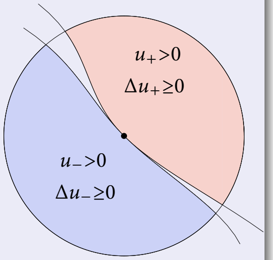

- This formula has been of fundamental importance in the regularity theory of free boundaries, especially in problems with two phases.

Original Elliptic Monotonicity Formula

- One of the applications of the monotonicity formula is the ability to produce estimates of the type

$$c_n|\nabla u_+(0)|^2|\nabla u_-(0)|^2 \le \varphi(0+) \le \varphi(1/2) \le C_n\|u_+\|^2_{L^2(B_1)}\|u_-\|^2_{L^2(B_1)}$$

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- For $\Sigma \subset \partial B_1$ define

$$\lambda(\Sigma) = \inf \frac{\int_\Sigma |\nabla_\theta f|^2}{\int_\Sigma f^2}, \quad f|_{\partial\Sigma} = 0$$

Define also $\alpha(\Sigma)$ so that $\lambda(\Sigma) = \alpha(\Sigma)(n-2+\alpha(\Sigma))$.

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Theorem (Friedland-Hayman 1976)

Let $\Sigma_\pm$ be disjoint open sets on ∂B_1. Then

$$\alpha(\Sigma_+) + \alpha(\Sigma_-) \geq 2.$$

Parabolic Monotonicity Formula

Theorem (Caffarelli 1993)

Let $u_\pm(x,s)$ *be two continuous functions in* $S_1 = \mathbb{R}^n \times (-1,0]$

$$u_\pm \geq 0, \quad (\Delta - \partial_s) u_\pm \geq 0, \quad u_+ \cdot u_- = 0 \quad in \quad S_1$$

then

$$\Phi(r, u_+, u_-) = \frac{1}{r^4} \int_{-r^2}^0 \int_{\mathbb{R}^n} |\nabla u_+|^2 G(x,-s)\,dx\,ds \int_{-r^2}^0 \int_{\mathbb{R}^n} |\nabla u_-|^2 G(x,-s)\,dx\,ds$$

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- Note that $u_\pm$ must be defined in an entire strip and we must have a moderate growth at infinity.

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Theorem (Beckner-Kenig-Pipher)

Let $\Omega_\pm$ be two disjoint open sets in $\mathbb{R}^n$. Then

$$\lambda(\Omega_+) + \lambda(\Omega_-) \geq 2$$

- The proof is reduced to the convexity result of Brascamp-Lieb 1976 for first eigenvalues of $-\Delta + V(x)$ with convex potential V as a function of the domain.

Localized Parabolic Formula

Theorem (Caffarelli 1993)

Let $u_\pm(x,s)$ be two continuous functions in $Q_1^- = B_1 \times (-1,0]$ such that

$$u_\pm \geq 0, \quad (\Delta - \partial_s)u_\pm \geq 0, \quad u_+ \cdot u_- = 0 \quad in \quad Q_1^-.$$

Let $\psi \in C_0^\infty(B_1)$ be a cutoff function such that

$$0 \leq \psi \leq 1, \quad \operatorname{supp}\psi \subset B_{3/4}, \quad \psi|_{B_{1/2}} = 1$$

then $\Phi(r) = \Phi(r, u_+\psi, u_-\psi)$ is **almost monotone** *in a sense that*

$$\Phi(0+) - \Phi(r) \leq Ce^{-c/r^2}\|u_+\|^2_{L^2(Q_1^-)}\|u_-\|^2_{L^2(Q_1^-)}.$$

Generalization: Caffarelli-Kenig Estimate

- Instead of the heat operator $\Delta - \partial_s$ consider now uniformly parabolic

$$\mathscr{L}u = \mathscr{L}_{\mathscr{A},b,c}u := \operatorname{div}(\mathscr{A}(x,s)\nabla u) + b(x,s)\cdot\nabla u + c(x,s)u - \partial_s u$$

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Theorem (Caffarelli-Kenig 1998)

Let $u_\pm(x,s)$ be two continuous functions in Q_1^- such that

$$u_\pm \geq 0, \quad \mathcal{L}u_\pm \geq 0, \quad u_+\cdot u_- = 0 \quad in \quad Q_1^-.$$

Let $\psi \in C_0^\infty(B_1)$ be a cutoff function as before. Then $\Phi(r) = \Phi(r, u_+\psi, u_-\psi)$ is **almost monotone** *in a sense that we have an estimate*

$$\Phi(r) \leq C_0 \left(\|u_+\|^2_{L^2(Q_1^-)} + \|u_-\|^2_{L^2(Q_1^-)} \right)^2, \quad r < r_0.$$

Generalization: Caffarelli-Jerison-Kenig Estimate

Theorem (Caffarelli-Jerison-Kenig 2002)

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- The proof is based on a sophisticated iteration scheme.
- The difficulties in CJK and CK estimates are of completely different nature
- The proof can be easily generalized to parabolic case (Edquist-$\mathscr{P}$ 2008).

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- However, we still have an estimate of the type

$$\varphi(0+) \le C\left(\|u_+\|_{L^2(B_1)}, \|u_-\|_{L^2(B_1)}\right)$$

which is able to produce an estimate

$$|\nabla u_+(0)||\nabla u_-(0)| \le C\left(\|u_+\|_{L^2(B_1)}, \|u_-\|_{L^2(B_1)}\right).$$

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  This is crucial in proving the optimal regularity in certain two-phase problems (and not only!)
- Under certain growth assumptions on u, such as $|u(x)| \le C|x|^\epsilon$ one can show the existence of $\varphi(0+)$. This is important in classification of blowup solutions.

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- Namely, do we have an almost monotonicity estimate for $u_\pm$ satisfying

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- We will see that the answer is positive when $\mathscr{A}$ is double Dini and b, c are uniformly bounded.

Main Results: Assumptions

- We consider the uniformly parabolic operator

$$\mathscr{L}_{\mathscr{A},b,c}u := \operatorname{div}(\mathscr{A}(x,s)\nabla u) + b(x,s)\cdot\nabla u + c(x,s)u - \partial_s u$$

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1. $\lambda|\xi|^2 \leq \mathcal{A}(x,s)\xi\cdot\xi \leq \frac{1}{\lambda}|\xi|^2$
2. $\|\mathcal{A}(x,s) - \mathcal{A}(0,0)\| \leq \omega\left((|x|^2+s)^{1/2}\right)$ with double Dini ω:

$$\int_0^1 \frac{1}{r}\int_0^r \frac{\omega(\rho)}{\rho}\,d\rho dr = \int_0^1 \frac{\omega(\rho)\log\frac{1}{\rho}}{\rho}\,d\rho < \infty$$

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- We make similar assumption on the uniformly elliptic operator

$$\ell_{\mathscr{A},b,c}u := \operatorname{div}(\mathscr{A}(x)\nabla u) + b(x)\cdot\nabla u + c(x)u$$

Global Parabolic Formula

Theorem (Matevosyan-$\mathscr{P}$ 2009)

Let $u_\pm(x,s)$ be two continuous functions in S_1 such that

$$u_\pm \geq 0, \quad \mathscr{L}_{\mathscr{A},b,c} u_\pm \geq -1, \quad u_+ \cdot u_- = 0 \quad \text{in} \quad S_1$$

Assume also that $u_\pm$ have moderate growth at infinity, so that

$$M_\pm^2 := \iint_{S_1} u_\pm(x,s)^2 e^{-x^2/32} dx ds < \infty.$$

Then the functional $\Phi(r) = \Phi(r, u_+, u_-)$ satisfies

$$\Phi(r) \leq C_\omega (1 + M_+^2 + M_-^2)^2, \quad \text{for } 0 < r \leq r_\omega.$$

Localized Parabolic Formula

Theorem (Matevosyan-$\mathscr{P}$ 2009)

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Elliptic Formula

Theorem (Matevosyan-$\mathscr{P}$ 2009)

Let $u_\pm(x)$ be two continuous functions in B_1 such that

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Proof: CJK Iteration Scheme for $\mathscr{L} = \Delta - \partial_s$

- Let $A^{\pm}(r) = \iint_{S_r} |\nabla u|^2 G(x, -s)\, dx\, ds, \quad S_r = \mathbb{R}^n \times (-r^2, 0]$

Proof: CJK Iteration Scheme for $\mathscr{L} = \Delta - \partial_s$

- Let $A^{\pm}(r) = \iint_{S_r} |\nabla u|^2 G(x,-s)\,dx\,ds, \quad S_r = \mathbb{R}^n \times (-r^2, 0]$
- Define $A_k^{\pm} = A^{\pm}(4^{-k})$, $b_k^{\pm} = 4^{4k} A_k^{\pm}$. Then $\Phi(4^{-k}) = 4^{4k} A_k^{+} A_k^{-}$.

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Proposition

There exists C_0 such that if $b_k^{\pm} \geq C_0$ then

$$4^4 A_{k+1}^{+} A_{k+1}^{-} \leq A_k^{+} A_k^{-} (1 + \delta_k) \quad \textit{with} \quad \delta_k = \frac{C_0}{\sqrt{b_k^{+}}} + \frac{C_0}{\sqrt{b_k^{-}}}.$$

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Proposition

There exists C_0 such that if $b_k^{\pm} \geq C_0$ and $4^4 A_{k+1}^+ \geq A_k^+$ then

$$A_{k+1}^- \leq (1 - \epsilon_0) A_k^-.$$

Proof: CJK Iteration Scheme for $\mathscr{L}_{\mathscr{A},b,c}$

Define

- $\theta(r) = Cr + \omega(r^{1/2}) + \left(\int_0^{r^2} \frac{\omega(\rho^{1/4})^2}{\rho} d\rho\right)^{1/2}$

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- $\widetilde{A}^{\pm}(r) = e^{c_0 g(r)} A^{\pm}(r), \quad \widetilde{\Phi}(r) = r^{-4} \widetilde{A}^{+}(r) \widetilde{A}^{-}(r)$

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Proposition

$\widetilde{A}_k^{\pm}$ satisfy the same iterative inequalities as $A_k^{\pm}$ in the case of $\mathscr{L} = \Delta - \partial_s$.

Proof: Key Technical Estimate

- Normalize $\mathscr{A}(0,0) = I$, $c = 0$.

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Proposition

Let $u \geq 0$ *satisfy* $\mathscr{L}_{\mathscr{A},b,0} u \geq -1$ *in* S_1. *Suppose also* $\iint_{S_1} u(x,s)^2 e^{-x^2/32} dx ds \leq 1$. *Then*

$$(1 - c_n\,\theta(r)) \iint_{S_r} |\nabla u|^2 G(x,-s)\,dx ds \leq$$

$$C_0 r^4 + C_n r^2 \left(\int_{\mathbb{R}^n} u(x,-r^2)^2 G(x,r^2)\,dx \right)^{1/2} + \frac{1}{2} \int_{\mathbb{R}^n} u(x,-r^2)^2 G(x,r^2)\,dx$$

for any $0 < r \leq r_\omega$, *where*

$$\theta(r) = Cr + \omega(r^{1/2}) + \left(\int_0^{r^2} \frac{\omega(\rho^{1/4})^2}{\rho} d\rho \right)^{1/2}.$$

Proof: Parabolic ⇒ Elliptic

- Add a "dummy" variable s

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- $\widetilde{u}_\pm$ satisfy now conditions of localized parabolic case with

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for $r < r_\omega$.

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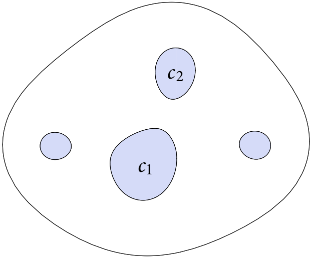
- Let u be a solution of the system in B_1

$$\begin{aligned} \operatorname{div}(a(|\nabla u|^2)\nabla u) &= f(x,u,\nabla u)\chi_\Omega, \\ |\nabla u| = 0 \quad &\text{on } \Omega^c, \end{aligned}$$

where Ω is an apriori unknown open set.



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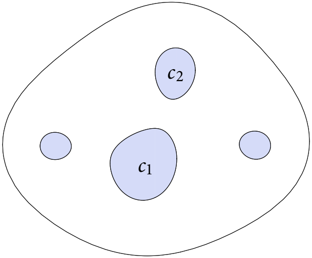

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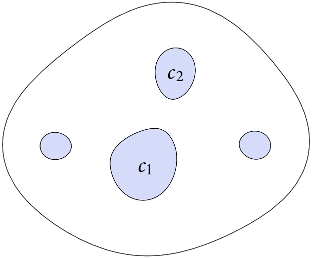

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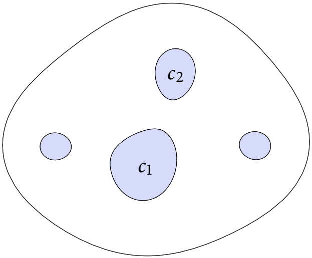

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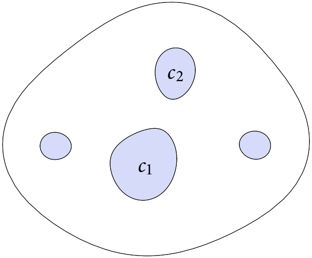

- Similar problem has been studied by Caffarelli-Salazar-Shahgholian 2004

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Theorem (Matevosyan-$\mathscr{P}$ 2009)

Under conditions above, $u \in C^{1,1}_{\text{loc}}(B_1)$ *and*

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- Generalizes a theorem of Shahgholian 2003 for

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Lemma

For any direction e the functions $w_\pm = (\partial_e u)^\pm = \max\{\pm\partial_e u, 0\}$ *satisfy*

$$w_\pm \geq 0, \quad \operatorname{div}(\mathcal{A}(x)\nabla w_\pm) + b(x)\nabla w_\pm + c(x)w_\pm \geq -M, \quad w_+ \cdot w_- = 0,$$

where

$$\begin{aligned} \mathcal{A}(x) &= a(|\nabla u(x)|^2)I + 2a'(|\nabla u(x)|^2)\nabla u(x) \otimes \nabla u(x), \\ b(x) &= -(\nabla_p f)(x, u(x), \nabla u(x)), \\ c(x) &= -(\partial_z f)(x, u(x), \nabla u(x)). \end{aligned}$$

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- $u \in W^{2,p}$, $p > n$, hence twice differentiable a.e.

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- take $e \perp \nabla u(x_0)$ and apply almost monotonicity formula to $w_\pm = (\partial_e u)^\pm$:

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- to obtain the estimate in the missing direction $e \parallel \nabla u(x_0)$, we use the equation.

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Let $u_\pm$ *satisfy* $u_\pm \geq 0$, $\mathcal{L}_{\mathcal{A},b,c} u_\pm \geq -1$, $u_+ \cdot u_- = 0$ *in* S_1, *and*

$$u_\pm(x,s) \leq \sigma((|x|^2 + |s|)^{1/2}) \quad \text{for } (x,s) \in Q_1^-$$

for a Dini modulus of continuity $\sigma(r)$. *Then* $\Phi(r) = \Phi(r, u_+\psi, u_-\psi)$ *satisfies*

$$\Phi(r) \leq [1 + \alpha(\rho)]\Phi(\rho) + C_{M,\psi,\sigma,\omega}\alpha(\rho), \quad 0 < r \leq \rho \leq r_\omega,$$

where $\alpha(r) = C_0 \left[r + \sigma(r^{1/2}) + \int_0^r \frac{\sigma(\rho^{1/2})}{\rho} d\rho + \int_0^r \frac{\theta(\rho)}{\rho} d\rho \right]$ *and*
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- This guaranties the existence of $\Phi(0+) = \lim_{r \to 0+} \Phi(r)$.

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- Problem is reduced to analyzing the case of equality for the original Alt-Caffarelli-Friedman montonicity formula (Caffarelli-Karp-Shahgholian 2000)