Monotonicity formulas and the singular set in the thin obstacle problem

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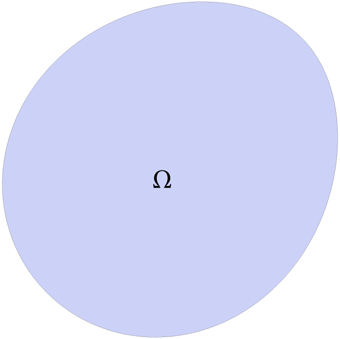

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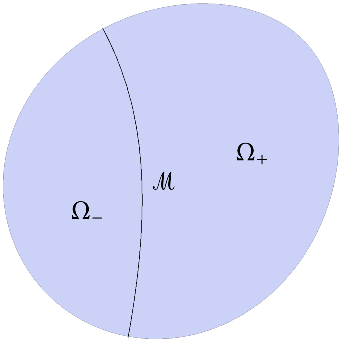

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    $g : \partial\Omega \to \mathbb{R}$ (**boundary values**),
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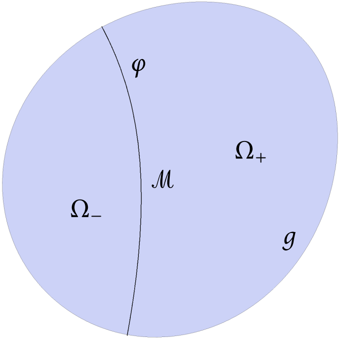

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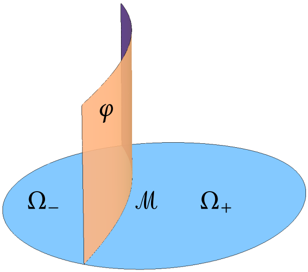

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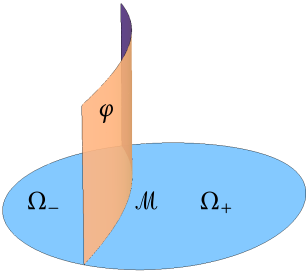

- Minimize the Dirichlet integral

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on the closed convex set

$$\mathfrak{K} = \{u \in W^{1,2}(\Omega) \mid u = g \text{ on } \partial\Omega, u \geq \varphi \text{ on } \mathcal{M} \cap \Omega\}.$$

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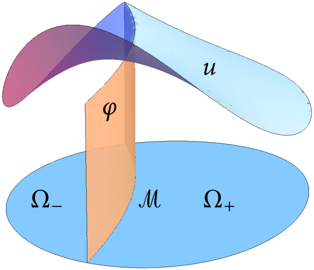

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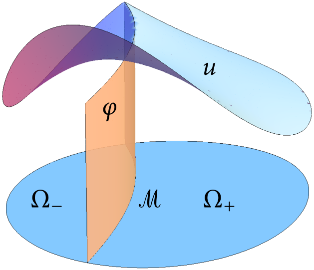

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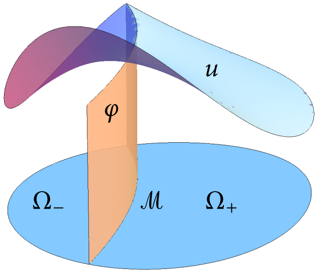

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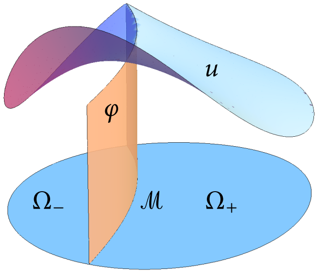

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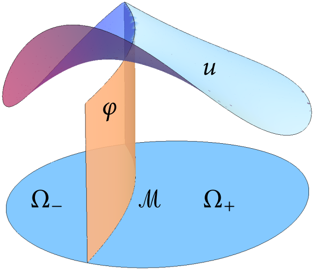

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- Main objects of study

$$
\begin{aligned}
\textit{Coincidence set}: &\quad \Lambda(u) := \{x \in \mathcal{M} \mid u = \varphi\} \\
\textit{Free Boundary}: &\quad \Gamma(u) := \partial_{\mathcal{M}} \Lambda(u)
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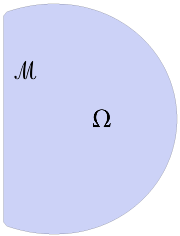

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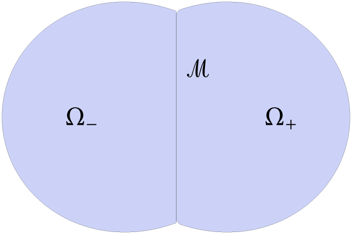

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- It models the flow of a saline concentration through a semipermeable membrane when the flow occurs in a preferred direction.
- It also arises in financial mathematics in situations in which the random variation of an underlying asset changes discontinuously.
- Obstacle problem for the fractional Laplacian $(-\Delta)^s$, $0 < s < 1$

$$u - \varphi \geq 0, \quad (-\Delta)^s u \geq 0, \quad (u - \varphi)(-\Delta)^s u \geq 0 \quad \text{in } \mathbb{R}^{n-1}.$$

The thin obstacle problem corresponds to $s = \frac{1}{2}$.

Part I

Zero thin obstacle: $\varphi = 0$

Outline

Normalization: class $\mathfrak{S}$

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Definition

We say u is a **normalized solution** of Signorini problem iff

$$\Delta u = 0 \quad \text{in } B_1^+$$
$$u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\,\partial_{x_n} u = 0 \quad \text{on } B_1'$$
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- *Notation*: $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, +\infty)$, $\quad B_1^+ := B_1 \cap \mathbb{R}^n_+$, $\quad B_1' := B_1 \cap (\mathbb{R}^{n-1} \times \{0\})$

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- More specifically:

$$\Delta u = 2(\partial_{x_n} u)\,\mathcal{H}^{n-1}\big|_{\Lambda(u)} \quad \text{in } \mathcal{D}'(B_1).$$

Rescalings and blowups

- For $u \in \mathfrak{S}$ and $r > 0$ consider **rescalings**

$$u_r(x) := \frac{u(rx)}{\left(\frac{1}{r^{n-1}} \int_{\partial B_r} u^2\right)^{\frac{1}{2}}}.$$

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- Limits of subsequences $\{u_{r_j}\}$ for some $r_j \to 0+$ are known as **blowups**.
- Generally the blowups may be different over different subsequences $r = r_j \to 0+$.

Almgren's frequency function

Theorem (Monotonicity of the frequency)

Let $u \in \mathfrak{S}$. Then the **frequency function**

$$r \mapsto N(r,u) := \frac{r\int_{B_r} |\nabla u|^2}{\int_{\partial B_r} u^2} \nearrow \quad \textit{for} \quad 0 < r < 1.$$

Moreover, $N(r,u) \equiv \kappa \iff x\cdot\nabla u - \kappa u = 0$ *in* B_1*, i.e.* u *is homogeneous of degree* κ *in* B_1*.*

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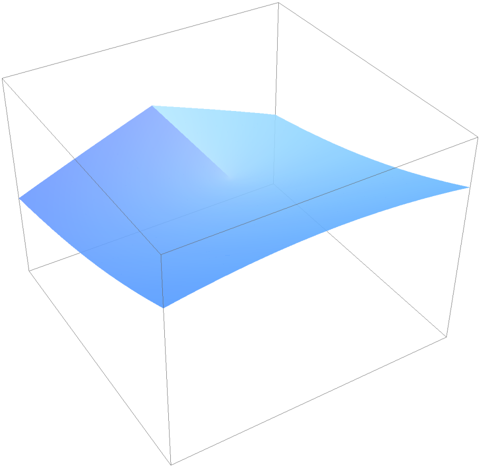

Figure: Solution of the thin obstacle problem $\mathrm{Re}(x_1 + i|x_2|)^{3/2}$

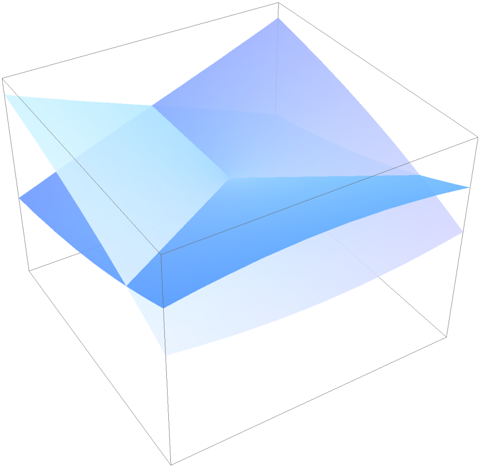

Figure: Multi-valued harmonic function $\mathrm{Re}(x_1 + ix_2)^{3/2}$

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Proof.

$N(r, u_0) = \lim_{r_j \to 0+} N(r, u_{r_j}) = \lim_{r_j \to 0+} N(rr_j, u) = N(0+, u)$ □

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- Achieved on $\hat{u}_{3/2}(x) = \mathrm{Re}(x_1 + i|x_n|)^{\frac{3}{2}}$

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- On the other hand, $0 \in \Gamma_\kappa(\hat{u}_\kappa)$ for

$$\hat{u}_\kappa(x) := \mathrm{Re}(x_1 + i|x_n|)^\kappa, \quad \kappa = 2 - \tfrac{1}{2}, 2, \ldots, 2m - \tfrac{1}{2}, 2m, \ldots$$

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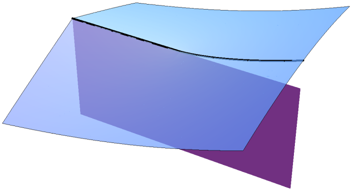
$$\hat{u}_\kappa(x) := \mathrm{Re}(x_1 + i|x_n|)^\kappa, \quad \kappa = 2 - \tfrac{1}{2}, 2, \ldots, 2m - \tfrac{1}{2}, 2m, \ldots$$

- In dimension 2, these are the **only possible values** of κ.
  Not known in higher dimensions.



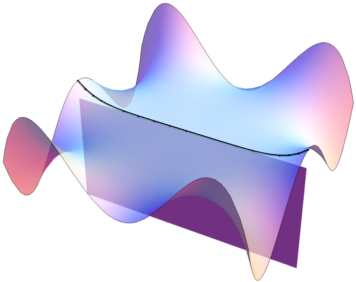


Figure: Graphs of $\mathrm{Re}(x_1 + i|x_2|)^{\frac{3}{2}}$ and $\mathrm{Re}(x_1 + i|x_2|)^6$

Regular free boundary points

- Of special interest is the case of the smallest possible value $\kappa = 2 - \frac{1}{2}$.

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Definition (Regular points)

For $u \in \mathfrak{S}$ we say that $x_0 \in \Gamma(u)$ is **regular** if $N^{x_0}(0+, u) = 2 - \frac{1}{2}$, i.e., if $x_0 \in \Gamma_{2-\frac{1}{2}}(u)$.

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- The following result was proved by [Athanasopoulos-Caffarelli-Salsa 2007].

Theorem (Regularity of the regular set)

Let $u \in \mathfrak{S}$, then the free boundary $\Gamma_{2-\frac{1}{2}}(u)$ is locally a $C^{1,\alpha}$ regular $(n-2)$-dimensional surface.

Singular free boundary points

Definition (Singular points)

Let $u \in \mathfrak{S}$. We say that x_0 is a **singular point** of the free boundary $\Gamma(u)$, if the coincidence set $\Lambda(u)$ has vanishing $(n-1)$-dimensional density at x_0, i.e.

$$\lim_{r\to 0+} \frac{\mathcal{H}^{n-1}(\Lambda(u)\cap B'_r(x_0))}{\mathcal{H}^{n-1}(B'_r(x_0))} = 0.$$

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- Also define

$$\Sigma_\kappa(u) := \Sigma(u) \cap \Gamma_\kappa(u).$$

Singular free boundary points: example

$u(x', 0) = x_1^2 x_2^2$

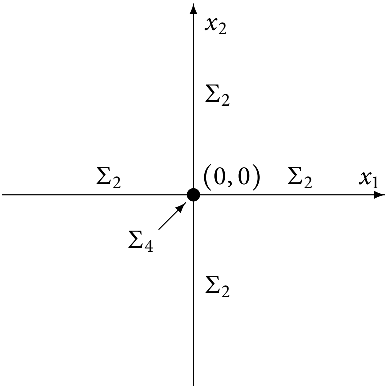

Figure: Free boundary for $u(x) = x_1^2 x_2^2 - \left(x_1^2 + x_2^2\right) x_3^2 + \frac{1}{3} x_3^4$ in $\mathbb{R}^3$ with zero thin obstacle on $\mathbb{R}^2 \times \{0\}$.

Singular free boundary points: blowups

- Any blowup u_0 at a singular point $x_0 \in \Sigma(u)$ belongs to the class $\mathfrak{P}_\kappa$ for $\kappa = N^{x_0}(0+, u)$:

$$\mathfrak{P}_\kappa = \{p_\kappa(x) \mid \Delta p_\kappa = 0,\ x \cdot \nabla p_\kappa - \kappa p_\kappa = 0,\ p_\kappa(x', 0) \geq 0\},$$

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- This implies $\kappa = 2m$, $m \in \mathbb{N}$.
- **Central question**: Are blowups unique at $x_0 \in \Sigma(u)$?
- Equivalent to Taylor's expansion:

$$u(x', x_n) = p_\kappa^{x_0}(x - x_0) + o(|x - x_0|^\kappa),$$

  with nonzero $p_\kappa^{x_0} \in \mathfrak{P}_\kappa$.

Historical development: classical obstacle problem

- Normalized solution of classical obstacle problem:

$$\Delta u = \chi_{\{u>0\}} \quad \text{in } B_1, \quad 0 \in \Gamma(u) = \partial\{u = 0\}$$

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Theorem (Taylor expansion at singular points)

At singular points one has the Taylor expansion

$$u(x) = p^{x_0}(x - x_0) + o(|x - x_0|^2)$$

where p^{x_0} is a nonnegative homogeneous quadratic polynomial with $\Delta p^{x_0} = 1$.

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Theorem (ACF monotonicity formula)

If $v_\pm \geq 0$ are continuous subharmonic functions such that $v_+ \cdot v_- = 0$, then

$$r \mapsto \Phi(r, v_\pm) := \frac{1}{r^4} \int_{B_r} \frac{|\nabla v_+|^2}{|x|^{n-2}} \int_{B_r} \frac{|\nabla v_-|^2}{|x|^{n-2}} \nearrow$$

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- Applied to $v_\pm = (\partial_e u)^\pm = \max\{\pm\partial_e u, 0\}$

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Theorem (Weiss' monotonicity formula)

If u is a solution of the classical obstacle problem, then

$$r \mapsto W(r) := \frac{1}{r^{n+2}} \int_{B_r} |\nabla u|^2 + 2u - \frac{2}{r^{n+3}} \int_{\partial B_r} u^2 \nearrow .$$

In fact,

$$\frac{d}{dr} W(r) = \frac{2}{r^{n+4}} \int_{\partial B_r} (x \cdot \nabla u - 2u)^2.$$

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Theorem (Monneau's monotonicity formula)

Let u be a solution of the classical obstacle problem and 0 is a singular free boundary point. Then the function

$$r \mapsto M(r, u, p) := \frac{1}{r^{n+3}} \int_{\partial B_r} (u-p)^2 \nearrow$$

for arbitrary nonnegative quadratic polynomial p with $\Delta p = 1$.

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- *Initial idea*: is there a Monneau type formula based on Almgen's?
- *Solution found*: there is a one-parameter family of monotonicity formulas $\{W_\kappa\}_{\kappa \geq 0}$ of Weiss type, which further generate a family of $\{M_\kappa\}_{\kappa = 2m}$ of Monneau type formulas.

Weiss type monotonicity formulas

Theorem (Weiss type monotonicity formulas)

Let $u \in \mathfrak{S}$ *and* $\kappa \geq 0$. *Then*

$$r \mapsto W_\kappa(r,u) := \frac{1}{r^{n-2+2\kappa}} \int_{B_r} |\nabla u|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r} u^2 \nearrow .$$

In fact,

$$\frac{d}{dr} W_\kappa(r,u) = \frac{2}{r^{n+2\kappa}} \int_{\partial B_r} (x \cdot \nabla u - \kappa\, u)^2 .$$

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- $W_\kappa \equiv const \iff u$ is homogeneous of degree κ.

Connection with Almgren's formula

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- Both follow from the same identities for $D'(r)$ and $H'(r)$:

$$H'(r) = \frac{n-1}{r}H(r) + 2D(r)$$
$$D'(r) = \frac{n-2}{r}D(r) + 2\int_{\partial B_r} u_\nu^2$$

Monneau type monotonicity formulas

Theorem (Monneau type monotonicity formulas)

Let $u \in \mathfrak{S}$ *with* $0 \in \Sigma_\kappa(u)$, $\kappa = 2m$, $m \in \mathbb{N}$. *Then for arbitrary* $p_\kappa \in \mathfrak{P}_\kappa$

$$r \mapsto M_\kappa(r, u, p_\kappa) := \frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} (u - p_\kappa)^2 \nearrow .$$

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- Every blowup at a singular point $0 \in \Sigma_k(u)$ is an element of $\mathfrak{P}_\kappa$.

Taylor expansion at singular points

Theorem (Taylor expansion at singular points)

Let $u \in \mathfrak{S}$. Then for any $x_0 \in \Sigma_\kappa(u)$ there exists a nonzero $p_\kappa^{x_0} \in \mathfrak{P}_\kappa$ such that

$$u(x) = p_\kappa^{x_0}(x - x_0) + o(|x - x_0|^\kappa).$$

Moreover, the mapping $x_0 \mapsto p_\kappa^{x_0}$ is continuous on $\Sigma_\kappa(u)$.

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Idea of the proof.

Assume $x_0 = 0$. Let p_κ be a blowup of u over a sequence $r_j \to 0$. Then

$$M_\kappa(r_j, u, p_\kappa) \to 0.$$

Monotonicity of $M_\kappa \quad \Rightarrow \quad M_\kappa(r, u, p_\kappa) \to 0 \quad \text{as } r \to 0.$ □

Structure of the singular set

Definition (Dimension at the singular point)

For $x_0 \in \Sigma_\kappa(u)$ denote

$$d_\kappa^{x_0} := \dim\{\xi \in \mathbb{R}^{n-1} \mid \xi \cdot \nabla_{x'} p_\kappa^{x_0} \equiv 0\},$$

which we call the **dimension** of $\Sigma_\kappa(u)$ at x_0.
For $d = 0, 1, \dots, n-2$ define

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- Note that since $p_\kappa^{x_0} \not\equiv 0$ on $\mathbb{R}^{n-1} \times \{0\}$ one has

$$0 \le d_\kappa^{x_0} \le n-2.$$

Structure of the singular set

Theorem (Structure of the singular set)

Let $u \in \mathfrak{S}$. Then every set $\Sigma_\kappa^d(u)$, $\kappa = 2m$, $m \in \mathbb{N}$, $d = 0, 1, \ldots, n-2$ is contained in a countable union of d-dimensional C^1 manifolds.

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- Implicit function theorem

Structure of the singular set: example

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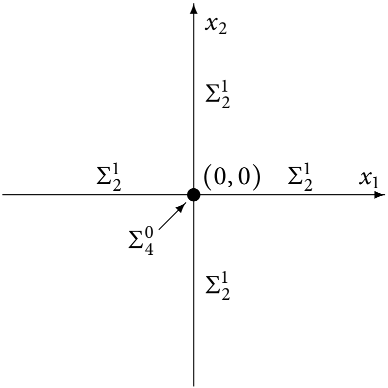

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Part II

Nonzero thin obstacle: $\varphi \neq 0$

Outline

Normalization: class $\mathfrak{S}^\varphi$

Definition

Let $\mathfrak{S}^\varphi$ be the class of solutions of the Signorini problem:

$$\Delta v = 0 \quad \text{in } B_1^+$$
$$v - \varphi \geq 0, \quad -\partial_{x_n} v \geq 0, \quad (v - \varphi)\,\partial_{x_n} v = 0 \quad \text{on } B_1'$$
$$0 \in \Gamma(v) = \partial\Lambda(v) = \partial\{v = \varphi\}.$$

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$$0 \in \Gamma(v) = \partial\Lambda(v) = \partial\{v = \varphi\}.$$

- *Rough idea:* Subtract φ from v
- *Proper way:*
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  - Define

$$u(x', x_n) := v(x', x_n) - Q(x', x_n) - (\varphi(x') - q(x'))$$

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Definition

We say $u \in \mathfrak{S}_k(M)$ iff

$$|\Delta u| \le M|x'|^{k-1} \quad \text{in } B_1^+$$
$$u \ge 0, \quad -\partial_{x_n} u \ge 0, \quad u\,\partial_{x_n} u = 0 \quad \text{on } B_1'$$
$$0 \in \Gamma(u) = \partial\Lambda(u) = \partial\{u = 0\}.$$

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- By allowing nonzero obstacles one sacrifices Almgren's frequency formula in its purest form. However, a modified version of it does hold.

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Theorem (Generalized frequency formula)

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$$r \mapsto \Phi_k(r,u) := (r + C_M r^2)\frac{d}{dr}\log\max\left\{H(r), r^{n-1+2k}\right\} \nearrow \quad \textit{for } 0 < r < r_M$$

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- Proof consists in estimating the error terms. The truncation of the growth of needed to absorb those terms.
- Most useful when $H(r) > r^{n-1+2k}$. In a sense the "precision " of the study is limited by regularity of the thin obstacle φ.

Classification of free boundary points

Definition

For $v \in \mathfrak{S}^{\varphi}$ define

$$\Gamma_{\kappa}^{(k)}(v) := \{x_0 \in \Gamma(v) \mid \Phi_k(0+, u_k^{x_0}) = n - 1 + 2\kappa\},$$

where $u_k^{x_0} \in \mathfrak{S}_k$ is obtained by properly subtracting the k-th Taylor's polynomial of φ at x_0.

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- Thus, for $\kappa < k$ we can define $\Gamma_\kappa(v) = \Gamma_\kappa^{(k)}(v)$
- The higher is the regularity of φ, the more values of κ we can study.

Generalized Weiss type monotonicity formulas

Theorem (Weiss type monotonicity formula)

Let $u \in \mathfrak{S}_k(M)$ and $\kappa \le k$. Then there exist C_M and $r_M > 0$ such that

$$W_\kappa(r,u) := \frac{1}{r^{n-2+2\kappa}} \int_{B_r} |\nabla u|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{\partial B_r} u^2$$
$$= \frac{1}{r^{n-2+2\kappa}} D(r) - \frac{\kappa}{r^{n-1+2\kappa}} H(r).$$

satisfies

$$\frac{d}{dr} W_\kappa(r) \ge -C_M \quad \text{for } 0 < r < r_M.$$

Generalized Monneau type monotonicity formulas

Theorem (Monneau type monotonicity formulas)

Let $u \in \mathfrak{S}_k(M)$ *and suppose that* $0 \in \Sigma_\kappa(u)$ *with* $\kappa = 2m < k$, $m \in \mathbb{N}$. *Then there exist* C_M *and* $r_M > 0$ *such that for any* $p_\kappa \in \mathfrak{P}_\kappa$

$$M_\kappa(r, u, p_\kappa) = \frac{1}{r^{n-1+2\kappa}} \int_{\partial B_r} (u - p_\kappa)^2$$

satisfies

$$\frac{d}{dr} M_\kappa(r, u, p_\kappa) \geq -C_M \left(1 + \|p_\kappa\|_{L^2(B_1)}\right) \quad \text{for } 0 < r < r_M.$$

Taylor expansion at singular points

Theorem (Taylor expansion at singular points)

Let $u \in \mathfrak{S}_k$ and $0 \in \Sigma_\kappa(u)$ for $\kappa = 2m < k$, $m \in \mathbb{N}$. Then there exist nonzero $p_\kappa \in \mathfrak{P}_\kappa$ such that

$$u(x) = p_\kappa(x) + o(|x|^\kappa).$$

Moreover, if $v \in \mathfrak{S}^\varphi$ with $\varphi \in C^{k,1}(B_1')$, $x_0 \in \Sigma_\kappa(v)$ and $u_k^{x_0}$ is obtained by translating to x_0, then in the Taylor expansion

$$u_k^{x_0}(x) = p_\kappa^{x_0}(x) + o(|x|^\kappa)$$

the mapping $x_0 \mapsto p_\kappa^{x_0}$ from $\Sigma_\kappa(v)$ to $\mathfrak{P}_\kappa$ is continuous.

Structure of the singular set

- For $\kappa < k$, precisely as before one defines the **dimension** $d_\kappa^{x_0}$ of $\Sigma_\kappa(v)$ at a point x_0 and denotes

$$\Sigma_\kappa^d(v) := \{x_0 \in \Sigma_\kappa(v) \mid d_\kappa^{x_0} = d\}, \quad d = 0, 1, \ldots, n-2.$$

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Theorem (Structure of the singular set)

Let $v \in \mathfrak{S}^\varphi$ with $\varphi \in C^{k,1}(B_1')$. Then every set $\Sigma_\kappa^d(v)$, $\kappa = 2m < k$, $m \in \mathbb{N}$, $d = 0, 1, \dots, n-2$ is contained in a countable union of d-dimensional C^1 manifolds.

Open problems

- It remains to study the set of *nonregular nonsingular* points, i.e. the set

$$\Gamma(u) \setminus (\Gamma_{2-\frac{1}{2}}(u) \cup \Sigma(u)) = \bigcup_{\kappa > 2-\frac{1}{2}} \Gamma_\kappa(u) \setminus \Sigma_\kappa(u).$$

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- However, the true picture may be much more complicated than that.