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On the independence of K-theory and stable rank for simple C^*-algebras

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Abstract. Jiang and Su and (independently) Elliott discovered a simple, nuclear, infinite-dimensional C^*-algebra $\mathscr{Z}$ having the same Elliott invariant as the complex numbers. For a nuclear C^*-algebra A with weakly unperforated K_*-group the Elliott invariant of $A \otimes \mathscr{Z}$ is isomorphic to that of A. Thus, any simple nuclear C^*-algebra A having a weakly unperforated K_*-group which does not absorb $\mathscr{Z}$ provides a counterexample to Elliott's conjecture that the simple nuclear C^*-algebras will be classified by the Elliott invariant. In the sequel we exhibit a separable, infinite-dimensional, stably finite instance of such a non-$\mathscr{Z}$-absorbing algebra A, and so provide a counterexample to the Elliott conjecture for the class of simple, nuclear, infinite-dimensional, stably finite, separable C^*-algebras.

1. Introduction

Elliott's classification of AF C^*-algebras ([2]) via the scaled, ordered K_0-group began what is now a widespread effort to classify nuclear C^*-algebras via the Elliott invariant. In the case of a stably finite, unital, simple C^*-algebra A this invariant consists of the group $\mathrm{K}_*A = \mathrm{K}_0A \oplus \mathrm{K}_1A$, the class of the unit of A in K_*A, an order structure on K_*A (an element $[p] \oplus x$ is positive if $[p]$ is positive in K_0A and x can be represented as the K_1-class of a unitary $u \in \mathrm{M}_l(A)$ such that uu^* is a sub-projection of p), the Choquet simplex of normalised traces $\mathrm{T}A$, and the pairing between K_0A and $\mathrm{T}A$ via evaluation. In this paper the invariant above will be denoted $\mathrm{Ell}(A)$. Let $\mathrm{sr}(A)$ be the stable rank of A, as defined by Rieffel in [9]. $\mathrm{Ell}(-)$ has been particularly successful in classifying simple C^*-algebras of stable rank one. Until now, it was not known whether this invariant would suffice for the classification of stably finite C^*-algebras of stable rank greater than one.

Recall that an ordered group (G, G^+) is said to be weakly unperforated if $x \notin G^+$ and $nx \in G^+$ for some natural number n implies that $nx = 0$. We recall that the Elliott invariant of a simple nuclear unital C^*-algebra A is isomorphic to that of $A \otimes \mathscr{Z}$ whenever K_*A is weakly unperforated ([4]). If $A \cong A \otimes \mathscr{Z}$, then we say that A is $\mathscr{Z}$-stable. Our main result is the following:

Theorem 1.1. *For each natural number $n \geqq 2$ there exists a simple, unital, nuclear, separable, infinite-dimensional, stably finite, non-$\mathscr{Z}$-stable C^*-algebra B_n such that $\mathrm{K}_* B_n$ is weakly unperforated and* $\mathrm{sr}(B_n) \in \{n+1, n+2\}$. *In particular,*

$$\mathrm{Ell}(B_n) \simeq \mathrm{Ell}(B_n \otimes \mathscr{Z}).$$

Thus, B_n and $B_n \otimes \mathscr{Z}$ constitute a counterexample to the Elliott conjecture for the class of simple, nuclear, infinite-dimensional, stably finite C^*-algebras. We note that the existence of B_n answers Question 1.5 of [4] negatively; the weak unperforation of the K_*-group does not imply that a simple, unital, nuclear, separable, infinite-dimensional C^*-algebra absorbs $\mathscr{Z}$.

The title of this paper derives from the fact that the algebra B_n of Theorem 1.1 has $\mathrm{sr}(B_n) \in \{n+1, n+2\}$ while, as we shall see, $\mathrm{sr}(B_n \otimes \mathscr{Z}) \leqq 2$. It is possible (but purely speculative) that finer invariants such as K-theory with coefficients, the semigroup of Murray-von Neumann equivalence classes of projections, or higher algebraic K-theory will recover stable rank, and so the independence of the title is only with respect to the notion of K-theory captured by $\mathrm{Ell}(-)$.

We conclude this section with an outline of the sequel. Section 2 lists several theorems from [3], which are applied in section 3 to construct the algebra B_n of Theorem 1.1. The general ideas of this latter section are also found in [3]. In section 4, B_n is shown to have the properties claimed in Theorem 1.1.

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2. Background and essential results

We begin by reviewing the definition of the generalised mapping torus, due to Elliott. Let C, D be C^*-algebras and let ϕ_0, ϕ_1 be $*$-homomorphisms from C to D. Then the generalised mapping torus of C and D with respect to ϕ_0 and ϕ_1 is

$$A := \{(c, d) \mid d \in C([0,1]; D), c \in C, d(0) = \phi_0(c), d(1) = \phi_1(c)\}.$$

We will denote A by $A(C, D, \phi_0, \phi_1)$ where appropriate for clarity. We now list (without proof) some theorems of [3] which will be used in the sequel.

Theorem 2.1 (Elliott and Villadsen [3], Theorem 2). *The index map $b_* : K_* C \to K_{1-*} SD = K_* D$ in the six term periodic sequence for the extension*

$$0 \to SD \to A \to C \to 0$$

is the difference

$$\mathrm{K}_* \phi_1 - \mathrm{K}_* \phi_0 : \mathrm{K}_* C \to \mathrm{K}_* D.$$

Thus, the six-term exact sequence may be written as the short exact sequence

$$0 \to \operatorname{Coker} b_{1-*} \to \mathrm{K}_* A \to \operatorname{Ker} b_* \to 0.$$

In particular, if b_{1-*} is surjective, then $\mathrm{K}_* A$ is isomorphic to its image, $\operatorname{Ker} b_*$, in $\mathrm{K}_* C$.

Suppose that cancellation holds for D—i.e., that cancellation holds in the semigroup of Murray-von Neumann equivalence classes of projections in D and in matrix algebras over D (equivalently, in $D \otimes \mathscr{K}$). It follows that if b_1 is surjective, so that $\mathrm{K}_0 A \subseteq \mathrm{K}_0 C$, then

$$(\mathrm{K}_0 A)^+ = (\mathrm{K}_0 C)^+ \cap \mathrm{K}_0 A.$$

The preceding conclusion also holds if cancellation is only known to hold for each pair of projections in $D \otimes \mathscr{K}$ obtained as the images under the maps ϕ_0 and ϕ_1 of a single projection in $C \otimes \mathscr{K}$. (In other words, if two such projections in $D \otimes \mathscr{K}$ have the same K_0 class then they should be equivalent, assuming as before that b_1 is surjective.)

Theorem 2.2 (Elliott and Villadsen [3], Theorem 3). *Let A_1 and A_2 be building block algebras as described above,*

$$A_i = A(C, D, \phi_0^i, \phi_1^i), \quad i = 1, 2.$$

Let there be given four maps between the fibres,

$$\gamma : C_1 \to C_2,$$
$$\delta, \delta' : D_1 \to D_2, \quad \textit{and},$$
$$\varepsilon : C_1 \to D_2,$$

such that δ, δ' and ε have mutually orthogonal images, and

$$\delta\phi_0^1 + \delta'\phi_1^1 + \varepsilon = \phi_0^2\gamma,$$
$$\delta\phi_1^1 + \delta'\phi_0^1 + \varepsilon = \phi_1^2\gamma.$$

Then there exists a unique map

$$\theta : A_1 \to A_2,$$

respecting the canonical ideals, giving rise to the map $\gamma : C_1 \to C_2$ between the quotients (or fibres at infinity), and such that for any $0 < s < 1$, if e_s denotes evaluation at s, and e_∞ the evaluation at infinity,

$$e_s\theta = \delta e_s + \delta' e_{1-s} + \varepsilon e_\infty.$$

Theorem 2.3 (Elliott and Villadsen [3], Theorem 4). *Let A_1 and A_2 be building block algebras as in Theorem 2.1. Let $\theta : A_1 \to A_2$ be a homomorphism constructed as in Theorem 2.2, from maps $\gamma : C_1 \to C_2$, $\delta, \delta' : D_1 \to D_2$, and $\varepsilon : C_1 \to D_2$.*

Let there be given a map $\beta : D_1 \to C_2$ such that the composed map $\beta\phi_1^1$ is a direct summand of the map γ, and such that the composed maps $\phi_0^2\beta$ and $\phi_1^2\beta$ are direct summands of the maps δ' and δ, respectively. Suppose that the decomposition of γ as the orthogonal sum of $\beta\phi_1^1$ and another map is such that the image of the second map is orthogonal to the image of β. (Note that this requirement is automatically satisfied if C_1, D_1, and the map $\beta\phi_1^1$ are unital.)

It follows that, for any $0 < t < 1/2$, the map $\theta : A_1 \to A_2$ is homotopic to a map $\theta_t : A_1 \to A_2$ differing from it only as follows: the map $e_\infty\theta_t$ has the direct summand βe_t instead of one of the direct summands $\beta\phi_0^1 e_\infty$ and $\beta\phi_1^1 e_\infty$ of $e_\infty\theta$, and for each $0 < s < 1$ the map $e_s\theta_t$ has either the direct summand $\phi_0^2\beta e_t$ instead of the direct summand $\phi_0^2\beta e_s$ of $e_s\theta$, or the direct summand $\phi_1^2\beta e_t$ instead of the direct summand $\phi_1^2\beta e_s$ of $e_s\theta$, or both.

Furthermore, let $\alpha : D_1 \to C_2$ be any map homotopic to β within the hereditary sub-C^*-algebra of C_2 generated by the image of β. Then the map θ_t is homotopic to a map $\theta_t' : A_1 \to A_2$ differing from θ_t only in the direct summands mentioned, and such that $e_\infty\theta_t'$ has the direct summand αe_t instead of βe_t, and for each $0 < s < 1$, $e_s\theta_t'$ has either $\phi_0^2\alpha e_t$ instead of $\phi_0^2\beta e_t$, or $\phi_1^2\alpha e_t$ instead of $\phi_1^2\beta e_t$.

Theorem 2.4 (Elliott and Villadsen [3], Theorem 5). *Let*

$$A_1 \xrightarrow{\theta_1} A_2 \xrightarrow{\theta_2} \cdots$$

be a sequence of separable building block C^*-algebras,

$$A_i = A(C_i, D_i, \phi_0^i, \phi_1^i), \quad i = 1, 2, \ldots$$

with each map $\theta_i : A_i \to A_{i+1}$ obtained by the construction of Theorem 2.2 (and thus respecting the canonical ideals). For each $i = 1, 2, \ldots$ let $\beta_i : D_i \to C_{i+1}$ be a map verifying the hypotheses of Theorem 2.3.

Suppose that for every $i = 1, 2, \ldots$, the intersection of the kernels of the boundary maps ϕ_0^i and ϕ_1^i from C_i to D_i is zero.

Suppose that, for each i, the image of each of ϕ_0^{i+1} and ϕ_1^{i+1} generates D_{i+1} as a closed two-sided ideal, and that this is in fact true for the restriction of ϕ_0^{i+1} and ϕ_1^{i+1} to the smallest direct summand of C_{i+1} containing the image of β_i. Suppose that the closed two-sided ideal of C_{i+1} generated by the image of β_i is a direct summand.

Suppose that, for each i, the maps $\delta_i' - \phi_0^i\beta_i$ and $\delta_i - \phi_1^i\beta_i$ from D_i to D_{i+1} are injective.

Suppose that, for each i, the map $\gamma_i - \beta_i\phi_1^i$ takes each non-zero direct summand of C_i into a subalgebra of C_{i+1} not contained in any proper closed two-sided ideal.

Suppose that, for each i, the map $\beta_i : D_i \to C_{i+1}$ can be deformed—inside the hereditary sub-C^*-algebra generated by its image—to a map $\alpha_i : D_i \to C_{i+1}$ with the following property: There is a direct summand of α_i, say $\bar{\alpha}_i$, such that $\bar{\alpha}_i$ is non-zero on an arbitrary given element x_i of D_i, and has image a simple sub-C^*-algebra of C_{i+1}, the closed two-sided ideal generated by which contains the image of β_i.

Choose a dense sequence (t_n) *in the open interval* $(0,1/2)$, *such that* $t_{2n} = t_{2n-1}$, $n = 1,2,\ldots$.

Choose a sequence of elements $x_3 \in D_3$, $x_5 \in D_5$, $x_7 \in D_7, \ldots$ (*necessarily non-zero*) *with the following property*: *For some countable basis for the topology of the spectrum of each of* $D_1, D_2, \ldots$, *and for some choice of non-zero element of the closed two-sided ideal associated to each of these* (*non-empty*) *open sets, under successive application of the maps* $\delta_i - \phi_1^{i+1}\beta_i$ *each one of these elements is taken into* x_j *for all* j *in some set* $S \subseteqq \{3,5,7,\ldots\}$ *such that* $\{t_j, j \in S\}$ *is dense in* $(0,1/2)$. *Choose* α_j *as above such that* $\bar{\alpha}_j(x_j) \neq 0$ *for some direct summand* $\bar{\alpha}_j$ *of* α_j *for each* $j \in \{3,5,7,\ldots\}$. *For each* $j \in \{4,6,8,\ldots\}$ *choose* α_j *with respect to the non-zero element* $(\delta'_{j-1} - \phi_0^j\beta_{j-1})(x_{j-1})$ *of* D_j. (*If* $j = 1$ *or* 2, *choose* $\alpha_j = \beta_j$.)

It follows that, if θ'_i *denotes the deformation of* θ_i *constructed in Theorem* 4, *with respect to the point* $t_i \in (0,1/2)$ *and the maps* α_i *and* β_i (*and a fixed homotopy of* β_i *to* α_i), *then the inductive limit of the sequence*

$$A_1 \xrightarrow{\theta'_1} A_2 \xrightarrow{\theta'_2} \cdots$$

is simple.

3. The construction of B_n

We now specify C^*-algebras $A_i = A_i(C_i, D_i, \phi_i^0, \phi_i^1)$ as in Theorem 2.1, and maps $\delta_i, \delta'_i, \gamma_i$, and β_i satisfying the hypotheses of Theorems 2.2, 2.3, and 2.4 in order to construct an inductive sequence

$$A_1 \xrightarrow{\theta'_1} A_2 \xrightarrow{\theta'_1} \cdots$$

whose limit will be the C^*-algebra B_n of Theorem 1.1.

Let D denote the closed unit disc in the complex numbers. Put

$$X_i = \mathrm{D}^n \times \mathrm{CP}^{n\sigma(1)} \times \mathrm{CP}^{n\sigma(2)} \times \cdots \times \mathrm{CP}^{n\sigma(i)}$$

—the $\sigma(i)$ are natural numbers to be specified—so that

$$X_{i+1} = X_i \times \mathrm{CP}^{n\sigma(i+1)},$$

and let

$$\pi_{i+1}^1 : X_{i+1} \to X_i, \quad \pi_{i+1}^2 : X_{i+1} \to \mathrm{CP}^{n\sigma(i+1)}$$

be the co-ordinate projections.

We will take $C_i = p_i(\mathrm{C}(X_i) \otimes \mathscr{K})p_i$, where p_i is a projection in $\mathrm{C}(X_i) \otimes \mathscr{K}$ to be specified. Let $D_i = C_i \otimes \mathrm{M}_{k_i \dim(p_i)}$, where k_i is a positive integer to be specified. Define maps

$$\mu_i, \nu_i : C_i \to C_i \otimes \mathrm{M}_{\dim(p_i)}$$

as follows:

$$\mu_i(a) = p_i \otimes a(x_i),$$
$$\nu_i(a) = a \otimes 1_{\dim(p_i)}.$$

For $t \in \{0,1\}$, we will take ϕ_i^t to be the direct sum of l_i^t copies of μ_i and $k_i - l_i^t$ copies of ν_i, where the l_i^t are non-negative integers to be specified. All that we mention now is that we should have $l_i^1 - l_i^0 \neq 0$. We need only specify the ϕ_i^t up to unitary equivalence, a fact we shall exploit below.

By Theorem 2.1 we have that for any $e \in \mathrm{K}_0(C_i)$,

$$\begin{aligned} b_0(e) &= (l_i^1 - l_i^0)\big(\mathrm{K}_0(\mu_i) - \mathrm{K}_0(\nu_i)\big) \\ &= (l_i^1 - l_i^0)\big(\dim(e) \cdot \mathrm{K}_0(p_i) - \dim(p_i) \cdot e\big). \end{aligned}$$

Since $l_i^1 - l_i^0 \neq 0$ and since $\mathrm{K}_0 C_i$ is a finitely generated free abelian group, we have that $\operatorname{Ker} b_0$ is the largest subgroup of $\mathrm{K}_0 C_i$ containing $\mathrm{K}_0(p_i)$ and isomorphic to the integers. In the sequel we will choose p_i so that $\mathrm{K}_0(p_i)$ in fact generates said subgroup. Since $\mathrm{K}_1 C_i = 0$ we have, by Theorem 2.1, that $\mathrm{K}_0 A_i$ is isomorphic as an ordered group to its image, $\operatorname{Ker} b_0$, in $\mathrm{K}_0 C_i$, considered as a sub ordered group. The latter (with the choice of p_i below) is isomorphic to the integers with the unique unperforated order structure, and the image of $[1_{A_i}]$ is $[p_i]$.

Let p_1 be a projection corresponding to the vector bundle

$$\theta_1 \times \xi_{n\sigma(1)},$$

over X_1, where θ_1 denotes the trivial line bundle of dimension one over D, ξ_k denotes the universal line bundle over CP^k for every natural number k, and $\sigma(1) = 1$. We now specify, inductively, the maps $\gamma_i : C_i \to C_{i+1}$. Consider first the map

$$\psi_i := \mathrm{id} \otimes 1$$

from $\mathrm{C}(X_i)$ to $\mathrm{C}(X_{i+1}) = \mathrm{C}(X_i \times \mathrm{CP}^{n\sigma(i+1)}) = \mathrm{C}(X_i) \otimes \mathrm{C}(\mathrm{CP}^{n\sigma(i+1)})$, where 1 denotes the unit of $\mathrm{C}(\mathrm{CP}^{n\sigma(i+1)})$ and id denotes the identity map from $\mathrm{C}(X_i)$ to itself.

Consider also the map

$$\beta_i' := \pi_{i+1}^{2*}(\xi_{n\sigma(i+1)}) \cdot e_{x_i}$$

from $\mathrm{C}(X_i)$ to $\mathrm{C}(X_{i+1}) \otimes \mathscr{K}$, where e_{x_i} denotes evaluation at x_i. All that we shall require of the x_i at this stage is that $\pi_{i+1}^1(x_{i+1}) = x_i$.

Now, inductively, let us take γ_i to be the map from C_i to $\mathrm{C}(X_{i+1}) \otimes \mathrm{M}_2(\mathscr{K})$ consisting of the direct sum of the following two maps: first, the restriction to $C_i \subseteq \mathrm{C}(X_i) \otimes \mathscr{K}$ of the tensor product of ψ_i with the identity map from $\mathscr{K}$ to $\mathscr{K}$, and second, the map from C_i to $\mathrm{C}(X_{i+1}) \otimes \mathrm{M}_{q_i}(\mathscr{K})$ consisting of the composition of the map ϕ_i^1 from C_i to D_i with the direct sum of q_i copies of the tensor product of the map β_i' with the identity map from $\mathscr{K}$ to $\mathscr{K}$ (restricted to $D_i \subseteqq \mathrm{C}(X_i) \otimes \mathscr{K}$), where q_i is to be specified. The induction consists in first considering the case $i = 1$ (as p_1 has already been chosen), then setting $p_2 = \gamma_1(p_1)$, so that C_2 is specified as the cut-down of $\mathrm{C}(X_2) \otimes \mathscr{K}$ by p_2, and continuing in this way.

With the maps γ_i defined as above, we have that p_i is a projection in $\mathrm{C}(X_i) \otimes \mathscr{K}$ corresponding to the vector bundle

$$\theta_1 \times \xi_n \times \sigma(2)\xi_{n\sigma(2)} \times \cdots \times \sigma(i)\xi_{n\sigma(i)},$$

where

$$\sigma(i) = \prod_{l=1}^{i-1} \bigl(\mathrm{mult}(\gamma_l) - 1\bigr).$$

Notice that by the Künneth formula (in [10], Chapter 5, for instance) the classes $[\theta_1]$, $[\xi_{n\sigma(1)}], \ldots, [\xi_{n\sigma(i)}]$ are independent in $\mathrm{K}^0(X_i)$ (we are abusing notation slightly here, using $[\xi_k]$ to represent the class of the induced bundle $\pi^*(\xi_k)$, where π is projection from X_i onto CP^k). Suppose that $[p_i] = ky$ for some $k \in \mathrm{Z}$, $y \in \mathrm{K}^0(X_i)$. It follows from independence that we have $[\xi_n] = ky'$, $y' \in \mathrm{K}^0(\mathrm{CP}^n)$, whence $k = \pm 1$. We conclude that $[p_i]$ itself generates the subgroup of rational multiples of $[p_i]$ in $\mathrm{K}^0 X_i$, as desired. Thus γ_i induces an isomorphism of ordered groups from $\operatorname{Ker} b_0$ at the i^{th} stage to $\operatorname{Ker} b_0$ at the $(i+1)^{\text{th}}$ stage.

Note that $\gamma_i - \beta_i \phi_i^1$ is non-zero, and so takes C_i into a subalgebra of C_{i+1} not contained in any proper closed two-sided ideal.

Next, we construct the maps $\delta_i, \delta_i' : D_i \to D_{i+1}$, with orthogonal images, such that

$$\delta_i \phi_i^0 + \delta_i' \phi_i^1 = \phi_{i+1}^0 \gamma_i,$$

$$\delta_i \phi_i^1 + \delta_i' \phi_i^0 = \phi_{i+1}^1 \gamma_i,$$

and $\phi_{i+1}^0 \beta_i$ and $\phi_{i+1}^1 \beta_i$ are direct summands of δ_i' and δ_i, respectively. To do this we shall have to modify ϕ_{i+1}^0 and ϕ_{i+1}^1 by inner automorphisms. This will not affect the K-theory of A.

Note that, up to unitary equivalence, we have

$$e_{x_{i+1}} \gamma_i = \mathrm{mult}(\gamma_i) e_{x_i},$$

where $\mathrm{mult}(\gamma_i)$ denotes the factor by which γ_i multiplies dimension. It follows that up to unitary equivalence

$$\begin{aligned} \mu_{i+1} \gamma_i &= p_{i+1} \otimes e_{x_{i+1}} \gamma_i \\ &= \gamma_i(p_i) \otimes \mathrm{mult}(\gamma_i) e_{x_i} \\ &= \mathrm{mult}(\gamma_i) \gamma_i (p_i \otimes e_{x_i}) \\ &= \mathrm{mult}(\gamma_i) \gamma_i \mu_i, \end{aligned}$$

and

$$\begin{aligned} \nu_{i+1}\gamma_i &= \gamma_i \otimes 1_{\dim(p_{i+1})} \\ &= \mathrm{mult}(\gamma_i)\gamma_i \otimes 1_{\dim(p_i)} \\ &= \mathrm{mult}(\gamma_i)\gamma_i\nu_i. \end{aligned}$$

Take δ_i and δ_i' to be r_i and s_i copies of γ_i, where r_i and s_i are integers to be specified. The conditions

$$\delta_i\phi_i^0 + \delta_i'\phi_i^1 = \phi_{i+1}^0\gamma_i$$

and

$$\delta_i\phi_i^1 + \delta_i'\phi_i^0 = \phi_{i+1}^1\gamma_i,$$

understood up to unitary equivalence imply that

$$r_i\gamma_i\big(l_i^t\mu_i - (k_i - l_i^t)\nu_i\big) + s_i\gamma_i\big(l_i^{1-t}\mu_i + (k_i - l_i^{1-t})\nu_i\big) = \big(l_{i+1}^t\mu_{i+1} + (k_{i+1} - l_{i+1}^t)\nu_{i+1}\big)\gamma_i,$$

again, up to unitary equivalence. As $\mathrm{K}_0(\mu_i)$ and $\mathrm{K}_0(\nu_i)$ are independent, the above equation is equivalent to the two equations

$$\begin{aligned} r_i l_i^t + s_i l_i^{1-t} &= \mathrm{mult}(\gamma_i) l_{i+1}^t, \\ (r_i + s_i)k_i &= \mathrm{mult}(\gamma_i)k_{i+1}. \end{aligned}$$

Choose $r_i = 2\,\mathrm{mult}(\gamma_i)$ and $s_i = \mathrm{mult}(\gamma_i)$, so that

$$k_{i+1} = 3k_i,$$

and

$$l_{i+1}^t = 2l_i^t + l_i^{1-t}.$$

Take $k_1 = 1$, $l_1^1 = 1$, and $l_1^0 = 0$. Then $l_i^1 - l_i^0 \neq 0$ for all i, as required.

Next, let us show that, up to unitary equivalence preserving the equations $\delta_i\phi_i^t + \delta_i'\phi_i^{1-t} = \phi_{i+1}^t\gamma_i$, $\phi_{i+1}^1\beta_i$ is a direct summand of $\delta_i = 2\,\mathrm{mult}(\gamma_i)\gamma_i$ and $\phi_{i+1}^0\beta_i$ is a direct summand of $\delta_i' = \mathrm{mult}(\gamma_i)\gamma_i$.

Note that $\phi_{i+1}^t\beta_i$ is a direct sum of l_{i+1}^t copies of $p_{i+1} \otimes \beta_i$ and $(k_{i+1} - l_{i+1}^t)$ copies of β_i, whereas δ_i and δ_i' contain, respectively, $q_i\,\mathrm{mult}(\gamma_i)$ and $2q_i\,\mathrm{mult}(\gamma_i)$ copies of β_i. Note also, that by [5], Theorem 8.1.2, a trivial projection of dimension $\dim(p_{i+1}) + \frac{1}{2}\dim X_{i+1}$ in $\mathrm{C}(X_{i+1}) \otimes \mathscr{K}$ contains a copy of p_{i+1}. Therefore, $2\dim(p_{i+1}) + 2\dim X_{i+1}$ copies of β_i contain a copy of $p_{i+1} \otimes \beta_i$ (since $2\dim(p_{i+1}) + 2\dim X_{i+1}$ copies of ξ_{i+1} contain a trivial projection of dimension $\dim(p_{i+1}) + \frac{1}{2}\dim X_{i+1}$). It follows that

$k_{i+1}(2\dim(p_{i+1}) + 2\dim X_{i+1})$ copies of β_i contain a copy of $\phi^t_{i+1}\beta_i$ when t is equal to either 0 or 1. By a copy of a given map from D_i to D_{i+1} we mean another map obtained from it by conjugating by a partial isometry in D_{i+1} with initial projection the image of the unit.

Note that

$$k_{i+1}(2\dim(p_{i+1}) + 2\dim X_{i+1}) = 6k_i \operatorname{mult}(\gamma_i)(\dim(p_i) + \dim X_i),$$

and that $k_i, \dim(p_i)$, and $\dim X_i$ have already been specified, and do not depend on q_i. It follows that, with

$$q_i \geqq 6k_i(\dim(p_i) + \dim X_i),$$

$q_i \operatorname{mult}(\gamma_i)$ copies of β_i contain a copy of $\phi^t_{i+1}\beta_i$ ($t \in \{0, 1\}$). In particular δ'_i and δ_i contain copies of $\phi^0_{i+1}\beta_i$ and $\phi^1_{i+1}\beta_i$, respectively.

With q_i as above, let us show that for each $t = 0, 1$ there exists a unitary $u_t \in D_{i+1}$ such that

$$(\operatorname{Ad} u_t)\phi^t_{i+1}\gamma_i = \phi^t_{i+1}\gamma_i,$$

with $(\operatorname{Ad} u_0)\phi^0_{i+1}\beta_i$ a direct summand of δ'_i and $(\operatorname{Ad} u_1)\phi^1_{i+1}\beta_i$ a direct summand of δ_i. In other words, for each $t = 0, 1$, we must show that the partial isometry constructed in the preceding paragraph, producing a copy of $\phi^t_{i+1}\beta_i$ inside δ_i or δ'_i may be chosen in such a way that it extends to a unitary element of D_{i+1}—which in addition commutes with the image of $\phi^t_{i+1}\gamma_i$.

Let us consider the case $t = 0$; the case $t = 1$ is similar. First note that the unit of the image of $\phi^0_{i+1}\beta_i$—the initial projection of the partial isometry transforming $\phi^0_{i+1}\beta_i$ into a direct summand of δ'_i—lies in the commutant of the image of $\phi^0_{i+1}\gamma_i$. Indeed, this projection is the image by $\phi^0_{i+1}\beta_i$ of the unit of D_i, which, by construction, is the image of the unit of C_i by ϕ^1_i. The property that $\beta_i\phi^1_i$ is a direct summand of γ_i implies that the image by $\beta_i\phi^1_i$ of the unit of C_i commutes with the image of γ_i. The unit of the image of $\phi^0_{i+1}\beta_i$ therefore commutes with the image of $\phi^0_{i+1}\gamma_i$, as desired.

The final projection of the above partial isometry also commutes with the image of $\phi^0_{i+1}\gamma_i$. Indeed, it is the unit of the image of a direct summand of δ'_i, and since D_i is unital it is the image of the unit of D_i by this direct summand; since C_i is unital and $\phi^1_i : C_i \to D_i$ is unital, the projection in question is the image of the unit of C_i by a direct summand of $\delta'_i\phi^1_i$. But $\delta'_i\phi^1_i$ is itself a direct summand of $\phi^0_{i+1}\gamma_i$, and so the projection in question is the image of the unit of C_i by a direct summand of $\phi^0_{i+1}\gamma_i$, and in particular commutes with the image of $\phi^0_{i+1}\gamma_i$.

Note that both direct summands of $\phi^0_{i+1}\gamma_i$ under consideration ($\phi^0_{i+1}\beta_i\phi^1_i$ and a copy of it) factor through the evaluation of C_i at the point x_i, and so are contained in the largest such direct summand of $\phi^0_{i+1}\gamma_i$; this largest direct summand, say π_i, is seen to exist by inspection of the construction of $\phi^0_{i+1}\gamma_i$. Since both projections under consideration (the images of $1 \in C_i$ by the two copies of $\phi^0_{i+1}\beta_i\phi^1_i$) are less than $\pi_i(1)$, to show that they are unitarily equivalent in the commutant of the image of $\phi^0_{i+1}\gamma_i$ (in D_{i+1}) it is sufficient to show that they are unitarily equivalent in the commutant of the image of π_i in $\pi_i(1)D_{i+1}\pi_i(1)$. Note that this image is isomorphic to $\mathrm{M}_{\dim(p_i)}(\mathrm{C})$. By construction, the two projections in question are Murray-von Neumann equivalent in D_{i+1} and hence in $\pi_i(1)D_{i+1}\pi_i(1)$, but all we shall use from this is that they have the same class in $\mathrm{K}^0 X_{i+1}$. Note that the dimension of these projections is $\big(k_{i+1}\dim(p_{i+1})\big)\big(k_i \dim(p_i)\big)$, and that the dimension of $\pi_i(1)$ is at least $l^0_{i+1}\big(\dim(p_{i+1})\big)^2$. Since the two projections under consideration commute with $\pi_i(C_i)$, and this is isomorphic to $\mathrm{M}_{\dim(p_i)}(\mathrm{C})$, to prove unitary equivalence in the commutant of $\pi_i(C_i)$ in $\pi_i(1)D_{i+1}\pi_i(1)$ it is sufficient to prove unitary equivalence of the product of these projections with a fixed minimal projection of $\pi_i(C_i)$, say e. Since $\mathrm{K}^0 X_{i+1}$ is torsion free, the products of the two projections under consideration with e still have the same class in $\mathrm{K}^0 X_{i+1}$. To prove that they are unitarily equivalent in $eD_{i+1}e$, it is sufficient (and necessary) to prove that both they and their complements inside e are Murray-von Neumann equivalent. Since both the cut-down projections and their complements inside e have the same class in $\mathrm{K}^0 X_{i+1}$, to prove that they (i.e., the two pairs) are equivalent it is sufficient, by [5], Theorem 8.1.5, to show that all four projections have dimension at least $\frac{1}{2}\dim X_{i+1}$ (note that $\dim X_i$ is even). Dividing the numbers above by $\dim(p_i)$ (the order of the matrix algebra), we see that the dimension of the first pair of projections is $k_{i+1}k_i\,\mathrm{mult}(\gamma_i)\dim(p_i)$, so that the dimension of the second pair of projections is at least

$$\mathrm{mult}(\gamma_i)\big(l^0_{i+1}\dim(p_{i+1}) - k_{i+1}k_i \dim(p_i)\big).$$

By construction, $\dim(p_i) = \frac{1}{2}\dim X_i$. Since $k_{i+1}k_i$ is non-zero for all i, the first inequality holds. Since l^0_{i+1}, the second inequality holds if $\mathrm{mult}(\gamma_i)$ is strictly greater than $k_{i+1}k_i$. Since $k_{i+1}k_i = 3k_i^2$, and k_i was chosen before q_i, we may modify our choice of q_i to ensure that $\mathrm{mult}(\gamma_i)$ is sufficiently large.

This shows that the two projections in D_{i+1} under consideration are unitarily equivalent by a unitary in the commutant of the image of $\phi^0_{i+1}\gamma_i$. Replacing ϕ^0_{i+1} by its composition with the corresponding inner automorphism, we may suppose that the two projections in question are equal. In other words $\phi^0_{i+1}\beta_i$ is unitarily equivalent to the cut-down of δ_i' by the projection $\phi^0_{i+1}\beta_i(1)$.

Now consider the compositions of these two maps with ϕ^1_i, i.e., $\phi^0_{i+1}\beta_i\phi^1_i$ and the cut-down of $\delta_i'\phi^1_i$ by the projection $\phi^0_{i+1}\beta_i(1)$. Since both of these maps can be viewed as the cut-down of $\phi^0_{i+1}\gamma_i$ by the same projection, they are in fact the same map.

Therefore, any unitary inside the cut-down of D_{i+1} by $\phi^0_{i+1}\beta_i(1)$ taking $\phi^0_{i+1}\beta_i$ into the cut-down of δ_i' by this projection—such a unitary is known to exist—must commute with the image of $\phi^0_{i+1}\beta_i\phi^1_i$, and hence with the image of $\phi^0_{i+1}\gamma_i$—since this commutes with the projection $\phi^0_{i+1}\beta_i(1)$. The extension of such a partial unitary to a unitary u_0 in D_{i+1} equal to one inside the complement of this projection then belongs to the commutant of the image of $\phi^0_{i+1}\gamma_i$, and transforms $\phi^0_{i+1}\beta_i$ into the cut-down of δ_i' by this projection, as desired.

Inspection of the construction will show that the maps $\delta_i' - \phi^0_i\beta_i$ and $\delta_i - \phi^1_i\beta_i$ are injective, as required in the hypotheses of Theorem 2.4.

Replacing ϕ_{i+1}^t with $(\mathrm{Ad}\, u_t)\phi_{i+1}^t$ and deforming the β_i to other point evaluations α_i which are non-zero on a given element (as we may, since X_i is connected), we have completed the construction of the desired inductive system (A_i, θ_i) satisfying the hypotheses of Theorem 2.4. Thus, the limit B_n of the inductive system with deformed finite stage maps, (A_i, θ_i'), is simple. Notice that $(\mathrm{K}_0 B_n, [1_{B_n}]) = (\mathbb{Z}, 1)$—the θ_i' are unital and $(\mathrm{K}_0 A_i, [1_{A_i}]) = (\mathbb{Z}, 1)$ for every i—and that B_n is separable, nuclear and stably finite since each of the A_i is ([1]).

4. The main result

In this section we prove Theorem 1.1 through a series of lemmas. We establish that $\mathrm{sr}(B_n) \in \{n+1, n+2\}$ (Lemma 4.1), that K_* is weakly unperforated (Lemma 4.3), and that B_n does not absorb $\mathscr{Z}$ (Lemma 4.4). Taken together, these results show that B_n is as claimed in Theorem 1.1.

Lemma 4.1.

$$\mathrm{sr}(B_n) \in \{n+1, n+2\}.$$

The proof will depend on some definitions and results which we review below.

For a unital C^*-algebra A we let

$$\mathrm{Lg}_s(A) = \{(a_1, \dots, a_s) \in A^s \mid a_1 A + \cdots + a_s A = A\}$$

for every natural number s, and recall that the stable rank of $A, \mathrm{sr}(A)$, is the least natural number s such that $\mathrm{Lg}_s(A)$ is dense in A^s. If no such natural number exists, we set $\mathrm{sr}(A) = \infty$ ([9]). Note that if (c_k, d_k) are elements of a generalised mapping torus $A(C, D, \phi_0, \phi_1)$ for $k \in \{1, 2, \dots, n\}$ such that

$$\mathrm{dist}\big((c_1, c_2, \dots, c_n), \mathrm{Lg}_n(C)\big) \geqq \delta,$$

then

$$\mathrm{dist}\big(((c_1, d_1), (c_2, d_2), \dots, (c_n, d_n)), \mathrm{Lg}_n(A)\big) \geqq \delta.$$

Indeed, one can check that

$$\|(c, d)\| := \max\Big\{\|c\|, \sup_{t \in [0,1]} \|d(t)\|\Big\}$$

defines the unique C^*-norm on $A(C, D, \phi_0, \phi_1)$. Thus, if $\mathrm{dist}(c, c') \geqq \delta$ for $c, c' \in C$, then $\mathrm{dist}\big((c, d), (c', d')\big) \geqq \delta$ for any $(c, d), (c', d') \in A(C, D, \phi_0, \phi_1)$.

For the remainder of this proof, any notation with subscript i refers, where applicable, to the corresponding object in section 3. In order to show that B_n has stable rank greater than n, we must exhibit n sequences of elements $A_i \ni a_{i,j} = \theta_{i1}'(a_{1,j}) = (c_{i,j}, d_{i,j})$, $1 \leqq j \leqq n$, $i \in \mathbb{N}$, such that

$$\operatorname{dist}\big(((c_{i,1},d_{i,1}),(c_{i,2},d_{i,2}),\dots,(c_{i,n},d_{i,n})),\operatorname{Lg}_n(A_i)\big) \geqq \delta > 0$$

for all i. From this it follows that

$$\operatorname{dist}\big((\theta'_{\infty 1}((c_{1,1},d_{1,1})),\dots,\theta'_{\infty 1}((c_{1,n},d_{1,n}))),\operatorname{Lg}_n(B_n)\big) \geqq \delta,$$

so that $\operatorname{sr}(B_n) > n$ by definition. (Here $\theta'_{\infty 1}$ denotes the inclusion of A_i into B_n.) By the definition of the norm on the A_i, it will be enough to show that

$$\operatorname{dist}\big((c_{i,1},c_{i,2},\dots,c_{i,n}),\operatorname{Lg}_n(C_i)\big) \geqq \delta > 0$$

for all i.

We now review Theorem 7 of [11]. Let $e(\cdot)$ denote the Euler class of a vector bundle. Suppose that C is a C^*-algebra of the form

$$(r+q)\big(\mathrm{C}(M\times \mathrm{D}^n)\otimes \mathscr{K}\big)(r+q),$$

where M is a smooth oriented manifold, and r and q are orthogonal projections in $\mathrm{C}(M\times \mathrm{D}^n)\otimes \mathscr{K}$ such that r corresponds to the trivial line bundle and q corresponds to a vector bundle α for which $e(\alpha)^n \neq 0$. Let $\pi : M\times \mathrm{D}^n \to \mathrm{D}^n$ be projection onto D^n, and let $f_j : \mathrm{D}^n \to \mathrm{D}$ be the j^{th} co-ordinate projection.

Theorem 4.2 (Villadsen [11], Theorem 7). *Let C, π and f_j be as above, and let $\tilde{c} = (c_1,\dots,c_n) \in C^n$ be such that $rc_jr = (f_j\circ\pi)r$ for all $1 \leqq j \leqq n$. Then,* $\operatorname{dist}\big(\tilde{c},\operatorname{Lg}_n(C)\big) \geqq 1$.

Proof of Lemma 4.1. We wish to apply Theorem 4.2 above to the algebras C_i, $i \geqq 1$. The sequel is similar to the proof of Theorem 8 in [11]. For all i, let r_i denote the sub-projection of the unit of C_i corresponding to the one-dimensional trivial sub-bundle of $\theta_1 \times \xi_n \times \cdots \times \sigma(i)\xi_{n\sigma(i)}$. Note that p_i considered as a vector bundle over X_i is the Whitney sum of r_i and a second vector bundle, say q_i, and this second vector bundle has $e(q_i)^n \neq 0$. Indeed,

$$q_i = \xi_n \times \sigma(2)\xi_{n\sigma(2)} \times \cdots \times \sigma(i)\xi_{n\sigma(i)},$$

and $e(\omega\oplus\gamma) = e(\omega)e(\gamma)$ for any two vector bundles ω and γ over a fixed base space so that

$$e(q_i)^n = e(\xi_n)^n e(\xi_{n\sigma(2)})^{n\sigma(2)}\cdots e(\xi_{n\sigma(i)})^{n\sigma(i)}.$$

(We are, as before, abusing notation slightly, using ξ_k to represent the bundle induced on X_i by ξ_k via projection from X_i onto CP^k.) Since the integral cohomology ring $\mathrm{H}^*(\mathrm{CP}^k)$ is generated by $e(\xi_k)$ with the relation $e(\xi_k)^{k+1} = 0$, we may conclude by the Künneth Theorem that $e(q_i)^n \neq 0$, as claimed. Each X_i is of the form $M_i \times \mathrm{D}^n$ for some smooth oriented manifold M_i, so the C_i have the same form as the algebra C of Theorem 4.2.

Note that for any element $c \in C_i$ there exists an element $(c,d)\in A_i$ for some suitable $d \in \mathrm{C}([0,1];D)$. Let $\pi_i : X_i \to D^n$ be the co-ordinate projection, and let $f_j : D^n \to D$ be projection onto the j^{th} co-ordinate. Let $a_{1,j} = (c_{1,j},d_{1,j})$ be elements of A_1 such that

$c_{1,j} = (f_j \circ \pi_1)r_1$, $1 \leqq j \leqq n$. For each $i \geqq 2$, put $a_{i,j} = \theta'_{i-1} \circ \theta'_{i-2} \circ \cdots \circ \theta'_1(a_{1,j})$. Write $a_{i,j} = (c_{i,j}, d_{i,j})$.

In section 3, the map γ_i was constructed as the direct sum of ψ_i and a second map. Let ψ_{i1} denote the composition $\psi_i \circ \psi_{i-1} \circ \cdots \circ \psi_1$. Note that $\psi_{i1}(r_1) = r_{i+1}$. By Theorem 2.3,

$$c_{i+1,j} = \psi_i(c_{i,j}) \oplus c'_{i+1,j},$$

where $c'_{i+1,j}$ is an element of the cut down of C_i by q_i; the deformation of θ_i to θ'_i is visible in the fibre at infinity only in the perturbation of the image of the second direct summand of γ_i—the image of ψ_i remains unchanged. Thus, by construction

$$r_{i+1}c_{i+1,j}r_{i+1} = \psi_{i1}(r_1)c_{i+1,j}\psi_{i1}(r_1) = \psi_{i1}(c_{1,j}) = (f_j \circ \pi_i)\psi_{i1}(r_1) = (f_j \circ \pi_i)r_{i+1}.$$

By Theorem 4.2 we conclude that

$$\operatorname{dist}\big((c_{i+1,1}, c_{i+1,2}, \ldots, c_{i+1,n}), \mathrm{Lg}_n(C_{i+1})\big) \geqq 1.$$

As noted above, this implies that the simple limit B_n has stable rank strictly greater than n.

We now show that $\mathrm{sr}(B_n) \leqq n+2$. Given an exact sequence $B \to A \to C$ of C^*-algebras, [9], Corollary 4.12, states that

$$\mathrm{sr}(A) \leqq \max\{\mathrm{sr}(B), \mathrm{sr}(C)+1\}.$$

Applying this formula to the exact sequence $\mathrm{S}D_i \to A_i \to C_i$ we have

$$\mathrm{sr}(A_i) \leqq \max\{\mathrm{sr}(\mathrm{S}D_i), \mathrm{sr}(C_i)+1\}.$$

It is known that

$$\mathrm{sr}\big(p\big(\mathrm{C}(X) \otimes \mathscr{K}\big)p\big) = \lceil \lceil \dim X/2 \rceil / \dim p \rceil + 1$$

whenever X a compact Hausdorff space and p is a projection in $\mathrm{C}(X) \otimes \mathscr{K}$ ([8]). Thus, $\mathrm{sr}(C_i) = \mathrm{sr}\big(p_i\big(\mathrm{C}(X_i) \otimes \mathscr{K}\big)p_i\big) = n+1$ by inspection of the dimensions of the p_i and X_i. Since $\mathrm{S}D_i$ is an ideal in $D_i \otimes \mathrm{C}([0,1])$, we have

$$\mathrm{sr}(\mathrm{S}D_i) \leqq \mathrm{sr}\big(D_i \otimes \mathrm{C}([0,1])\big) \leqq \mathrm{sr}(D_i) + 1$$

by [9], Corollary 7.2. [9], Theorem 6.1 states that

$$\mathrm{sr}\big(\mathrm{M}_n(A)\big) \leqq \big\lceil \big(\mathrm{sr}(A) - 1\big)/n \big\rceil + 1,$$

so that $\mathrm{sr}(D_i) = \mathrm{sr}(\mathrm{M}_{k_i \dim p_i} \otimes C_i) \leqq n+1$ for all i. We conclude that $\mathrm{sr}(A_i) \leqq n+2$, so that $\mathrm{sr}(B_n) \leqq n+2$ by [9], Theorem 5.1. Combining this with the fact that $\mathrm{sr}(B_n) \geqq n+1$ yields Lemma 4.1. $\square$

Lemma 4.3. *The ordered group* $\mathrm{K}_*B_n = \mathrm{K}_0B_n \oplus \mathrm{K}_1B_n$ *is weakly unperforated. Its order structure is the strict one coming from the first direct summand* $(\mathrm{K}_0B_n, \mathrm{K}_0B_n^+) = (\mathbb{Z}, \mathbb{Z}^+)$.

Proof. Since $(\mathrm{K}_0 B_n, \mathrm{K}_0 B_n^+)$ is weakly unperforated it will be enough to show that every element in $\mathrm{K}_1 B_n$ is the K_1-class of a unitary element in B_n. Since $\mathrm{K}_* B_n$ is the inductive limit of the $\mathrm{K}_* A_i$, it will suffice to prove this assertion for all A_i with i sufficiently large. By the formulas and discussion in the proof of Lemma 4.1, we know that $\mathrm{sr}\big(\mathrm{M}_{\dim p_i}(\mathrm{S}C_i)\big) = 2$ for all i sufficiently large. Assume that i is so large for the remainder of the proof.

From [1] and [9] we know that there is a bijection between elements of $\mathrm{K}_1 \mathrm{S}D_i$ and the K_1-classes of unitaries in $\mathrm{M}_3 \otimes \widetilde{\mathrm{M}_{\dim p_i}(\mathrm{S}C_i)}$. Furthermore, any unitary in this latter algebra is homotopic to a unitary in $\widetilde{\mathrm{M}_{3\dim p_i}(\mathrm{S}C_i)}$. Unitaries in $\widetilde{\mathrm{M}_{3\dim p_i}(\mathrm{S}C_i)}$ give rise to unitaries in $\mathrm{S}D_i$, since $3 \leqq k_i$ for all i. Thus, every element of $\mathrm{K}_1 \mathrm{S}D_i$ can be represented as the K_1-class of a unitary. The map $\mathrm{K}_1 \iota$ induced by the inclusion $\iota : \mathrm{S}D_i \to A_i$ is surjective (as $\mathrm{K}_1 C_i = 0$) and the desired conclusion for A_i follows from functoriality. □

Lemma 4.4. *For $n \geqq 2$, B_n and $B_n \otimes \mathscr{Z}$ are not isomorphic.*

Proof. We proceed by showing that $\mathrm{sr}(B_n \otimes \mathscr{Z}) \leqq 2$, so that $\mathrm{sr}(B_n) \neq \mathrm{sr}(B_n \otimes \mathscr{Z})$.

The algebra $\mathscr{Z}$ is an inductive limit of prime dimension drop algebras $\mathrm{I}[p_i, p_i q_i, q_i]$, $i = 1, 2, \ldots$, where $p_i \to \infty$ and $q_i \to \infty$ as $i \to \infty$ (cf. [6]). For any C^*-algebra A the algebra $\mathrm{I}[p_i, p_i q_i, q_i] \otimes A$ is a full algebra of operator fields, so by [8], Theorem 1.1, we have

$$\mathrm{sr}(\mathrm{I}[p_i, p_i q_i, q_i] \otimes A) \leqq \sup_{t \in [0,1]} \big\{\mathrm{sr}\big(A_t \otimes \mathrm{C}([0,1])\big)\big\},$$

where A_t is the fibre of $\mathrm{I}[p_i, p_i q_i, q_i] \otimes A$ at $t \in [0,1]$. Since each such fibre is one of $\mathrm{M}_{p_i}(A), \mathrm{M}_{q_i}(A)$, or $\mathrm{M}_{p_i q_i}(A)$ we may rewrite our estimate above as

$$\begin{aligned}\mathrm{sr}(\mathrm{I}[p_i, p_i q_i, q_i] \otimes A) \leqq \max\big\{&\mathrm{sr}\big(\mathrm{M}_{p_i q_i}\big(A \otimes \mathrm{C}([0,1])\big)\big),\\ &\mathrm{sr}\big(\mathrm{M}_{q_i}\big(A \otimes \mathrm{C}([0,1])\big)\big), \mathrm{sr}\big(\mathrm{M}_{p_i}\big(A \otimes \mathrm{C}([0,1])\big)\big)\big\}.\end{aligned}$$

By [9], Corollary 7.2, we have $\mathrm{sr}(A \otimes C[0,1]) \leqq \mathrm{sr}(A) + 1$. By [9], Theorem 6.1, we have that $\mathrm{sr}\big(\mathrm{M}_n(A)\big) \leqq \big\lceil \big(\mathrm{sr}(A) - 1\big)/n \big\rceil + 1$. Thus, there exists $i_0 \in \mathrm{N}$ such that $\mathrm{sr}\big(\mathrm{M}_{p_i q_i}\big(A \otimes \mathrm{C}([0,1])\big)\big), \mathrm{sr}\big(\mathrm{M}_{q_i}\big(A \otimes \mathrm{C}([0,1])\big)\big)$ and $\mathrm{sr}\big(\mathrm{M}_{p_i}\big(A \otimes \mathrm{C}([0,1])\big)\big)$ are all less than or equal to two for $i \geqq i_0$. We conclude that

$$\mathrm{sr}(\mathrm{I}[p_i, p_i q_i, q_i] \otimes A) \leqq 2$$

for all $i \geqq i_0$. Finally, $B_n \otimes \mathscr{Z}$ is an inductive limit of algebras of the form $\mathrm{I}[p_i, p_i q_i, q_i] \otimes B_n$, all but finitely many of which have stable rank less than or equal to two. By [9], Theorem 5.1, the limit $B_n \otimes \mathscr{Z}$ must have stable rank less than or equal to two, as claimed. □

Thus, we have established Theorem 1.1. In closing, we note that given two natural numbers n and m one may carry out the construction of section 3 to produce algebras B_n and B_m which, if the parameters q_i are chosen to be the same for both constructions, will have isomorphic Elliott invariants. This shows that one can produce simple, nuclear, infinite-dimensional, stably finite counterexamples to the Elliott conjecture which lie entirely outside the class of $\mathscr{Z}$ absorbing C^*-algebras. The explicit calculation of $\mathrm{Ell}(B_n)$ and $\mathrm{Ell}(B_m)$ is long and not particularly illuminating. We leave it to the reader.

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