

# Minimal dynamics and the classification of C\*-algebras

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Let  $X$  be an infinite, compact, metrizable space of finite covering dimension and  $\alpha : X \rightarrow X$  a minimal homeomorphism. We prove that the crossed product  $C(X) \rtimes_{\alpha} \mathbb{Z}$  absorbs the Jiang-Su algebra tensorially and has finite nuclear dimension. As a consequence, these algebras are determined up to isomorphism by their graded ordered K-theory under the necessary condition that their projections separate traces. This result applies, in particular, to those crossed products arising from uniquely ergodic homeomorphisms.

From its earliest days the theory of operator algebras has been entwined with dynamics, and some of the most important developments in the subject revolve around this interaction. The group-measure space construction of Murray and von Neumann provided the first examples of non-type-I factors; recently we have seen connections with orbit equivalence and associated rigidity phenomena in the remarkable works of Popa and Giordano et al. (see refs. 1 and 2, respectively). In this note we consider crossed product C\*-algebras arising from topological dynamical systems, and we announce a maximally general theorem concerning the degree to which they are determined by their graded ordered K-theory.

## Principal Theorems

**Theorem A.** Let  $\mathcal{C}$  denote the class of C\*-algebras having the following properties:

- i. every  $A \in \mathcal{C}$  has the form  $C(X) \rtimes_{\alpha} \mathbb{Z}$  for some infinite, compact, finite-dimensional, metrizable space  $X$  and minimal homeomorphism  $\alpha : X \rightarrow X$ ;
- ii. the projections of every  $A \in \mathcal{C}$  separate traces.

If  $A, B \in \mathcal{C}$  and there is a graded ordered isomorphism  $\phi : K_{*}(A) \rightarrow K_{*}(B)$ , then there is a \*-isomorphism  $\Phi : A \rightarrow B$  which induces  $\phi$ .

This result was conjectured by G. A. Elliott (21) in 1990 as part of his wider program to classify separable amenable C\*-algebras. The hypotheses of minimality for  $\alpha$ , finite-dimensionality for  $X$ , and the separation of traces by projections are all known to be necessary; the necessity of finite-dimensionality for  $X$  was established recently by Giol and Kerr (3). If one imposes unique ergodicity on  $\alpha$ , then condition ii is unnecessary. Our result is the culmination of a sequence of earlier important results due to Elliott and Evans (4), H. Lin and Phillips (5), and the second named author (6, 7). It covers irrational rotation algebras (which have many projections) as well as the (projectionless) C\*-algebras associated to minimal homeomorphisms of odd spheres considered in ref. 8.

Our proof uses deep relations between topological, algebraic, and homological regularity properties of C\*-algebras. In many cases, like for large classes of simple nuclear C\*-algebras, such conditions tend to be equivalent. In fact, *Theorem A* may be regarded as a particularly satisfactory incarnation of this phenomenon.

In general, finding useful notions of the regularity properties in question requires a substantial amount of interpretation. For example, one might ask a C\*-algebra to be finite dimensional when regarded as a noncommutative topological space. In this context we will use the nuclear dimension from ref. 9, which is based on using completely positive approximations of the identity map as noncommutative partitions of unity. It generalizes the usual covering dimension of locally compact Hausdorff spaces to the realm of nuclear C\*-algebras; its close cousin, the decomposition rank (see

ref. 10), has already proved to be a very powerful tool in efforts to further Elliott's classification program, and there is much evidence to suggest that the nuclear dimension will be similarly important.

An algebraic regularity property would typically provide enough space within the C\*-algebra to decouple certain relations (or procedures) using only inner automorphisms. A property known as  $\mathcal{Z}$ -stability (i.e., absorbing the Jiang-Su algebra  $\mathcal{Z}$  tensorially) has turned out to be extremely useful in this respect. (For example, one can show that unital, separable,  $\mathcal{Z}$ -stable C\*-algebras always satisfy Kadison's similarity question.)  $\mathcal{Z}$ -stability is also necessary for K-theoretic rigidity results akin to *Theorem A*, both in general and in the case of tracial algebras (see refs. 11 and 12, respectively). We refer the reader to refs. 13 and 14 for a complete discussion of the Jiang-Su algebra and its relevance to Elliott's program.

As for homological regularity properties, we shall mostly be interested in situations where the order structure of homological invariants such as K-theory (or the Cuntz semigroup) is determined by a suitable subset of the C\*-algebra dual like the trace space (or by the simplex of dimension functions).

The proof of *Theorem A* illustrates the interplay of all three types of such regularity conditions. We will establish  $\mathcal{Z}$ -stability in *Theorem B* and finite nuclear dimension in *Theorem C*; K-theoretic rigidity makes its appearance in the proof of *Theorem F*. The latter (which is based on a general classification result developed by the second named author in ref. 7) will yield the actual classification statement of *Theorem A*; in order to apply it, we reduce the problem to a setting similar to the one considered in ref. 5, where an abundance of projections was assumed.

The bulk of our effort is concentrated on proving the following result.

**Theorem B.** Let  $X$  be an infinite, compact, finite-dimensional metrizable space and  $\alpha : X \rightarrow X$  a minimal homeomorphism. It follows that  $(C(X) \rtimes_{\alpha} \mathbb{Z}) \otimes \mathcal{Z} \cong C(X) \rtimes_{\alpha} \mathbb{Z}$ .

Notice that we do not require projections to separate traces in this theorem. It is conjectured that *Theorem A* continues to hold in the absence of condition ii, provided that one augments the invariant  $K_{*}$  by the simplex of tracial states (here identified with the  $\alpha$ -invariant Borel probability measures on  $X$ ). We expect that *Theorem B* will prove crucial to the solution of this conjecture, too.

The techniques that we develop in proving *Theorem B* also allow us to bound the nuclear dimension of the crossed products we consider.

**Theorem C.** Let  $X$  be an infinite, compact, finite-dimensional metrizable space and  $\alpha : X \rightarrow X$  a minimal homeomorphism. It follows that the nuclear dimension of  $C(X) \rtimes_{\alpha} \mathbb{Z}$  is at most  $2\dim(X) + 1$ .

We note that the C\*-algebras of *Theorem A* have nuclear dimension at most 2, and that the same is most likely true of the algebras considered in *Theorem C*. This improved bound, however, relies on the existence of a special inductive limit decomposition for the crossed product, and this, in turn, relies on the classification theorem itself.

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**Proofs: An Outline**

In the remainder of this note we outline the proofs of *Theorems A, B, and C*; they use much of the same machinery.

Let  $X$  and  $\alpha$  be as in *Theorem A*, and let  $Y \subseteq X$  be closed. Define  $A_Y = C^*(C(X), C_0(X \setminus Y)u)$ , where  $u$  is the unitary operator implementing the action of  $\alpha: ufu^* = f \circ \alpha^{-1}$  for each  $f \in C(X)$ . It is shown in ref. 15 that if the interior of  $Y$  is not empty, then  $A_Y$  is a recursive subhomogeneous  $C^*$ -algebra (see ref. 16). These algebras have a tractable structure. In particular, the main result of ref. 17 shows that the decomposition rank of any such  $A_Y$  is at most  $\dim(X)$ . Combining this fact with an inductive limit argument, and extending the analysis of ref. 5, we arrive at the following result.

**Proposition D.** *Let  $X$  and  $\alpha$  be as in Theorem A, and let  $x_0, x_1 \in X$  have disjoint orbits. It follows that  $A_Y$  is a unital simple separable  $C^*$ -algebra of decomposition rank at most  $\dim(X)$  for  $Y = \{x_0\}, \{x_1\}, \{x_0, x_1\}$ . Applying the main result of ref. 6, we also have  $A_Y \otimes \mathcal{Z} \cong A_Y$  for these  $Y$ .*

The next step is to show that the finite sets appearing both in the definition of the nuclear dimension and in a common criterion for the property of  $\mathcal{Z}$ -stability may be replaced with finite sets whose elements are contained in one of the algebras  $A_Y$  of *Proposition D* (for some suitable  $x_0$  and  $x_1$ ).

**Proposition E.** *Let  $X$  and  $\alpha$  be as in Theorem A. Let there be given a finite subset  $\mathcal{F}$  of  $C(X) \rtimes_{\alpha} \mathbb{Z}$  and  $\epsilon > 0$ . It follows that there are  $x_0, x_1 \in X$  with disjoint orbits and a positive function  $h \in C(X)$  of norm at most one satisfying  $h(x_0) = 0, h(x_1) = 1$ , and  $\|[h, f]\| < \epsilon$  for each  $f \in \mathcal{F}$ .*

From *Proposition E* one can show easily that  $h\mathcal{F}$  is almost contained in  $A_{\{x_0\}}$  and that  $(1-h)\mathcal{F}$  is almost contained in  $A_{\{x_1\}}$ . We may therefore replace  $\mathcal{F}$  with finite subsets of  $A_{\{x_0\}}$  and  $A_{\{x_1\}}$  at small expense in norm, as desired. Pushing this technique, one can also show that if  $\{d_0, d_{1/2}, d_1\}$  is a more or less arbitrary partition of unity in  $C^*(h)$ , then for any  $a \in \mathcal{F}$  there are  $a_0 \in A_{\{x_0\}}, a_1 \in A_{\{x_1\}}$ , and  $a_{1/2} \in A_{\{x_0, x_1\}}$  such that  $a \approx a_0 + a_{1/2} + a_1$  and  $a_i \approx d_i a, i = 0, 1/2, 1$ .

Let us address *Theorem B*. It will suffice to find a unital embedding of  $\mathcal{Z}$  into  $A := C(X) \rtimes_{\alpha} \mathbb{Z}$  whose image almost commutes with a specified finite subset  $\mathcal{F}$  of  $A$ . We have  $\mathcal{Z}$ -stable subalgebras

$$A_{\{x_0\}} \supseteq A_{\{x_0, x_1\}} \subseteq A_{\{x_1\}}$$

of  $C(X) \rtimes_{\alpha} \mathbb{Z}$ , and from the comments following *Proposition E* we may assume that  $\mathcal{F}$  is a union of elements from these three

subalgebras. Each of these subalgebras admits a unital embedding of  $\mathcal{Z}$  whose image commutes with those elements of  $\mathcal{F}$  contained in the said subalgebra. Using an almost central partition of unity contained in  $C^*(h)$  (where  $h$  comes from *Proposition E*) we can patch these three embeddings of  $\mathcal{Z}$  together to prove that  $A$  admits an embedding of a  $C^*$ -algebra bundle whose base space is finite-dimensional, whose fibres are all isomorphic to  $\mathcal{Z}$ , and whose image almost commutes with  $\mathcal{F}$ . It then follows from the main result of ref. 18 that this bundle admits a unital embedding of  $\mathcal{Z}$ , giving the desired embedding of  $\mathcal{Z}$  into  $A$ .

To prove *Theorem C*, we require the following: For each finite subset  $\mathcal{F}$  of  $A$  and  $\epsilon > 0$  there are a finite-dimensional  $C^*$ -algebra  $B$ , a completely positive contraction  $\phi : A \rightarrow B$ , and a completely positive map  $\psi : B \rightarrow A$  such that  $\psi \circ \phi$  approximates the identity map on  $A$  to within  $\epsilon$  on  $\mathcal{F}$ , and  $\psi$  is a direct sum of at most  $2\dim(X) + 2$  completely positive orthogonality preserving contractions. From *Proposition D* we know that such approximations exist for the subalgebras  $A_{\{x_0\}}$  and  $A_{\{x_1\}}$ , with  $2\dim(X) + 2$  replaced by  $\dim(X) + 1$  (this is a consequence of the decomposition rank estimate in *Proposition D*). From the comments following *Proposition E*, we may assume that  $\mathcal{F}$  is contained in  $A_{\{x_0\}} \cup A_{\{x_1\}}$ . Then (without delving into details), one may sum the approximations for  $\mathcal{F} \cap A_{\{x_i\}}, i = 0, 1$ , and bootstrap one's way up to maps from and to  $A$  (as opposed to its subalgebras) using Arveson's Extension Theorem and a simple sum.

Finally, for *Theorem A*, we apply a generalization, due to Lin and Niu, of the main result of ref. 7. The algebra  $\mathcal{U}_l$  below is the tensor product of countably many copies of  $M_l(\mathbb{C})$ .

**Theorem F.** *Let  $A, B$  be unital simple separable nuclear  $C^*$ -algebras which absorb the Jiang-Su algebra  $\mathcal{Z}$  tensorially (7, 19). Suppose that for any prime  $l$ , the tensor products  $\mathcal{U}_l \otimes A$  and  $\mathcal{U}_l \otimes B$  have tracial rank zero and satisfy the Universal Coefficients Theorem. Also suppose that there is a graded ordered isomorphism  $\phi : K_*(A) \rightarrow K_*(B)$ . It follows that there is a  $*$ -isomorphism  $\Phi : A \rightarrow B$  inducing  $\phi$ .*

With  $A, B$  as in *Theorem A*, our *Theorem B* provides the  $\mathcal{Z}$ -stability hypothesis of *Theorem F*. To complete the proof of *Theorem A*, then, we must show that  $A \otimes \mathcal{U}_l$  has tracial rank zero. This result follows from the combination of the main result of ref. 20 with the arguments of Lin and Phillips from section 4 of ref. 5 (again, in combination with ref. 20). It is here that Berg's technique makes its appearance.

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